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A TORSION GRAPH DETERMINED BY EQUIVALENCE CLASSES OF TORSION ELEMENTS AND ASSOCIATED PRIME IDEALS

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Abstract. In this paper, we define the torsion graph determined by equivalence classes of torsion elements and denote it by $A_E(M)$. The vertex set of $A_E(M)$ is the set of equivalence classes $\{[x] \mid x \in T(M)^*\}$, where two torsion elements $x, y \in T(M)^*$ are equivalent if $ann(x) = ann(y)$. Also, two distinct classes $[x]$ and $[y]$ are adjacent in $A_E(M)$, provided that $ann(x)ann(y)M = 0$. We shall prove that for every torsion finitely generated module M over a Dedekind domain R, a vertex of $A_E(M)$ has degree two if and only if it is an associated prime of M.

1. Introduction

Throughout this paper, all rings are commutative with identity and all modules are unitary. An element x of an R-module M is called a torsion element if it has a non-zero annihilator in R. Let $T(M)$ be the set of torsion elements of M. It is clear that if R is an integral domain, then $T(M)$ is a submodule of M. We call $T(M)$ the torsion submodule of M. If $T(M) = M$, then M is called a torsion module. For every subset X of R (or M), we define $X^* = X - \{0\}$. Recall that a prime ideal P is an associated prime of R (or M) if $P = ann(x)$ for some non-zero element $x \in R$ (or $x \in M$). The set of all associated primes of a ring R (or R-module M) is denoted by $Ass(R)$ (or $Ass(M)$). It is well known that for a finitely generated module M over a Noetherian ring R, $Ass(R)$ and $Ass(M)$ are both finite.

A Dedekind domain is a Noetherian integrally closed domain in which every non-zero prime ideal is maximal. If R is a Dedekind domain, then for every non-zero prime ideal P of R, R_P is a DVR [2, Theorem 9.3]. Also every non-zero ideal I of a Dedekind domain R can be uniquely expressed by $I = P_1^{n_1} \cdots P_r^{n_r}$, where P_i $(1 \leq i \leq r)$ are prime ideals of R containing I [2, Corollary 9.4].

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A graph G consists of a set of vertices $V(G)$ and a set of edges $E(G)$ consisting of pairs of vertices. We consider simple graphs, that is, graphs without loops and parallel edges. Two vertices of a graph are said to be connected if there is a path between them. A graph G is connected if any two distinct vertices are connected. The distance $d(x, y)$ between connected vertices x and y is the length of a shortest path from x to y; if there is no such a path we write $d(x, y) = \infty$. The diameter of a connected graph G is the supremum of the distances between vertices. The diameter is 0 if the graph consists of a single vertex. For a graph G, the degree $\deg(v)$ of a vertex v in G is the number of edges incident to v. We denote the complete graph with n vertices and a complete bipartite graph with two parts of sizes m and n, by K_n and $K_{m,n}$, respectively. The complete bipartite graph $K_{1,n}$ is called a star graph. A cycle graph C_n is a path from v_1 to v_n such that $v_1 = v_n$. Two graphs G and H are isomorphic if there is a bijection f from $V(G)$ onto $V(H)$ such that two vertices x and y of G are adjacent if and only if the vertices $f(x)$ and $f(y)$ of H are adjacent. A colour-partition of a graph G is a partition of $V(G)$ into colour-classes $V_1, \ldots, V_l$ such that no V_i $(1 \leq i \leq l)$, contains a pair of adjacent vertices. In other words, the induced subgraphs $\langle V_i \rangle$ have no edges. The chromatic number of G, denoted by $\nu(G)$, is the least natural number l for which such a partition is possible. A subset X of the vertices of G is called a clique if the induced subgraph on X is a complete graph. The clique number of G is $\chi(G) = n$ if G contains a clique with n elements and no clique has more than n elements. If the sizes of the cliques are not bounded, then $\chi(G) = \infty$. We always have $\chi(G) \leq \nu(G)$.

The notion of a zero-divisor graph $G(R)$ of a ring R was introduced by I. Beck in [3]. The vertices of the graph $G(R)$ are the elements of R and two distinct vertices r and s are adjacent provided that $rs = 0$. The first simplification of Beck's zero-divisor graph $\Gamma(R)$ was introduced by D. F. Anderson and P. S. Livingston in [1]. This zero-divisor graph helps us study the algebraic properties of rings using graph theoretical tools. S. B. Mulay [8] introduced the zero-divisor graph $\Gamma_E(R)$ associated with a ring. For a ring R, two zero-divisors $r, s \in Z(R)^*$ are said to be equivalent if $ann(r) = ann(s)$, where $Z(R)$ is the set of all zero-divisors of R. The equivalence class of r is denoted by $[r]$. The set of vertices of the graph $\Gamma_E(R)$ is the set of equivalence classes $\{[r] \mid r \in Z(R)^*\}$. Distinct classes $[r]$ and $[s]$ are adjacent in $\Gamma_E(R)$ provided that $rs = 0$ in R.

