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MULTI-VALUED HICKS CONTRACTIONS IN b-MENGER SPACES

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Abstract. In this work, we will generalize the notion of multivalued $(\nu, \mathcal{C})$-contraction mapping in b-Menger spaces and we shall give a new fixed point result of this type of mappings. As a consequence of our main result, we obtained the corresponding fixed point theorem in fuzzy b-metric spaces. Also, an example will be given to illustrate the main theorem in ordinary b-metric spaces.

1. Introduction

The b-Menger space is a new concept which was introduced recently by Mbarki et al. in [5] as a generalization of Menger spaces and many topological properties and fixed point theorems have been proved especially for single

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valued mappings [5, 6, 7, 4]. Hadžić in [10] presented the concept of $(\nu, \mathcal{C})$-contraction for multivalued mappings as a generalization of $\mathcal{C}$-contraction which was presented before by Hicks in [2], a fixed point theorem was proved by using the notions of H-type and weakly demicompacts functions. A generalization of the results proved by Hadžić was presented by Mihet in [8].

Our main goal on this work is to prove a new fixed point theorem for multivalued mapping satisfying $(\nu, \mathcal{C})$-contractive condition in b-Menger spaces. As an extend of these results, we also obtain the analogous fixed point theorem in the fuzzy b-metric spaces [9]. Our main results generalize and improve upon the finding of Hadžić [10] and Mihet [8].

This paper is structured as the following. In Section 2, we recall some basic definitions and topological proprieties from b-Menger spaces that will be used throughout the work. In Section 3, we will generalize the notion of $(\nu, \mathcal{C})$-contraction in the sense of multivalued mappings in b-Menger spaces which was introduced before in Menger spaces by Mihet. After that we shall prove the existence of a fixed point theorem with an extend to the fuzzy b-metric spaces. Furthermore, an example will be given in usual b-metric spaces to illustrate our result.

2. Preliminaries

We will now present some fundamental notations, definitions and topological proprieties of the b-Menger spaces. For more information, we mention [5] to the readers.

Definition 2.1. A mapping $\xi : [0, \infty) \to [0, 1]$ is called a distance distribution function if the following conditions are verified

(1) ξ is left continuous on $[0, \infty)$,
(2) ξ is non-decreasing,
(3) $\xi(0) = 0$ and $\xi(\infty) = 1$.

We represent by Δ^+ the class of all distance distribution functions. The subset $D^+ \subset \Delta^+$ is the set $D^+ = \left\{\xi \in \Delta^+ : \lim_{x\to\infty} \xi(x) = 1\right\}$.

As a specific element of D^+ is the Heavyside function ϵ_0 given as

$$\epsilon_0(x) = \begin{cases} 0, & \text{if } \ x = 0, \\ 1, & \text{if } \ x > 0. \end{cases}$$

Definition 2.2. ([11]) A mapping $\daleth : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a triangular norm (shortly t-norm) if for each $u, v, w \in [0, 1]$ the following conditions are satisfied

(1) $\daleth(u,v) = \daleth(v,u)$;
(2) $\daleth(u,\daleth(v,w)) = \daleth(\daleth(u,v),w)$;
(3) $\daleth(u,v) \leq \daleth(u,w) \quad$ for $\; v \leq w$;
(4) $\daleth(u,1) = \daleth(1,u) = u$.

Among the most used t-norms: $\daleth_M(u,v) = \min(u,v)$ and $\daleth_L(u,v) = \max(u+v-1,0)$.

Definition 2.3. Let $\daleth$ be a t-norm and $\{u_n\}_{n\in\mathbb{N}^*}$ is a sequence in $[0,1]$. Then $\daleth_{i=1}^{n} u_i$ is defined recurrently by

$$\daleth_{i=1}^{1} u_i = u_1 \quad \textit{and} \quad \daleth_{i=1}^{n} u_i = \daleth(\daleth_{i=1}^{n-1} u_i, u_n), \quad \forall n \geq 2.$$

We are able to extend $\daleth$ to enumerable infinitary operation by taking $\daleth_{i=1}^{\infty} u_i$ for any sequence $\{u_i\}_{i\in\mathbb{N}^*}$ as $\lim\limits_{n\to\infty} \daleth_{i=1}^{n} u_i$.

