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ON THE ERGODIC SHADOWING PROPERTY THROUGH UNIFORM LIMITS

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Abstract. In this paper, we study some dynamics of the uniform limits of sequences in dynamical systems on a noncompact metric space. We show that if a sequence of homeomorphisms on a noncompact metric space has the uniform ergodic shadowing property, then the uniform limit also has the ergodic shadowing property. Then we apply this result to nonwandering maps.

1. Introduction and preliminaries

Fakhari and Ghane [2] introduced the notion of the ergodic shadowing property for a continuous map on a compact metric space which is equivalent to the map being topologically mixing and has the shadowing property. Then, they defined some kind of specification property and investigated its relation to the ergodic shadowing property.

Fedeli and Le Donne [3] studied the dynamical behaviour of the uniform limit of a sequence of continuous maps on a compact metric space satisfying topological transitivity or other related properties. Then Rego [6] proved that if a sequence of continuous maps on a compact metric space has the uniform shadowing property, then the uniform limit has also the shadowing property. Also, the author showed that if a sequence of continuous maps with the uniform shadowing property is topological transitivity, topological mixing, or nonwandering, then so is the limit. Koo and Lee showed that if a sequence of homeomorphisms on a compact metric space is uniformly expansive with the uniform shadowing property, then the uniform limit is expansive with the shadowing property, and so topologically stable(see [4, Theorem 1.2]). Moreover, some

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dynamics of the uniform limits of sequences in dynamical systems on compact metric spaces has been studied by many authors(see [7, 8]).

To present well-known notions of dynamical systems and our results about the ergodic shadowing property for the class of homeomorphisms on noncompact metric spaces, let us recall basic notions of dynamical systems which is used in this paper.

Let X be a metric space equipped with a metric d and $f : X \to X$ be a homeomorphism. For $\delta > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ of points in X is called a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. Given $\varepsilon > 0$, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ is called to be *ε-shadowed* by $x \in X$ if $d(f^i(x), x_i) < \varepsilon$ for all $i \in \mathbb{Z}$. We recall that a homeomorphism $f : X \to X$ is said to have the *shadowing property* if for every $\varepsilon > 0$ there is $\delta > 0$ such that each δ-pseudo orbit of f is ε-shadowed by some point of X. Various concepts related to the shadowing property play an important role in the qualitative theory of hyperbolic dynamical systems on compact metric spaces(see [1, 9, 10]).

We also recall the notion of the ergodic shadowing property for the class of homeomorphisms on a metric space. Given a sequence $\xi = \{x_i\}_{i\in\mathbb{Z}}$ in X and $\delta > 0$, put

$$\mathrm{Npo}(\xi, f, \delta) = \{i \in \mathbb{Z} \mid d(f(x_i), x_{i+1}) \geq \delta\}$$

and

$$\mathrm{Npo}_m(\xi, f, \delta) = \mathrm{Npo}(\xi, f, \delta) \cap [-(m-1), m-1],$$

where $[-(m-1), m-1] = \{i \in \mathbb{Z} \mid -(m-1) \leq i \leq m-1\}$. For a point $x \in X$, put

$$\mathrm{Ns}(\xi, x, f, \delta) = \{i \in \mathbb{Z} \mid d(f^i(x), x_i) \geq \delta\}$$

and

$$\mathrm{Ns}_m(\xi, x, f, \delta) = \mathrm{Ns}(\xi, x, f, \delta) \cap [-(m-1), m-1].$$

A sequence $\xi = \{x_i\}_{i\in\mathbb{Z}}$ in X is called a *δ-ergodic pseudo orbit* of f if $\mathrm{Npo}(\xi, f, \delta)$ has density zero, which means that

$$\lim_{m\to\infty} \frac{\sharp \mathrm{Npo}_m(\xi, f, \delta)}{2m-1} = 0,$$

where $\sharp$ denotes the cardinal number. A δ-ergodic pseudo orbit ξ is said to be *ε-ergodic shadowed* by a point x in X if $\lim_{m\to\infty} \frac{\sharp \mathrm{Ns}_m(\xi, x, f, \varepsilon)}{2m-1} = 0$. A homeomorphism $f : X \to X$ has the *ergodic shadowing property* if for any $\varepsilon > 0$ there is $\delta > 0$ such that any δ-ergodic pseudo orbit of f can be ε-ergodic shadowed by a point in X.

