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TAYLORS SERIES IN TERMS OF THE MODIFIED CONFORMABLE FRACTIONAL DERIVATIVE WITH APPLICATIONS

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Abstract. This study depends on the modified conformable fractional derivative definition to generalize and proves some theorems of the classical power series into the fractional power series. Furthermore, a comprehensive formulation of the generalized Taylor's series is derived within this context. As a result, a new technique is introduced for the fractional power series. The efficacy of this new technique has been substantiated in solving some fractional differential equations.

1. Introduction

The field of study known as fractional calculus has garnered a significant amount of interest in recent years as a result of its capacity to offer explanations that are both more accurate and comprehensive descriptions of various

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phenomena in science and engineering. The creation of effective and trustworthy methods of solving fractional differential equations (FDEs) is one subject that has recently attracted much attention. The Riemann-Liouville and Caputo fractional derivatives are this field's most often used definitions. These definitions and their characteristics are referred to in [3, 7, 9, 10, 12, 13].

Recently, Khalil and his co-authors provided a new definition for both fractional derivative and integral, known as Conformable fractional derivative (CFD) and Conformable fractional Integral (CFI) [8]. These new definitions have satisfied some theorems such as the Mean Value Theorem, Rolle Theorem and all linear and non-linear derivative rules such as product, quotient and chain. In the same vein, Abdeljawad organized and generalized the core definitions and notions of the CFD and CFI [1]. In addition to this, he outlined the definition of the fractional Laplace transform, established a Taylor power series represented by utilizing the CFD, and proved related theorems and results.

The conformable fractional derivative has garnered considerable attention from researchers due to its versatile applications, Notably, Abu Hammad and Khalil [2] introduced the concept of conformable fractional Fourier series, utilizing it to solve specific partial fractional differential equations. Additionally, Gonzalez et al. [6] generalized significant results of classical power series to fractional power series using the Conformable Fractional Power Series. Furthermore, Syouri [14] proposed a conformable method for fractional differential transformations and established the proof for basic properties of differintegrals. Simultaneously, Unal and Gokdogan [16] defined a solution for Conformable Fractional Ordinary Differential Equations using the Differential Transform Method.

In the same context, some properties have not been defined. These properties are commutative properties, and the derivatives of higher order do not line up well with the derivatives of sequential order. To solve these problems, El-Ajou modified the CFD and re-defined the definition as "Modified Conformable Fractional Derivative (MCFD)" [4]. The following section provides some basic definitions and properties related to the MCFD.

The power series has applications in various scientific fields, including computational science, physics, chemistry, etc. By employing power expansions, scientists can perform approximate studies of complex systems, facilitating easier analysis and interpretation of results. Furthermore, fractional power series (FPS) has become a fundamental tool in the study of elementary functions, particularly in the fractional calculus approach, providing a robust and efficient means of tackling intricate problems in diverse scientific domains, as many authors generalize some theorems related to the classical power series

(CPS) into the FPS. Among them are El-Ajou et al. [5] provided a new result for generalizing fractional power series along with the Caputo definition. While Odibat and Shawagfeh [11] have presented a new generalized Taylor's formula. Trujillo et al.[15] have introduced the generalized Taylor's formula along with the Riemann-Liouville fractional derivative. Recently, Abdeljawad [1] has formulated a novel generalized Taylor's formula utilizing CFD, which is presented as follows:

$$f(h)=\sum_{k=0}^{\infty}\frac{(T_{\alpha}^{h_0}f)^k(h_0)}{(k)!\alpha^k}(h-h_0)^{k\alpha}, \tag{1.1}$$

where $0<\alpha\leq 1,\ h_0\leq h<h_0+R$ and $R>0$

This study introduces a new comprehensive approach to FPS by employing MCFD. Some important theorems related to the CPS will be constructed and generalized for the new FPS using MCFD. This novel technique is applied to obtain solutions for some FDEs. Additionally, the derived generalized Taylor's formula is applicable for values of β within the range of $m-1<\beta\leq m$, where $\beta\in\chi_{\alpha}$.

This research is organized into different sections. In the next section, we will present some of the essential definitions and theorems that will be utilized throughout this study. In the third section, some theorems related to the FPS are mentioned and proved. The last section provides a series of solutions for FDEs using the new technique to demonstrate the proposed method's practical applications and efficiency.

2. Preliminaries

This section provides all the important definitions and theorems used in this study. See [1, 4, 5, 8] for more details and proofs.

2.1. Conformable Fractional Derivative and Integral.

Definition 2.1. ([1]) The CFD of order $\alpha\in(m-1,m]$ starting from a of a function $f:[a,\infty)\to\mathbb{R}$ is defined by

$$\left(T_a^{\alpha}f\right)(h)=\lim_{\epsilon\to 0}\frac{f^{(m-1)}\left(h+\epsilon(h-a)^{1-\alpha}\right)-f^{(m-1)}(h)}{\epsilon},\ h>x \tag{2.1}$$

$\left(T_a^{\alpha}f\right)(a)=\lim_{t\to a^+}\left(T_a^{\alpha}f\right)(h)$ provided the limits exist and $f(h)$ is $(m-1)-$differentiable at $h>a$.

Theorem 2.2. ([1, 8]) *Let $\alpha\in(m-1,m], m\in\mathbb{N}$, λ is a constant. Then,*

(1) $T_a^{\alpha}(f(h)+g(h))=T_a^{\alpha}f(h)+T_a^{\alpha}g(h)$.