The operation $\daleth^n(u)$ is defined by

$$\daleth^n(u) = \begin{cases} 1, & \text{if } n = 0 \\ \daleth(\daleth^{n-1}(u),u), & \text{otherwise.} \end{cases}$$

Definition 2.4. ([1]) A t-norm $\daleth$ is said of H-type if the family $\{\daleth^n(x)\}_{n\in\mathbb{N}}$ is equi-continuous at the point $x = 1$, which means that

$$\forall \epsilon \in (0,1), \;\; \exists \lambda \in (0,1): \; t > 1-\lambda \quad \Rightarrow \quad \daleth^n(t) > 1-\epsilon \;\; \textit{for all } n \geq 1.$$

Definition 2.5. A t-norm $\daleth$ is said to be k-convergent if for all $k \in (0,1)$ we have

$$\lim_{n\to\infty} \daleth_{i=n}^{\infty}(1-k^i) = 1.$$

We should note if $\daleth$ is k-convergent then,

$$\forall \delta \in (0,1), \;\; \exists v \in \mathbb{N}: \;\; \daleth_{i=1}^{n}(1-k^{v+i}) > 1-\delta, \; \forall n \in \mathbb{N}.$$

As well, if the t-norm $\daleth$ is k-convergent then $\sup_{0\leq x<1} \daleth(x,x) = 1$.

Definition 2.6. A quadruple $(\mathcal{E}, \digamma, \daleth, s)$ is called a b-Menger space if $\mathcal{E}$ is an arbitrary set, $\digamma$ is a mapping from $\mathcal{E}\times\mathcal{E}$ into Δ^+, $\daleth$ is a t-norm and $s \geq 1$ is a real number, such that for all $a,b,c \in \mathcal{E}$ and $u,v > 0$ we have

(1) $\digamma_{a,a} = \epsilon_0$,
(2) $\digamma_{a,b} \neq \epsilon_0 \;\; if \; a \neq b$,
(3) $\digamma_{a,b} = \digamma_{b,a}$,
(4) $\digamma_{a,b}(s(u+v)) \geq \daleth(\digamma_{a,c}(u), \digamma_{c,b}(v))$.

It is obvious that a Menger space is also a b-Menger space with the constant $s = 1$. Mbarki et al. in [5] proved that if $(\mathcal{E}, \digamma, \daleth, s)$ is a b-Menger space with a continuous t-norm $\urcorner$, then $(\mathcal{E}, \digamma, \urcorner, s)$ is a Hausdorff topological space in the topology induced by the family of (ϵ, λ)-neighborhoods

$$\mathcal{N} = \{N_p(\epsilon, \lambda) : \ p \in \mathcal{E}, \ \epsilon > 0 \ \ and \ \ \lambda > 0\},$$

where

$$\mathcal{N}_p(\epsilon, \lambda) = \{q \in \mathcal{E} : \ \digamma_{p,q}(\epsilon) > 1 - \lambda\}.$$

Definition 2.7. Let $(\mathcal{E}, \digamma, \urcorner, s)$ be a b-Menger space with $\urcorner$ is a continuous t-norm, a sequence $\{u_n\}$ in $\mathcal{E}$ is

(1) Convergent to $u \in \mathcal{E}$ if for any given $\epsilon > 0$ and $\lambda > 0$ there exist $n_{\epsilon,\lambda} \in \mathbb{N}$ such that $\digamma_{u_n,u}(\lambda) > 1 - \epsilon$, whenever $n \geq n_{\epsilon,\lambda}$.
(2) A Cauchy sequence if for any $\epsilon > 0$ and $\lambda > 0$ there exist $n_{\epsilon,\lambda} \in \mathbb{N}$ such that $\digamma_{u_n,u_m}(\lambda) > 1 - \epsilon$, whenever $n, m \geq n_{\epsilon,\lambda}$.

A b-Menger space $(\mathcal{E}, \digamma, \urcorner, s)$ is complete if each Cauchy sequence in $\mathcal{E}$ is convergent to some point into $\mathcal{E}$.