We denote by $H(X)$ the set of all homeomorphisms $X \to X$ from a metric space X to itself. We say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ of $H(X)$

has the *uniform ergodic shadowing property* if every f_n has the ergodic shadowing property and the constants δ_n are the same, i.e., $\delta_n = \delta$ for all $n \in \mathbb{N}$.

We say that a point $x \in X$ is *nonwandering* if for each neighborhood U of x in X there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $f^n(U) \cap U \neq \emptyset$. The set of all nonwandering points of f is called the nonwndering set denoted by $\Omega(f)$. We say that f is nonwandering if $\Omega(f) = X$. A *δ-chain* from x to y of length n is a finite sequence $x_0 = x, x_1, \ldots, x_n = y$ such that $d(f(x_i), x_{i+1}) < \delta$ for $i = 0, \ldots, n-1$. We say that a point $x \in X$ is *chain-recurrent* if for any $\delta > 0$ there is a δ-chain from x to itself. We denote by the set of all chain-recurrent points of f as $CR(f)$. We say that f is chain-recurrent if $CR(f)$=X. We note that if a homeomorphism $f : X \to X$ of a compact metric space X has the shadowing property, then $\Omega(f) = CR(f)$(see [1, Theorem 3.1.2]).

In this paper, we study variant shadowing properties and recurrences of the uniform limit of a sequence of homeomorphisms on a metric space. Thus, we show that if a sequence of homeomorphisms on a noncompact metric space has the uniform ergodic shadowing property, then the uniform limit has also the ergodic shadowing property. Furthermore, we apply this result to nonwandering maps. Now, we state a main result of the paper.

Theorem 1.1. *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of $H(X)$ which converges uniformly to an $f \in H(X)$. If $\{f_n\}_{n\in\mathbb{N}}$ has the uniform ergodic shadowing property, then the uniform limit f has the ergodic shadowing property.*

2. Proof of Theorem 1.1

In this section, we give a proof of our main result about the ergodic shadowing property of the uniform limit of a sequence of $H(X)$. Then we present some results related to Theorem 1.1.

First, we give a proof of Theorem 1.1.

Proof of Theorem 1.1. Fix $\varepsilon > 0$. Since $\{f_n\}_{n\in\mathbb{N}}$ has the uniform ergodic shadowing property, choose $\delta > 0$ such that for each $n \in \mathbb{N}$, we have that every δ-ergodic pseudo orbit of f_n is $\frac{\varepsilon}{2}$-ergodic shadowed by some point $y_n \in X$. We claim that every $\frac{\delta}{2}$-ergodic pseudo orbit of f is ε-ergodic shadowed by a point in X. Let $\xi = \{x_i\}_{i\in\mathbb{Z}}$ be any $\frac{\delta}{2}$-ergodic pseudo orbit of f. Since $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly to f, there exists $n_0 \in \mathbb{N}$ sufficiently large such that $d(f_n(x), f(x)) < \frac{\delta}{2}$ for each $x \in X$ and all $n \geq n_0$. Let $J_1 = \{i \in \mathbb{Z} \mid d(f(x_i), x_{i+1}) < \frac{\delta}{2}\}$. Then $J_1 = \mathbb{Z} \setminus \mathrm{N_{po}}(\xi, f, \frac{\delta}{2})$ and we have that for each $i \in J_1$

$$\begin{aligned} d(f_n(x_i), x_{i+1}) &\leq d(f_n(x_i), f(x_i)) + d(f(x_i), x_{i+1}) \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta, \text{ for all } n \geq n_0, \end{aligned}$$

which means that $\xi = \{x_i\}_{i \in \mathbb{Z}}$ is a δ-ergodic pseudo orbit of f_n for each $n \geq n_0$. From the uniform ergodic shadowing property of $\{f_n\}$, we have that any δ-ergodic pseudo orbit ξ of f_n is $\frac{\varepsilon}{2}$-ergodic shadowed by $y_n \in X$ for each $n \geq n_0$, i.e.,

$$\lim_{m \to \infty} \frac{\sharp \mathrm{Ns}_m(\xi, y_n, f_n, \frac{\varepsilon}{2})}{2m - 1} = 0, \text{ for each } n \geq n_0.$$