(2) $T_a^\alpha(\lambda f(h)) = \lambda T_a^\alpha f(h)$.
(3) $T_a^\alpha f(h) = (h-a)^{m-\alpha} f^{(m)}(h)$.
(4) $T_a^\alpha f(\lambda) = 0$.
(5) $T_a^\alpha (h-a)^\gamma = \prod_{k=0}^{m-1}(\gamma-\alpha)(h-a)^{\gamma-\alpha}$.

Definition 2.3. ([1]) The CFI of order $\alpha \in (m-1, m]$ starting from a of a function $f : [a, \infty) \to \mathbb{R}$ is defined by

$$I_a^\alpha f(h) = \frac{1}{(m-1)!}\int_a^h \frac{(h-x)^{m-1} f(x)}{(x-a)^{m-\alpha}} dx, \quad \alpha > 0, \quad h > a \tag{2.2}$$
$$I_a^0 f(x) = f(x)$$

Theorem 2.4. ([1]) *If* $\alpha \in (m-1, m], m \in \mathbb{N}, \lambda$ *is a constant and* $\alpha+\gamma-m > -1$, *then*

(1) $I_a^\alpha (h-a)^\gamma = \frac{\Gamma(1+\gamma+\alpha-m)}{\Gamma(1+\gamma+\alpha)}(h-a)^{\gamma+\alpha}$,
(2) $I_a^\alpha(\lambda) = \lambda\frac{\Gamma(1+\alpha-m)}{\Gamma(1+\alpha)}(h-a)^\alpha$.

Lemma 2.5. ([1, 8]) *If* $\alpha \in (m-1, m], m \in \mathbb{N}$ *and* $f : [a, \infty) \to \mathbb{R}$ *is* $(m-1)$-*differentiable, then*

(1) $T_a^\alpha I_a^\alpha f(h) = f(h)$,
(2) $I_a^\alpha T_a^\alpha f(h) = f(h) - \sum_{k=0}^{m-1} f^{(k)}(a)\frac{(h-a)^k}{k!}, \quad h > a$.

Remark 2.6. ([1, 4]) For $\alpha, \beta \in (m-1, m]$ and the function $f : [a, \infty) \to \mathbb{R}$,

(1) the Commutative Property do not satisfy by the CFDs,
that is, $T_a^\alpha T_a^\beta \neq T_a^\beta T_a^\alpha$,
(2) the Commutative Property do not satisfy by the CFIs,
that is, $I_a^\alpha I_a^\beta \neq I_a^\beta I_a^\alpha$.

2.2. **Modified Conformable Fractional Derivative and Integral.** In the CFD and CFI properties, $T^\alpha T^\beta \neq T^\beta T^\alpha$ and $I^\alpha I^\beta \neq I^\beta I^\alpha$, where T^α and I^α are CFD and CFI, respectively. Also, $T^\beta \neq T^{n\alpha}$ and $I^\beta \neq I^{n\alpha}$, where $\beta = n\alpha$ for some $n \in \mathbb{N}$.

To modify the statement as mentioned above, El-Ajou [4] modified the definition of the CFD and CFI, defined as follows:

Definition 2.7. ([4]) Let $\langle \alpha : 0 < \alpha \leqslant 1\rangle$ be a cyclic subgroup of $(\mathbb{R}, +)$ generated by α. The real number β is said to be of the class α if $\beta \in \langle \alpha : 0 < \alpha \leqslant 1\rangle$.

Note that if $\beta \in (m-1, m], m \in \mathbb{N}$, then we have

$$\beta \in \left\langle \alpha : \frac{m-1}{m} < \alpha \leqslant 1, \alpha = \frac{\beta}{m} \right\rangle = \chi_\alpha,$$

which is the main idea of the MCFD.

Definition 2.8. ([4]) The MCFD of order $\beta \in \chi_\alpha$ of a function $f : [a, \infty) \to \mathbb{R}$ is defined by

$$\mathcal{T}_a^\beta f(h) = f^{(\beta)}(h) = \lim_{\epsilon \to 0} \frac{f^{((m-1)\alpha)}\left(h + \epsilon (h-a)^{1-\alpha}\right) - f^{((m-1)\alpha)}(h)}{\epsilon}, h > a, \tag{2.3}$$

$\mathcal{T}_a^\beta f(a) = \lim_{h \to a^+} \mathcal{T}_a^\beta f(h)$, provided the limits exist and $f(h)$ is $(m-1)\alpha$-differentiable at $h > a$.

Definition 2.9. ([4]) The n-sequential MCFD of order $\beta \in \chi_\alpha$ of a function $f : [a, \infty) \to \mathbb{R}$ is defined by:

$$\mathcal{T}_a^{n\beta} f(h) = f^{(n\beta)}(h) = \lim_{\epsilon \to 0} \frac{f^{((nm-1)\alpha)}\left(h + \epsilon (h-a)^{1-\alpha}\right) - f^{((nm-1)\alpha)}(h)}{\epsilon}, h > a \tag{2.4}$$

$\mathcal{T}_a^{n\beta} f(a) = \lim_{h \to a^+} \mathcal{T}_a^{n\beta} f(h)$, provided the limits exist and $f(h)$ is $(nm-1)\alpha$ - differentiable at $h > a$.

Lemma 2.10. ([4]) *Let $\beta \in \chi_\alpha$ and $\zeta \in \mathbb{R}$. Then*

(1) $\mathcal{T}_a^\beta f(h) = (h-a)^{1-\alpha} \frac{d}{dt} \left(\mathcal{T}_a^{(m-1)\alpha} f(h) \right)$,

(2) $\mathcal{T}_a^{n\beta} f(h) = (h-a)^{1-\alpha} \frac{d}{dt} \left(\mathcal{T}_a^{(nm-1)\alpha} f(h) \right)$,

(3) $\mathcal{T}_a^\beta (h-a)^\zeta = \prod_{k=0}^{m-1} (\zeta - k\alpha)(h-a)^{\zeta - \beta}$,

(4) $\mathcal{T}_a^{n\beta} (h-a)^\zeta = \prod_{k=0}^{nm-1} (\zeta - k\alpha)(h-a)^{\zeta - n\beta}$.