Definition 2.8. A quadruple $(\mathcal{E}, R, \urcorner, s)$ is said to be a fuzzy b-metric space if $\mathcal{E}$ is an arbitrary nonempty set, $\urcorner$ is a continuous t-norm, $s \geq 1$ is a real number and R is a fuzzy set on $\mathcal{E} \times \mathcal{E} \times (0, \infty)$ such that following conditions are verified:

(1) $R(u, v, 0) = 0$,
(2) $R(u, v, r) = 1$ for all $r > 0$ if and only if $u = v$,
(3) $R(u, v, r) = R(v, u, r)$,
(4) $R(u, w, s(r + q)) \geq \urcorner(R(u, v, r), R(v, w, q))$,
(5) $R(u, v, .) : [0, \infty) \longrightarrow [0, 1]$ is left-continuous and nondecreasing for all $u, v, w \in \mathcal{E}$ and $r, q > 0$.

When we take $s = 1$ then $(\mathcal{E}, R, \urcorner, 1)$ became a fuzzy metric space in the form of Kramosil and Michalek [3].

Afterward, we assume for the b-Menger space $(\mathcal{E}, \digamma, \urcorner, s)$ that the t-norm $\urcorner$ is continuous, and we represent by $C(\mathcal{E})$ the class of all nonempty closed subsets of $\mathcal{E}$, also we consider the gauge functions among the class Γ of all mapping $\nu : [0, \infty] \to [0, \infty]$ such that $\nu(u) < u$ for all $u > 0$.

Definition 2.9. Let $(\mathcal{E}, \digamma, \urcorner, s)$ be a b-Menger space, a multi-valued mapping $f : \mathcal{E} \to C(\mathcal{E})$ is continuous if for each $\epsilon > 0$ we can found $\lambda \in (0, 1)$ such that

$$\digamma_{x,y}(\lambda) > 1 - \lambda \quad \Rightarrow \quad \forall p \in fx, \ \exists q \in fy : \ \ \digamma_{p,q}(\epsilon) > 1 - \epsilon.$$

3. Main result

Here we will prove a new fixed point theorem of multivalued $(\nu, \mathcal{C})$-contraction in b-Menger spaces, and before stating the principal result, we introduce the next definition.

Definition 3.1. Let $(\mathcal{E}, F, \urcorner, s)$ be a b-Menger space and $\nu : [0, \infty) \to [0, \infty)$. A mapping $f : \mathcal{E} \to C(\mathcal{E})$ is said to be a multi-valued $(\nu, \mathcal{C})$-contraction if for each $x, y \in \mathcal{E}$, and $u > 0$ we have

$$F_{x,y}(u) > 1 - u \quad \Rightarrow \quad \forall p \in fx \; \exists q \in fy : \quad F_{p,q}(\nu(u)) > 1 - \nu(u).$$

Theorem 3.2. *Let $(\mathcal{E}, F, \urcorner, s)$ be a complete b-Menger space and $f : \mathcal{E} \to C(\mathcal{E})$ a multi-valued $(\nu, \mathcal{C})$-contraction in which the series $\sum_{n=1}^{\infty} s^n \nu^n(u)$ is convergent for some $u > 1$ with $\nu \in \Gamma$. If $\lim_{n\to\infty} \urcorner_{i=1}^{\infty}(1 - \nu^{n+i-1}(u)) = 1$, then f admits a fixed point.*

Proof. Let take $p_0 \in \mathcal{E}$ and $p_1 \in f(p_0)$. Since $u > 1$ we obtain that $F_{p,q}(u) > 1 - u$ for each $p, q \in \mathcal{E}$, then we get $F_{p_0,p_1}(u) > 1 - u$. And by using the contractivity relation we obtain that there exists $p_2 \in f(p_1)$ such that

$$F_{p_1,p_2}(\nu(u)) > 1 - \nu(u).$$

Hence, inductively we can construct a sequence $\{p_n\}$ that satisfy

$$p_{n+1} \in f(p_n) \;\; \textit{and} \;\; F_{p_n,p_{n+1}}(\nu^n(u)) \geq 1 - \nu^n(u), \quad \forall n \in \mathbb{N}. \tag{3.1}$$

Subsequently, we show that $\{p_n\}$ is a Cauchy sequence.