We follow the ideas from Mulay, Spiroff and Wickham in [9], who studied the graph of equivalence classes of zero-divisors of a ring R. This graph has some advantages over the earlier zero-divisor graph $\Gamma(R)$. In many cases, $\Gamma_E(R)$ is finite even when $\Gamma(R)$ is infinite. In addition, there are no complete graphs $\Gamma_E(R)$ with three or more vertices, since the graph collapses into a single point. Every vertex in this graph either corresponds to an associated prime or is connected to one.

In this paper we extend this concept to modules, i.e., we define a graph and derive relationships between the associated primes of M and its graph-theoretic properties. In [6], the concept of the zero-divisor graph for a ring has been extended to a module and the authors defined the torsion graph $\Gamma(M)$ of an R-module M as one whose vertices are the non-zero torsion elements of M and two distinct vertices x and y are adjacent if $[x : M][y : M]M = 0$. Here we define a graph whose set of vertices is the set of equivalence classes $\{[x] \mid x \in T(M)^*\}$, and two distinct torsion elements $x, y \in T(M)^*$ are equivalent if $ann(x) = ann(y)$. Also, two distinct classes $[x]$ and $[y]$ are adjacent provided that $ann(x)ann(y)M = 0$. This graph will be denoted by $A_E(M)$. We say an ideal I of R is an annihilating-ideal if there exists a non-zero ideal J of R such that $IJ = (0)$. We denote the set of annihilating-ideals of R by $\mathbb{A}(R)$. By the annihilating-ideal graph $\mathbb{AG}(R)$ of R we mean the graph with vertices $\mathbb{A}(R)^* = \mathbb{A}(R) - \{0\}$ such that there is an edge between vertices I and J if and only if $I \neq J$ and $IJ = (0)$ [4, 5]. For an R-module M (a ring R), we denote the set of all $ann(x)$ such that $0 \neq x \in M(R)$, by $\Omega_R(M)$ $(\Omega(R))$. There is a natural bijective map from $\Omega_R(M)$ (or $\Omega(R)$) to the set of vertices of $A_E(M)$ (or $A_E(R)$) given by $I \to [x]$, where $I = ann(x)$. We will slightly abuse terminology and refer to $[x]$ as an element of $\Omega_R(M)$ $(\Omega(R))$.

In Section 1, we define the graph $A_E(M)$, discuss the relation between the associated primes of M and the vertices of $A_E(M)$ and prove some basic results about $A_E(M)$ and $A_E(R)$. In Section 2, we show that a vertex of $A_E(M)$ has degree two if and only if it is an associated prime of M. We then determine $\nu(A_E(M))$, where $|V(A_E(M))| > 1$ and prove that the chromatic number of $A_E(M)$ equals its clique number.

2. The definition and some results about $A_E(M)$ and $A_E(R)$

Let M be an R-module. For every $x, y \in M$, we say that $x \sim y$ if $ann(x) = ann(y)$. The relation " $\sim$ " is an equivalence relation. The equivalence class of x is denoted by $[x]$.

Definition. The graph of equivalence classes of torsion elements of an R-module M, denoted by $A_E(M)$, is the graph whose vertices are the classes of elements in $T(M)^*$. Also, each pair of distinct classes $[x]$ and $[y]$ are joined by an edge if $ann(x)ann(y)M = 0$.

Proposition 2.1. (i) *Let R be a Noetherian ring and M be an R-module. If $V(A_E(M)) = \emptyset$, then R is an integral domain.*

(ii) *If R is an integral domain and M is a faithful R-module, then $E(A_E(M)) = \emptyset$.*

Proof. (i) Since R is a Noetherian ring, $Ass(M) \neq \emptyset$. Let $P \in Ass(M)$ be such that $ann(x) = P \neq 0$. Thus $x \in T(M)^*$ and $[x] \in V(A_E(M))$, which is a contradiction. So we have $P = 0 \in Spec(R)$ and R is an integral domain.