Remark 2.11. ([4]) If β and $\mu \in \chi_\alpha$, then

(1) $\mathcal{T}_a^\beta \mathcal{T}_a^\mu = \mathcal{T}_a^{\mu+\beta} = \mathcal{T}_a^\mu \mathcal{T}_a^\beta$,

(2) $\mathcal{T}_a^{n\beta} = \mathcal{T}_a^\beta \mathcal{T}_a^\beta \dots \mathcal{T}_a^\beta (n\text{-times }) = \mathcal{T}_a^\alpha \mathcal{T}_a^\alpha \dots \mathcal{T}_a^\alpha (nm\text{-times })$.

Definition 2.12. ([4]) A real function $f(h), h > a$ is said to be in the space $C_\mu, \mu \in \mathbb{R}$ if there exists a real number $\kappa > \mu$, such that $f(h) = (h-a)^\kappa f_1(h)$, where $f_1(h) \in C[a, \infty)$, which is the main idea of the MCFI.

Definition 2.13. ([4]) The MCFI of order $\beta \in \chi_\alpha$ of a function $f(h) \in C_\mu, \mu \geqslant -\alpha, h > a$ is defined by

$$J_a^\beta f(h) = \frac{1}{\alpha^{m-1}(m-1)!}\int_a^h \frac{((h-a)^\alpha - (y-a)^\alpha)^{m-1} f(y)}{(y-a)^{1-\alpha}} dy, \tag{2.5}$$
$$J_a^0 f(y) = f(y).$$

Definition 2.14. ([4]) The n-sequential MCFI of order $\beta \in \chi_\alpha$ of a function $f(h) \in C_\mu$, $\mu \geqslant -\alpha$, $h > a$ is defined by

$$J_a^{n\beta} f(h) = J_a^{nm\alpha} f(h)$$
$$= \frac{1}{\alpha^{(nm-1)}(nm-1)!}\int_x^h \frac{((h-a)^\alpha - (y-a)^\alpha)^{(nm-1)} f(y)}{(y-a)^{1-\alpha}} dy. \tag{2.6}$$

Lemma 2.15. *For $\beta \in \chi_\alpha$, $n \in \mathbb{N}$, $f \in C_\mu$, $\mu \geqslant -\alpha$ and $h > a$, then*

(1) $J_a^\beta (h-a)^\zeta = \dfrac{(h-a)^{\zeta+\beta}}{\prod_{k=1}^{m}(\zeta + k\alpha)}$,

(2) $J_a^{n\beta} (h-a)^\zeta = \dfrac{(h-a)^{\zeta+n\beta}}{\prod_{k=1}^{nm}(\zeta + k\alpha)}$,

(3) $\mathcal{T}_a^{n\beta} J_a^{n\beta} f(h) = f(h)$,

(4) $J_a^{n\beta} \mathcal{T}_a^{n\beta} f(h) = f(h) - \sum_{r=0}^{nm-1} f^{(r\alpha)}(a^+)\dfrac{(h-a)^{r\alpha}}{r!\alpha^r}$.

Remark 2.16. ([4]) Let $\beta, \mu \in \chi_\alpha$. Then

(1) $J_a^\beta J_a^\mu = J_a^{\beta+\mu} = J_a^\mu J_a^\beta$,
(2) $J_a^{n\beta} = J_a^\beta J_a^\beta \dots J_a^\beta (n-$ times).

2.3. **Fractional power series.**

Definition 2.17. ([5]) A power series expansion of the form

$$\sum_{i=0}^{\infty} c_i (h-h_0)^{i\beta} = c_0 + c_1(h-h_0)^\beta + c_2(h-h_0)^{2\beta} + c_3(h-h_0)^{3\beta} + \dots, \tag{2.7}$$

where $m-1 < \beta \leq m,\ h > h_0 \geq 0$ is called a FPS about h_0 and c_is are the coefficients of the series.

For a special case, when $h_0 = 0$ the expansion $\sum\limits_{i=0}^{\infty} c_i h^{i\beta}$ is called a fractional Maclaurin series. The FPS representation of Eq.(2.7) is always convergent when $h = h_0$.

Theorem 2.18. ([5]) *Consider the FPS* $\sum\limits_{i=0}^{\infty} c_i h^{i\beta}$, $h > 0$. *We have the following two cases.*

(1) *If the FPS* $\sum\limits_{i=0}^{\infty} c_i h^{i\beta}$ *converges when* $h = v > 0$, *then it's converges absolutely for all* h *with* $0 \leq h < v$.
(2) *If the FPS* $\sum\limits_{i=0}^{\infty} c_i h^{i\beta}$ *diverges when* $h = w > 0$, *then it's diverges absolutely for all* h *with* $h > w$.

Theorem 2.19. ([5]) *For the FPS* $\sum\limits_{i=0}^{\infty} c_i h^{i\beta}$, *there are three possibilities.*

(1) *The series converges only when* $h = 0$.
(2) *The series converges for each* $h \geq 0$.
(3) *The series converges for* $0 \leq h < R$ *and diverges for* $h > R$ *where* R *is a positive real number.*

Note: The number R is called the radius of convergence of the FPS. For that in case (1), the radius of convergence is defined as $R = 0$, but in case (2), it is defined as $R = \infty$.

3. Main results

In this section, we introduce a new technique for the FPS to prove and generalize Taylor's series and some theorems of the CPS to the FPS by depending on the MCFD definition.