Let $\epsilon > 0$ and $\lambda > 0$. Since $\lim_{n\to\infty} \urcorner_{i=1}^{\infty}(1 - \nu^{n+1-i}(u)) = 1$, there exists $n_1 \in \mathbb{N}$ such that

$$\urcorner_{i=1}^{\infty}(1 - \nu^{n+i-1}(u)) > 1 - \epsilon, \quad \forall n \geq n_1.$$

On the other hand we have that the series $\sum_{n=1}^{\infty} s^n \nu^n(u)$ is convergent, so there exists $n_2 \in \mathbb{N}$ such that

$$\sum_{n=n_2}^{\infty} s^n \nu^n(u) < \lambda.$$

We take $j = \max(n_1, n_2)$, then for each $n \geq j$ and $l \in \mathbb{N}$ we have

$$F_{p_n,p_{n+l}}(\lambda) \geq F_{p_n,p_{n+l}}\left(\sum_{i=n}^{n+l-1} s^i \nu^i(u)\right).$$

And by the b-Menger triangle inequality we get

$$F_{p_n,p_{n+l}}(\lambda) \geq \urcorner\left(F_{p_n,p_{n+1}}(s^{n-1}\nu^n(u)), F_{p_{n+1},p_{n+l}}(\sum_{i=n+1}^{n+l-1} s^{i-1}\nu^i(u))\right).$$

Continuing in this way, we obtain

$$\begin{aligned}
\digamma_{p_n,p_{n+l}}(\lambda) &\geq \daleth^l\Big(\digamma_{p_n,p_{n+1}}(s^{n-1}\nu^n(u)), \digamma_{p_{n+1},p_{n+2}}(s^{n-1}\nu^{n+1}(u)), ..., \\
&\qquad \digamma_{p_{n+l-1},p_{n+l}}(s^{n-1}\nu^{n+l-1}(u))\Big) \\
&\geq \daleth^l\Big(\digamma_{p_n,p_{n+1}}(\nu^n(u)), \digamma_{p_{n+1},p_{n+2}}(\nu^{n+1}(u)), ..., \\
&\qquad \digamma_{p_{n+l-1},p_{n+l}}(\nu^{n+l-1}(u))\Big) \\
&\geq \daleth^l\Big(1-\nu^n(u), 1-\nu^{n+1}(u), ..., 1-\nu^{n+l-1}(u)\Big) \\
&\geq \daleth_{i=1}^{\infty}\big(1-\nu^{n+i-1}(u)\big) \\
&> 1-\epsilon.
\end{aligned}$$

Finally we conclude that $\{p_n\}$ is a Cauchy sequence, and from that $\mathcal{E}$ is complete, then it follows that $\{p_n\}$ converges to some $h \in \mathcal{E}$.

It left to show that $h \in fh$. As fh is closed, then it suffice to show that $h \in \overline{fh}$, which is mean to prove that for every $\lambda > 0$ and $\epsilon > 0$ there exist $y \in fh$ such that $\digamma_{h,y}(\epsilon) > 1-\lambda$. From the condition that $\daleth$ is continuous it follow that $\sup_{0\leq a\leq 1}\daleth(a,a) = 1$, which is implies that for every $\lambda > 0$ there exists $\theta \in (0,1)$ such that

$$\daleth(1-\theta, 1-\theta) > 1-\lambda.$$

Let $\epsilon > 0$ be given such that $\frac{\epsilon}{2s} < \theta$ and $n_1' \in \mathbb{N}$ satisfying

$$\digamma_{p_n,h}(\frac{\epsilon}{2s}) > 1-\frac{\epsilon}{2s} > 1-\theta, \quad \forall n \geq n_1'.$$

Since $p_{n+1} \in f(p_n)$, we get by the $(\nu, \mathcal{C})$-contraction that there exists $y \in fh$ such that for all $n \geq n_1'$ we have

$$\begin{aligned}
\digamma_{p_{n+1},y}(\frac{\epsilon}{2s}) &> \digamma_{p_{n+1},y}\Big(\nu(\frac{\epsilon}{2s})\Big) \\
&\geq 1-\nu(\frac{\epsilon}{2s}) \\
&\geq 1-\frac{\epsilon}{2s} \\
&\geq 1-\theta.
\end{aligned}$$

From that $\lim\limits_{n\to\infty} p_{n+1} = h$, it follows that there exists $n_2' \in \mathbb{N}$ that satisfy

$$\digamma_{h,p_{n+1}}(\frac{\epsilon}{2s}) > 1-\theta, \quad \forall n \geq n_2'.$$