Say, for $n = n_0$ sufficiently large, let $J_2 = \{i \in \mathbb{Z} \mid d(f_{n_0}(y_{n_0}), x_i) < \frac{\varepsilon}{2}\}$. Fix $i \in \mathbb{Z}$. Note that $\{f_n^i\}_{n \in \mathbb{N}}$ converges f^i. On the other hand, we have that $J_2 = \mathbb{Z} \setminus \mathrm{Ns}(\xi, f_{n_0}, \frac{\varepsilon}{2})$ and for each $i \in J_2$

$$d(f^i(y_{n_0}), x_i) \leq d(f^i(y_{n_0}), f^i_{n_0}(y_{n_0})) + d(f^i_{n_0}(y_{n_0}), x_i) < \varepsilon,$$

which implies that

$$\lim_{m \to \infty} \frac{\sharp \mathrm{Ns}_m(\xi, y_{n_0}, f, \varepsilon)}{2m - 1} = 0.$$

Hence the uniform limit f has also the ergodic shadowing property. This completes the proof. □

We say that a metric space X is *s-relatively compact* if every bounded subset of X is relatively compact. For the proof of Corollary 2.4, we need some lemmas.

LEMMA 2.1. *[5, Proposition 3.8] Let X be an s-relatively compact metric space and let $f : X \to X$ be a homeomorphism. If f has the ergodic shadowing property, then f has the shadowing property.*

Rego obtained some results about chain transitivity, chain-mixing and chain recurrence for the uniform limit of a sequence of continuous maps on a compact metric space(see [6, Theorem 4.1.4]). We also obtain the similar result about chain recurrence of the uniform limit of a sequence in $H(X)$ when X is a noncompact space.

LEMMA 2.2. *Let X be a metric space and let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of homeomorphisms converging uniformly to an $f \in H(X)$. If f_n is chain-recurrent for every $n \in \mathbb{N}$, then f is chain-recurrent.*

Proof. Let $x, y \in X$ and fix $\varepsilon > 0$. Since f_n is chain-recurrent for every $n \in \mathbb{N}$, there exist finite $\frac{\varepsilon}{2}$-pseudo orbits $\{x_i^n\}_{i=0}^{k_n}$ of f_n starting on x and ending on x. Thus we can see that $\{x_i^{n_0}\}_{i=0}^{k_{n_0}}$ is an ε-pseudo orbit of f starting on x and ending on x if we take n_0 sufficiently large since

$$\begin{aligned} d(f(x_i^{n_0}), x_{i+1}^{n_0}) &\leq d(f_{n_0}(x_i^{n_0}), f(x_i^{n_0})) + d(f(x_i^{n_0}), x_{i+1}^{n_0}) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\ i = 0, \cdots, k_{n_0}. \end{aligned}$$

Hence f is chain-recurrent. □

We obtain the following result for the class of homeomorphisms on a metric space.

LEMMA 2.3. *If a homeomorphism $f : X \to X$ of a metric space has the shadowing property, then $\Omega(f) = CR(f)$.*

Proof. The proof is almost analogous to the proof of Lemma 3.4.3 in [11] and we shall omit it.

□

We obtain the following result that slightly improves Theorem D in [6] for the class of homeomorphisms on a noncompact metric space.

COROLLARY 2.4. *Let X be an s-relatively compact metric space and $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of nonwandering homeomorphisms on X which converges uniformly to an $f \in H(X)$. If $\{f_n\}_{n\in\mathbb{N}}$ has the uniform ergodic shadowing property, then the uniform limit f is nonwandering.*

Proof. Suppose that f_n is nonwandering for every $n \in \mathbb{N}$. Then f_n is chain-recurrent. From Lemma 2.2, we see that f is chain-recurrent. Furthermore, the uniform limit f has the ergodic shadowing property by Theorem 1.1. Thus f has the shadowing property by Lemma 2.1, and so $\Omega(f) = CR(f)$ by Lemma 2.3. Hence f is nonwandering. □

REMARK 2.5. There exists an expansive homeomorphism $f : X \to X$ of a compact metric space with the shadowing property such that $X \neq CR(f)$(see [1, Remark 3.1.9]).

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