Theorem 3.1. *Suppose that* $f(h) = \sum\limits_{i=0}^{\infty} c_i (h - h_0)^{i\beta}$ *is a FPS function representation at* h_0, *where* $m - 1 < \beta \leq m$, $0 \leq h_0 \leq h < h_0 + R$ *and* $R > 0$ *is a radius of convergence, then for* $\beta \in \chi_\alpha, n \in \mathbb{N}, f \in C_\mu, \mu \geqslant -\alpha$ *and* $h > h_0$, *we have*

$$\mathcal{T}_{h_0}^{n\beta} f(h) = \sum_{i=1}^{\infty} c_i \left(\prod_{k=0}^{nm-1} \left(i\beta - \frac{k\beta}{m} \right) (h - h_0)^{(i-n)\beta} \right), \tag{3.1}$$

$$J_{h_0}^{n\beta} f(h) = \sum_{i=0}^{\infty} c_i \left(\frac{(h - h_0)^{(i+n)\beta}}{\prod_{k=0}^{nm-1} \left(i\beta + \frac{k\beta}{m} \right)} \right). \tag{3.2}$$

Proof. Let $f(h)=\sum\limits_{i=0}^{\infty}c_i(h-h_0)^{i\beta}$, for $m-1<\beta\leq m,\ \beta\in\chi_\alpha,\ 0\leq h_0\leq h<h_0+R$ and $R>0$ be a radius of convergence. Then, by Lemma 2.10,

$$\begin{aligned}\mathcal{T}_{h_0}^{n\beta}f(h)&=\mathcal{T}_{h_0}^{n\beta}\sum_{i=0}^{\infty}c_i(h-h_0)^{i\beta}\\&=\sum_{i=0}^{\infty}c_i\mathcal{T}_{h_0}^{n\beta}\left((h-h_0)^{i\beta}\right)\\&=\sum_{i=1}^{\infty}c_i\left(\prod_{k=0}^{nm-1}(i\beta-\frac{k\beta}{m})(h-h_0)^{(i-n)\beta}\right).\end{aligned}$$

For the another part, by Lemma 2.15, we can defined by

$$\begin{aligned}J_{h_0}^{n\beta}f(h)&=J_{h_0}^{n\beta}\sum_{i=0}^{\infty}c_i(h-h_0)^{i\beta}\\&=\sum_{i=0}^{\infty}c_iJ_{h_0}^{n\beta}\left((h-h_0)^{i\beta}\right)\\&=\sum_{i=0}^{\infty}c_i\left(\frac{(h-h_0)^{(i+n)\beta}}{\prod_{k=0}^{nm-1}\left(i\beta+\dfrac{k\beta}{m}\right)}\right).\end{aligned}$$

□

Theorem 3.2. *For $m-1<\beta\leq m,\ \beta\in\chi_\alpha$. Suppose that a FPS function f representation at h_0 have the form*

$$f(h)=\sum_{i=0}^{\infty}c_i(h-h_0)^{i\beta},\tag{3.3}$$

where $0\leq h_0\leq h<h_0+R$ and $R>0$ be a radius of convergence. Then the coefficients c_n defined by

$$c_n=\frac{\mathcal{T}_{h_0}^{n\beta}f(h_0)}{\prod\limits_{k=0}^{(nm)-1}\left(n\beta-\frac{k\beta}{m}\right)}.$$

Proof. Assume that $f(h)=\sum\limits_{i=0}^{\infty}c_i(h-h_0)^{i\beta}$ is an arbitrary function. If we put $h=h_0$ into Eq. (3.3), then we get $c_0=f(h_0)$.

Now by using Theorem 3.1, we have

$$
\begin{aligned}
\mathcal{T}_{h_0}^{\beta} f(h) = c_1 \prod_{k=0}^{m-1} \left[\beta - \frac{k\beta}{m}\right] + c_2 \prod_{k=0}^{m-1} \left[2\beta - \frac{k\beta}{m}\right] (h - h_0)^{\beta} \\
+ c_3 \prod_{k=0}^{m-1} \left[3\beta - \frac{k\beta}{m}\right] (h - h_0)^{2\beta} + \dots \\
+ c_n \prod_{k=0}^{m-1} \left[n\beta - \frac{k\beta}{m}\right] (h - h_0)^{(n-1)\beta} + \dots,
\end{aligned}
$$

this implies that

$$
\mathcal{T}_{h_0}^{\beta} f(h_0) = c_1 \prod_{k=0}^{m-1} \left[\beta - \frac{k\beta}{m}\right] + 0 + 0 + 0 + \dots,
$$

that is,

$$
c_1 = \frac{\mathcal{T}_{h_0}^{\beta} f(h_0)}{\prod_{k=0}^{m-1} \left[\beta - \frac{k\beta}{m}\right]}.
$$

Again using Theorem 3.1, we have

$$
\begin{aligned}
\mathcal{T}_{h_0}^{2\beta} f(h) = c_2 \prod_{k=0}^{(2m)-1} \left[2\beta - \frac{k\beta}{m}\right] + c_3 \prod_{k=0}^{(2m)-1} \left[3\beta - \frac{k\beta}{m}\right] (h - h_0)^{\beta} + \dots \\
+ c_n \prod_{k=0}^{(2m)-1} \left[n\beta - \frac{k\beta}{m}\right] (h - h_0)^{(n-2)\beta} + \dots,
\end{aligned}
$$

this implies that

$$
\mathcal{T}_{t_0}^{2\beta} f(h_0) = c_2 \prod_{k=0}^{(2m)-1} \left[2\beta - \frac{k\beta}{m}\right] + 0 + 0 + \dots,
$$

that is,

$$
c_2 = \frac{\mathcal{T}_{h_0}^{2\beta} f(h_0)}{\prod_{k=0}^{(2m)-1} \left[2\beta - \frac{k\beta}{m}\right]}.
$$