By taking $j' = \max(n'_1, n'_2)$, then for every $n \geq j'$ we get

$$\begin{aligned} \digamma_{h,y}(\epsilon) &\geq \urcorner\left(\digamma_{h,p_{n+1}}(\frac{\epsilon}{2s}), \digamma_{p_{n+1},y}(\frac{\epsilon}{2s})\right) \\ &\geq \urcorner(1-\theta, 1-\theta) \\ &\geq 1-\lambda. \end{aligned}$$

Therefore the theorem is proved. □

Example 3.3. Let (G, d) be a complete separable b-metric space with a constant $s = 2$, (Ω, Π, P) be a probability space and $\mathcal{E}$ the space of measurable mappings from Π to G. We assume that $(\mathcal{E}, \digamma, \urcorner_L, 2)$ is a complete b-Menger space, where

$$\digamma_{\alpha,\beta}(u) = P\left(w \in \Omega,\ d(\alpha(w), \beta(w)) < u\right) \quad for\ \alpha, \beta \in \mathcal{E}.$$

Indeed, it's obvious that $\digamma_{\alpha,\beta}$ satisfies the conditions (1), (2) and (3) of definition 6, it left to prove that the b-Menger triangular inequality hold. Since

$$\frac{1}{2}d\left(\alpha(v), \chi(v)\right) \leq d\left(\alpha(v), \beta(v)\right) + d\left(\beta(v), \chi(v)\right), \quad \forall\ \alpha, \beta, \chi \in \mathcal{E} \quad and \quad v \in \Omega,$$

it follows that

$$H \cap L \subset \left\{v \in \Omega,\ d\left(\alpha(v), \chi(v)\right) < x + y\right\},$$

where $H = \left\{v \in \Omega,\ d\left(\alpha(v), \beta(v)\right) < \frac{1}{2}x\right\}$ and $L = \left\{v \in \Omega,\ d\left(\beta(v), \chi(v)\right) < \frac{1}{2}y\right\}$.

From that $P(H \cap L) = P(H) + P(L) - P(H \cup L)$, it implies that $P(H \cap L) \geq P(H) + P(L) - 1$. Hence

$$\digamma_{\alpha,\chi}(x+y) \geq \max\left(\digamma_{\alpha,\beta}(\frac{x}{2}) + \digamma_{\beta,\chi}(\frac{y}{2}) - 1, 0\right) = \urcorner_L\left(\digamma_{\alpha,\beta}(\frac{x}{2}), \digamma_{\beta,\chi}(\frac{y}{2})\right).$$

Therefore $(\mathcal{E}, \digamma, \urcorner_L, 2)$ is a b-Menger space.

It clear that if (G, d) is complete then $(\mathcal{E}, \digamma, \urcorner_L, 2)$ is complete. Let d_2 be the function defined by

$$d_2(\alpha, \beta) = \sup\left\{u \geq 0, P(v \in \Omega,\ d(\alpha(v), \beta(v)) > u) > u\right\}, \quad \forall \alpha, \beta \in \mathcal{E}.$$

It's not hard to show that d_2 is a b-metric with the constant $s = 2$.
We consider $f : \mathcal{E} \to C(\mathcal{E})$ a multivalued mapping that verify

$$\sup_{p \in f(\alpha)} \inf_{q \in f(\beta)} d_2(p, q) < \nu\left((d_2(\alpha, \beta)\right),$$

where $\nu \in \Gamma$ is strictly increasing and the series $\sum_{n=1}^{\infty} 2^n \nu^n(u)$ is convergent. It's clear that for every $\alpha, \beta \in \mathcal{E}$ we have

$$d_2(\alpha, \beta) = \sup\left\{x > 0,\ \digamma_{\alpha,\beta}(x) < 1 - x\right\}.$$

So if we suppose that $\digamma_{\alpha,\beta}(x) > 1 - x$, then we get $d_2(\alpha, \beta) < x$. And by the monotonicity of ν we obtain that

$$\nu\left(d_2(\alpha, \beta)\right) < \nu(x).$$

Therefore,

$$\sup_{p\in f(\alpha)} \inf_{q\in f(\beta)} d_2(p,q) < \nu(x).$$

That means for every $p \in f(\alpha)$ there exists $q \in f(\beta)$ such that $d_2(p,q) < \nu(x)$, which implies $\digamma_{p,q}(\nu(x)) > 1 - \nu(x)$. Hence, by Theorem 3.2 f admits a fixed point.