By continuing in the same way for (n-times) and substitute $h = h_0$ we can get

$$c_n = \frac{\mathcal{T}_{h_0}^{n\beta} f(h_0)}{\prod_{k=0}^{(nm)-1} \left[n\beta - \dfrac{k\beta}{m} \right]}.$$

This completes the proof. □

To see the pattern and discover the general formula for Generalization Taylor's series for the MCFD by substituting of

$$c_n = \frac{\mathcal{T}_{h_0}^{n\beta} f(h_0)}{\prod_{k=0}^{(nm)-1} \left[n\beta - \frac{k\beta}{m} \right]}, \ n = 0, 1, 2, \ldots$$

into the series representation of Eq.(3.3) it will obtained by

$$f(h) = \sum_{n=0}^{\infty} \frac{\mathcal{T}_{h_0}^{n\beta} f(h_0)}{\prod_{k=0}^{(nm)-1} \left[n\beta - \dfrac{k\beta}{m} \right]} (h - h_0)^{n\beta}, \tag{3.4}$$

where $m - 1 < \beta \leq m, \beta \in \chi_\alpha$ and $0 \leq h_0 \leq h < h_0 + R$.

Corollary 3.3. *Suppose that the Generalized Taylor's series of the function f representation at h_0 has the form*

$$f(h) = \sum_{n=0}^{\infty} \frac{\mathcal{T}_{h_0}^{n\beta} f(h_0)}{\prod_{k=0}^{(nm)-1} \left[n\beta - \dfrac{k\beta}{m} \right]} (h - h_0)^{n\beta}, \ m - 1 < \beta \leq m, h_0 \leq h < h_0 + R.$$

If we substituting $m = 1$ in the previous function, then we have

$$f(h) = \sum_{k=0}^{\infty} \frac{\mathcal{T}_{h_0}^{k\beta} f(h_0)}{(k)!\beta^k} (h - h_0)^{k\beta} \ \ h_0 \leq h < h_0 + R.$$

The obtained result of Taylor's series is in line with the findings of Abdeljawad [1].

Theorem 3.4. *Suppose that a FPS function $f(h) = \sum\limits_{i=0}^{\infty} c_i (h - h_0)^{i\beta}$, for $i = 0, 1, 2, \ldots, n+1, \ \beta \in \chi_\alpha, f \in C_\mu, \mu \geqslant -\alpha$ and $h > h_0$. Then the function f could be represented by*

$$f(h) = J_{h_0}^{(n+1)\beta} \mathcal{T}_{h_0}^{(n+1)\beta} f(h) + \sum_{i=0}^{n} \frac{\mathcal{T}_{h_0}^{i\beta} f(h_0)}{\prod_{k=0}^{(im)-1} \left[i\beta - \frac{k\beta}{m} \right]} (h - h_0)^{i\beta} \ , \ h_0 \leq h \leq h_0 + R.$$

Proof. From the operator of $J_{h_0}^{\beta}, \mathcal{T}_{h_0}^{\beta}$ and Lemma 2.15, we can find that

$$\begin{aligned}
&J_{h_0}^{(n+1)\beta}\mathcal{T}_{h_0}^{(n+1)\beta}f(h)\\
&= J_{h_0}^{n\beta}\left[\left(J_{h_0}^{\beta}\mathcal{T}_{h_0}^{\beta}\right)\mathcal{T}_{h_0}^{n\beta}f(h)\right]\\
&= J_{h_0}^{n\beta}\left[\mathcal{T}_{h_0}^{n\beta}f(h)-\mathcal{T}_{h_0}^{n\beta}f(h_0)\right]\\
&= J_{h_0}^{n\beta}\mathcal{T}_{h_0}^{n\beta}f(h)-J_{h_0}^{n\beta}\mathcal{T}_{h_0}^{n\beta}f(h_0)\\
&= J_{h_0}^{n\beta}\mathcal{T}_{h_0}^{n\beta}f(h)-\left(\frac{\mathcal{T}_{h_0}^{n\beta}f(h_0)}{\prod_{k=0}^{(nm)-1}\left[n\beta-\frac{k\beta}{m}\right]}(h-h_0)^{n\beta}\right)\\
&= J_{h_0}^{(n-1)\beta}\left[\left(J_{h_0}^{\beta}\mathcal{T}_{h_0}^{\beta}\right)\mathcal{T}_{h_0}^{(n-1)\beta}f(h)\right]-\left(\frac{\mathcal{T}_{h_0}^{n\beta}f(h_0)}{\prod_{k=0}^{(nm)-1}\left[n\beta-\frac{k\beta}{m}\right]}(h-h_0)^{n\beta}\right)\\
&= J_{h_0}^{(n-1)\beta}\left[\mathcal{T}_{h_0}^{(n-1)\beta}f(h)-\mathcal{T}_{h_0}^{(n-1)\beta}f(h_0)\right]-\left(\frac{\mathcal{T}_{h_0}^{n\beta}f(h_0)}{\prod_{k=0}^{(nm)-1}\left[n\beta-\frac{k\beta}{m}\right]}(h-h_0)^{n\beta}\right)\\
&= J_{h_0}^{(n-1)\beta}\mathcal{T}_{h_0}^{(n-1)\beta}f(h)-\left(\frac{\mathcal{T}_{h_0}^{(n-1)\beta}f(h_0)}{\prod_{k=0}^{(nm)-1}\left[(n-1)\beta-\frac{k\beta}{m}\right]}(h-h_0)^{(n-1)\beta}\right)\\
&\quad-\left(\frac{\mathcal{T}_{h_0}^{n\beta}f(h_0)}{\prod_{k=0}^{(nm)-1}\left[n\beta-\frac{k\beta}{m}\right]}(h-h_0)^{n\beta}\right)\\
&= J_{h_0}^{(n-2)\beta}\left[\left(J_{h_0}^{\beta}\mathcal{T}_{h_0}^{\beta}\right)\mathcal{T}_{h_0}^{(n-2)\beta}f(h)\right]-\left(\frac{\mathcal{T}_{h_0}^{(n-1)\beta}f(h_0)}{\prod_{k=0}^{(nm)-1}\left[(n-1)\beta-\frac{k\beta}{m}\right]}(h-h_0)^{(n-1)\beta}\right)\\
&\quad-\left(\frac{\mathcal{T}_{h_0}^{n\beta}f(h_0)}{\prod_{k=0}^{(nm)-1}\left[n\beta-\frac{k\beta}{m}\right]}(h-h_0)^{n\beta}\right).
\end{aligned}$$

If we continue repeating of this process (n-times), then we can find that

$$J_{h_0}^{(n+1)\beta}\mathcal{T}_{h_0}^{(n+1)\beta}f(h)=f(h)-\sum_{i=0}^{n}\frac{\mathcal{T}_{h_0}^{i\beta}f(h_0)}{\prod\limits_{k=0}^{(im)-1}\left[i\beta-\dfrac{k\beta}{m}\right]}(h-h_0)^{i\beta}.$$

Hence, we have

$$f(h)=J_{h_0}^{(n+1)\beta}\mathcal{T}_{h_0}^{(n+1)\beta}f(h)+\sum_{i=0}^{n}\frac{\mathcal{T}_{h_0}^{i\beta}f(h_0)}{\prod\limits_{k=0}^{(im)-1}\left[i\beta-\dfrac{k\beta}{m}\right]}(h-h_0)^{i\beta}\ ,\ h_0\le h\le h_0+R.$$

□

Theorem 3.5. *If* $|\mathcal{T}_{h_0}^{(n+1)\beta}f(h)|\le M$ *on* $h_0\le h\le d$*, where* $m-1<\beta\le m$*,* $\beta\in\chi_\alpha, f\in C_\mu, \mu\geqslant-\alpha$*, then the reminder* $R_n(h)$ *of the Generalized Taylor's series will be satisfies the inequality*

$$|R_n|\le\frac{M(h-h_0)^{(i+n)\beta}}{\prod_{k=0}^{nm-1}\left(i\beta+\dfrac{k\beta}{m}\right)},\ h_0\le h\le d. \tag{3.5}$$

Proof. Assume that $\mathcal{T}_{h_0}^{i\beta}f(h)$ exist for $i=0,1,2,\ldots,n+1$ and that

$$|\mathcal{T}_{h_0}^{(n+1)\beta}f(h)|\le M \text{ on } h_0\le h\le d. \tag{3.6}$$

From the definition of the reminder

$$R_n(h)=f(h)-\sum_{i=0}^{n}\frac{\mathcal{T}_{h_0}^{i\beta}f(h_0)}{\prod\limits_{k=0}^{(im)-1}\left[i\beta-\frac{k\beta}{m}\right]}(h-h_0)^{i\beta}.$$

One can get it $R_n(h_0)=\mathcal{T}_{h_0}^{\beta}R_n(h_0)=\mathcal{T}_{h_0}^{2\beta}R_n(h_0)=\cdots=\mathcal{T}_{h_0}^{n\beta}R_n(h_0)=0$ and $\mathcal{T}_{h_0}^{(n+1)\beta}R_n(h)=\mathcal{T}_{h_0}^{(n+1)\beta}f(h),\ h_0\le h\le d$. By Eq.(3.6) that $|\mathcal{T}_{h_0}^{(n+1)\beta}f(h)|\le M$. Hence,

$$-M\le\mathcal{T}_{h_0}^{(n+1)\beta}f(h)\le M,\ \ h_0\le h\le d.$$

On the other hand, we have

$$J_{h_0}^{(n+1)\beta}(-M)\le J_{h_0}^{(n+1)\beta}\mathcal{T}_{h_0}^{(n+1)\beta}f(h)\le J_{h_0}^{(n+1)\beta}(M). \tag{3.7}$$

Since from Theorem 3.4, we obtain $J_{h_0}^{(n+1)\beta}\mathcal{T}_{h_0}^{(n+1)\beta}f(h)=R_n(h)$, and substituting that in Eq.(3.7), we can find the inequality

$$\frac{-M\ (h-h_0)^{(i+n)\beta}}{\prod_{k=0}^{nm-1}\left(i\beta+\dfrac{k\beta}{m}\right)}\le R_n\le\frac{M\ (h-h_0)^{(i+n)\beta}}{\prod_{k=0}^{nm-1}\left(i\beta+\dfrac{k\beta}{m}\right)},\ h_0\le h\le d,$$

which is equivalent to $|R_n| \leq \dfrac{M\ (h-h_0)^{(i+n)\beta}}{\prod_{k=0}^{nm-1}\left(i\beta+\dfrac{k\beta}{m}\right)},\ h_0 \leq h \leq d.$ □

4. Applications

In this part, we solve some FDEs by using the new technique for the FPS. In the first application, we solve fractional Airy differential equation. In the second application, we solve homogeneous FDEs with initial conditions.