Remark 3.4. It should mark that in our proof of Theorem 3.2, the condition of H-type was not necessary as in [10]. Also note that if f is $(\nu, \mathcal{C})$-contraction then f is a continuous multi-valued mapping. In fact, let $\epsilon > 0$ be given and $\lambda \in (0,1)$ be such that $\nu(\lambda) < \epsilon$, if $\digamma_{x,y}(\lambda) > 1 - \lambda$ then for each $p \in fx$ there exists $q \in fy$ such that $\digamma_{p,q}(\nu(\lambda)) > 1 - \nu(\lambda)$, which implies that $\digamma_{p,q}(\epsilon) \geq \digamma_{p,q}(\nu(\lambda)) > 1 - \nu(\lambda) > 1 - \epsilon$. Hence f is continuous.

In what follows, we give some consequences of Theorem 3.2. Taking up $s = 1$, we get the result proved by Mihet in [8].

Corollary 3.5. ([8]) *Let $(\mathcal{E}, \digamma, \daleth)$ be a complete Menger space with $\sup_{0\leq x<1} \daleth(x,x) = 1$ and $f : \mathcal{E} \to C(\mathcal{E})$ a multi-valued $(\nu, \mathcal{C})$-contraction in which the series $\sum_{n=1}^{\infty} \nu^n(u)$ is convergent for some $u > 1$. If $\lim\limits_{n\to\infty} \daleth_{i=1}^{\infty}(1 - \nu^{n+i-1}(u)) = 1$, then f admits a fixed point.*

If we take $\nu(u) = ku$ with $u > 0$ and $k \in (0,1)$, then a $(\nu, \mathcal{C})$-contraction becomes a $\mathcal{C}$-contraction, i.e., for any $x, y \in \mathcal{E}$ and $u > 0$ we have:

$$\digamma_{x,y}(u) > 1 - u \quad \Rightarrow \quad \forall p \in fx,\ \exists q \in fy\ : \digamma_{p,q}(ku) > 1 - ku.$$

Corollary 3.6. *Let $(\mathcal{E}, \digamma, \daleth, s)$ be a complete b-Menger space and $f : \mathcal{E} \to C(\mathcal{E})$ a multivalued $\mathcal{C}$-contraction with $k \in (0, \frac{1}{s})$. If $\daleth$ is k-convergent then there exists $h \in \mathcal{E}$ such that $h \in fh$.*

Since the condition $\digamma_{p,q}(\infty) = 1$ have been not used in the proof of Theorem 3.2, we give the corresponding result in fuzzy b-metric spaces where $\digamma_{x,y}(u) = R(x,y,u)$ for all $u > 0$.

Corollary 3.7. *Let $(\mathcal{E}, R, \daleth, s)$ be a complete fuzzy b-metric space with a continuous t-norm $\daleth$ and $f : \mathcal{E} \to C(\mathcal{E})$ be a multi-valued $(\nu, \mathcal{C})$-contraction in which the series $\sum_{n=1}^{\infty} s^n \nu^n(u)$ is convergent for some $u > 1$ with $\nu \in \Gamma$. If $\lim\limits_{n\to\infty} \daleth_{i=1}^{\infty}(1 - \nu^{n+i-1}(u)) = 1$ for all $u > 0$, then f admits a fixed point.*

4. Conclusion

In this work, we defined the multi-valued $(\nu, \mathcal{C})$-contraction mapping and proved a fixed point theorem in b-Menger space which is a recent space of the literature. As a consequence of our result, we obtained Hicks's theorem for multivalued $\mathcal{C}$-contraction in b-Menger spaces by modifying the assumptions on the constant $k \in (0, \frac{1}{s})$ with the extended version on the b-fuzzy metric spaces. Also an example was presented in a particular b-metric space to support the results thus obtained. The results presented develop and generalize in a sense the fixed point theorems for multi-valued $(\nu, \mathcal{C})$-contraction proved by Hadžić and Mihet.

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