4.1. **Application.** Consider the fractional Airy differential equation

$$\mathcal{T}_{h_0}^{2\beta} y(h) - \beta^2 h^\beta y\,(h) = 0\ m-1 < \beta \leq m,\ h \geq 0. \tag{4.1}$$

Solution: By using the FPS expansion (2.7), we assume that the solution $y(h)$ of Equations (4.1) can be defined by

$$y(h) = \sum_{i=0}^{\infty} c_i h^{i\beta}. \tag{4.2}$$

From formula (3.1), we can obtain

$$\begin{aligned}\mathcal{T}_{h_0}^{2\beta} y(h) &= \sum_{i=2}^{\infty} c_i \left[\prod_{k=0}^{2m-1}\left(i\beta-\frac{k\beta}{m}\right) h^{(i-2)\beta}\right]\\ &= \sum_{i=0}^{\infty} c_{i+2}\left[\prod_{k=0}^{2m-1}\left((i+2)\beta-\frac{k\beta}{m}\right) h^{i\beta}\right].\end{aligned} \tag{4.3}$$

By substituting Eq.(4.2) and (4.3) into both sides of Eq.(4.1), yields that

$$\sum_{i=0}^{\infty} c_{i+2}\left[\prod_{k=0}^{2m-1}\left((i+2)\beta-\frac{k\beta}{m}\right) h^{i\beta}\right] - \beta^2 h^\beta \sum_{i=0}^{\infty} c_i h^{i\beta} = 0$$

and

$$c_2 \prod_{k=0}^{2m-1}\left(2\beta-\frac{k\beta}{m}\right)+\sum_{i=1}^{\infty}\left(c_{i+2}\left[\prod_{k=0}^{2m-1}\left((i+2)\beta-\frac{k\beta}{m}\right)\right]-\beta^2\sum_{i=1}^{\infty} c_{i-1}\right) h^{i\beta} = 0. \tag{4.4}$$

Now, by equating coefficients of $h^{i\beta}$ to zero in both sides of Eq (4.4). Then we have the following

$$c_2 = 0;\ c_{i+2} = \frac{\beta^2\ c_{i-1}}{\left[\prod_{k=0}^{2m-1}\left((i+2)\beta-\frac{k\beta}{m}\right)\right]}. \tag{4.5}$$

Hence, we obtain the following: $c_2 = c_5 = c_8 = c_{11} = \ldots = 0$ and

$$c_{(3i)} = \frac{\beta^{2i}\ c_0}{\left[\prod_{k=0}^{2m-1}\left((3i)\beta - \frac{k\beta}{m}\right)\right]\cdots\left[\prod_{k=0}^{2m-1}\left((3)\beta - \frac{k\beta}{m}\right)\right]},\ i = 1, 2, 3, 4, \ldots,$$

$$c_{(3i+1)} = \frac{\beta^{2i}\ c_1}{\left[\prod_{k=0}^{2m-1}\left((3i+1)\beta - \frac{k\beta}{m}\right)\right]\cdots\left[\prod_{k=0}^{2m-1}\left((4)\beta - \frac{k\beta}{m}\right)\right]},\ i = 1, 2, 3, 4, \ldots$$

If we compile all of these coefficient values and put them back into the Eq. (4.2). Then the solution of Eq.(4.1) takes the general form:

$$y(h) = c_0 y_1(h) + c_1 y_2(h), \tag{4.6}$$

where

$$y_1(h) = 1 + \sum_{i=1}^{\infty} \frac{\beta^{2i}\ h^{3i\beta}}{\left[\prod_{i=0}^{2m-1}\left(3i)\beta - \frac{k\beta}{m}\right)\right]\cdots\left[\prod_{k=0}^{2m-1}\left((3)\beta - \frac{k\beta}{m}\right)\right]}, \tag{4.7}$$

$$y_2(h) = 1 + \sum_{i=1}^{\infty} \frac{\beta^{2i}\ h^{(3i+1)\beta}}{\left[\prod_{k=0}^{2m-1}\left((3i+1)\beta - \frac{k\beta}{m}\right)\right]\cdots\left[\prod_{k=0}^{2m-1}\left((4)\beta - \frac{k\beta}{m}\right)\right]}. \tag{4.8}$$

4.2. **Application.** Consider the following homogeneous FDE

$$\mathcal{T}_{h_0}^{\beta} y(h) = \lambda y(h),\ m-1 < \beta \leq m,\ \ h \geq h_0 \tag{4.9}$$

subject to the non-homogeneous initial conditions

$$y^{(r)}(h_0) = a_0,\ r = 0, 1, 2, \ldots, m-1, \tag{4.10}$$

where λ and a_0 are constants.

Solution: To achieve our goal, by using the FPS expansion (2.7), we suppose that this solution takes the form

$$y(h) = \sum_{i=0}^{\infty} c_i (h - h_0)^{i\beta}. \tag{4.11}$$

From formula (3.3), we can obtain

$$\mathcal{T}_{h_0}^{\beta} y(h) = \sum_{i=1}^{\infty} c_i \left(\prod_{k=0}^{m-1}\left[i\beta - \frac{k\beta}{m}\right](h - h_0)^{(i-1)\beta}\right). \tag{4.12}$$

Substituting the expansion formulas of Eq.(4.11) and (4.12) back into both sides of Eq. (4.9), respectively, yields that

$$\sum_{i=0}^{\infty} c_{i+1}\left(\prod_{k=0}^{m-1}\left[(i+1)\beta-\frac{k\beta}{m}\right](h-h_0)^{i\beta}\right)-\lambda\sum_{i=0}^{\infty} c_i(h-h_0)^{i\beta}=0. \quad (4.13)$$

Equating the coefficients of $(h-h_0)^{i\beta}$ in both sides of Eq. (4.13) will leads to the following

$$c_{i+1}=\frac{\lambda c_i}{\prod_{k=0}^{m-1}\left[(i+1)\beta-\frac{k\beta}{m}\right]}, \quad i=0,1,2,\ldots. \quad (4.14)$$

Considering the initial conditions of Eq. (4.10) one can obtain $c_0=a_0$ and

$$c_1=\frac{\lambda a}{\prod_{k=0}^{m-1}\left[\beta-\frac{k\beta}{m}\right]},$$

$$c_2=\frac{\lambda c_1}{\prod_{k=0}^{m-1}\left[2\beta-\frac{k\beta}{m}\right]}=\frac{\lambda^2 a}{\prod_{k=0}^{m-1}\left[2\beta-\frac{k\beta}{m}\right]\prod_{k=0}^{m-1}\left[\beta-\frac{k\beta}{m}\right]},$$

$$c_3=\frac{\lambda c_2}{\prod_{k=0}^{m-1}\left[3\beta-\frac{k\beta}{m}\right]}=\frac{\lambda^3 a}{\prod_{k=0}^{m-1}\left[3\beta-\frac{k\beta}{m}\right]\prod_{k=0}^{m-1}\left[2\beta-\frac{k\beta}{m}\right]\prod_{k=0}^{m-1}\left[\beta-\frac{k\beta}{m}\right]},$$

$$\vdots$$

$$c_n=\frac{\lambda^n a}{\prod_{k=0}^{m-1}\left[n\beta-\frac{k\beta}{m}\right]\cdots\prod_{k=0}^{m-1}\left[2\beta-\frac{k\beta}{m}\right]\prod_{k=0}^{m-1}\left[\beta-\frac{k\beta}{m}\right]}. \quad (4.15)$$

For the conduct of proceedings in the solution, collecting the previous results and substituting Eq. (4.15) into Eq. (4.11), to formulate the solution of Eqs. (4.9) and (4.10) in the following form

$$y(h)=a_0\left(1+\sum_{i=1}^{\infty}\frac{\lambda^i\,(h-h_0)^{i\beta}}{\prod_{k=0}^{m-1}\left[i\beta-\frac{k\beta}{m}\right]\cdots\prod_{k=0}^{m-1}\left[2\beta-\frac{k\beta}{m}\right]\prod_{k=0}^{m-1}\left[\beta-\frac{k\beta}{m}\right]}\right).$$

5. Conclusions

This study utilized the MCFD definition to generalize and prove some theorems of the CPS into the FPS. We construct and prove several important theorems related to the CPS into the FPS. Moreover, a comprehensive formulation of the generalized Taylor's series is derived within this context. The practical utility of this new technique is underscored by its successful application in the resolution of FDEs. The presented formula may simplify and modify some methods used to solve linear and nonlinear FDEs, such as the differential transform method and variation of parameters method.

References

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math., **279** (2015), 57–66, doi.org/10.5923/j.ajcam.20140406.01.

[2] I. Abu Hammad and R. Khalil, *Fractional Fourier series with applications*, Amer. J. Comput. Appl. Math., **4**(6) (2014), 187–191, doi.org/10.5923/j.ajcam.20140406.01.

[3] A. T. Alabdala, B. N. Abood, S. S. Redhwan and S. Alkhatib, *Caputo delayed fractional differential equations by Sadik transform*, Nonlinear Funct. Anal. Appl., **28**(2) (2023), 439-448.

[4] A. El-Ajou, *A modification to the conformable fractional calculus with some applications*, Alexandria Eng. J., **59**(4) (2020), 2239–2249, doi.org/10.1016/j.aej.2020.02.003.

[5] A. El-Ajou, O.A. Arqub, Z.A. Zhour and S. Momani, *New results on fractional power series: theories and applications*, Entropy, **15**(12) (2013), 5305–5323, doi.org/10.3390/e15125305.

[6] F.M. Gonzalez, I. Martinez Vidal, M. Kaabar and S. Paredes, *Some new results on conformable fractional power series*, Asia Pacific J. Math., **7**(31) (2020), 1–14, doi.org/10.28924/APJM/7-31.

[7] F. M. Ismaael, *An investigation on the existence and uniqueness analysis of the fractional nonlinear integro-differential equations*, Nonlinear Funct. Anal. Appl., **28**(1) (2023), 237-249

[8] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math., **264** (2014), 65–70, doi.org/10.1016/j.cam.2014.01.002.

[9] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of the Fractional Differential Equations*, **204** North-Holland Mathematics Studies, Elsevier, Amsterdam, 2006.

[10] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[11] Z.M. Odibat and N.T. Shawagfeh, *Generalized Taylors formula*, Appl. Math. Comput., **186**(1) (2007), 286–293, doi.org/10.1016/j.amc.2006.07.102.

[12] K. Oldham and J. Spanier, *The Fractional Calculus, Theory and Applications of Differentiation and Integration of Arbitrary Order*, Academic Press, USA, 1974.

[13] I. Podlubny, *Fractional Differential Equations*, Academic Press, USA, 1999.

[14] S.T. Syouri, M. Mamat, I.M. Alghrouz and I.M. Sulaiman, *Conformable fractional differintegral method for solving fractional equations*, AQU Researchers Publications, **2020**, https://dspace.alquds.edu/handle/20.500.12213/5684.

[15] J.J. Trujillo, M. Rivero and B. Bonilla, *On a RiemannLiouville generalized Taylor's formula*, J. Math. Anal. Appl., **231**(1) (1999), 255–265, doi.ouvrg/10.1006/jmaa.1998.6224.

[16] E. Unal and A. Gokdogan, *Solution of conformable fractional ordinary differential equations via differential transform method*, Optik-Int. J. Light Electron Opt, **128** (2017), 264–273, doi.org/10.1016/j.ijleo.2016.10.031.