(ii) Now let $[x]$ and $[y] \in V(A_E(M))$ be adjacent. Then $ann(x)ann(y) = 0$. Since R is an integral domain, $ann(x) = 0$ or $ann(y) = 0$, which is a contradiction. So $E(A_E(M)) = \emptyset$. □

The converse of the first part of Proposition 2.1 is not true in general. For example for $\mathbb{Z}$-module $\mathbb{Z}_n$, $V(A_E(\mathbb{Z}_n)) \neq \emptyset$, but $\mathbb{Z}$ is an integral domain.

In the following we give an example to illustrate the second part of the Proposition 2.1.

Example 2.2. The $\mathbb{Z}$-module $\frac{\mathbb{Q}}{\mathbb{Z}}$ is faithful and for every $\frac{a}{b} + \mathbb{Z} \in \frac{\mathbb{Q}}{\mathbb{Z}}$ such that $(a, b) = 1$, we have $ann(\frac{a}{b} + \mathbb{Z}) = b\mathbb{Z}$. Thus $V(A_E(M)) = \{[\frac{1}{b} + \mathbb{Z}] \mid b \in \mathbb{N}\}$ and we have $E(A_E(M)) = \emptyset$. Note that $|V(A_E(M))| = \infty$ but $E(A_E(M)) = \emptyset$.

Proposition 2.3. *Let M be an R-module and $ann(M) = P \in Spec(R)$. Then $A_E(M)$ is a star graph or $E(A_E(M)) = \emptyset$. Furthermore, if $P \in Max(R)$, then $|V(A_E(M))| = 1$.*

Proof. Assume that $E(A_E(M)) \neq \emptyset$. Then there exist $[x], [y] \in V(A_E(M))$ such that $ann(x)ann(y) \subseteq P$. Since P is a prime ideal, we have $ann(x) \subseteq P$ or $ann(y) \subseteq P$. By assumption, $P \subseteq ann(x)$ and $P \subseteq ann(y)$, hence $ann(x) = P$ or $ann(y) = P$. Now suppose that $ann(x) = P$. For every $[z] \in V(A_E(M))$, $ann(x)ann(z) \subseteq P$ and hence $A_E(M)$ is a star graph. If $P \in Max(R)$, then for every $x \in T(M)^*$, $P \subseteq ann(x) \neq R$. So $P = ann(x)$ and $|V(A_E(M))| = 1$. □

Proposition 2.4. *Let M be a module over a Noetherian ring R with $m, m' \in Ass(M)$ such that m and m' are the only maximal elements of $\Omega_R(M)$. If m and m' are adjacent or there exists $x \in T(M)^*$ such that $ann(x) = ann(M)$, then $A_E(M)$ is connected and $diam(A_E(M)) \leq 2$.*

Proof. Since R is Noetherian, for every $z \in T(M)^*$ we have $ann(z) \subseteq m$ or $ann(z) \subseteq m'$. Since $mm' \subseteq ann(M)$, hence $[z]$ is adjacent to m or m'. Therefore $A_E(M)$ is connected. Let $[z]$ be adjacent to m and $[w]$ adjacent to m'. We have $ann(z)m \subseteq m'$ and $ann(w)m' \subseteq m$, hence $ann(z) \subseteq m'$ and $ann(w) \subseteq m$. So $ann(z)ann(w) \subseteq mm' \subseteq ann(M)$ and hence $[z]$ and $[w]$ are adjacent. Therefore, $diam(A_E(M)) \leq 2$. Now let $x \in T(M)^*$ be such that $ann(x) = ann(M)$. Since every vertex $[y]$ is adjacent to $[x]$, hence $A_E(M)$ is connected and $diam(A_E(M)) \leq 2$. □

Proposition 2.5. *Let $B = \{P \in Ass(M) \mid P$ is minimal in $Ass(M)\}$.*

(i) *If $P, Q \in Ass(M) \setminus B$, then P and Q are not adjacent.*

(ii) *If $|B| \geq 3$, then no two elements of $Ass(M)$ are adjacent.*

(iii) *If $B = \{P, Q\}$, then the only elements of $Ass(M)$ that can be adjacent are P and Q.*

(iv) *If $A_E(M)$ is a complete graph, then $|Ass(M)| \leq 2$.*

Proof. (i) Case 1: $B \neq \emptyset$. Suppose that T is minimal in $Ass(M)$. If P and Q are adjacent, then $PQ \subseteq T$ and we have $P \subseteq T$ or $Q \subseteq T$, which is a contradiction.

Case 2: $B = \emptyset$. Since $P \notin B$, there exists $T \in Ass(M)$ such that $T \subset P$. Suppose P and Q are adjacent. Then $PQ \subseteq T$ and we have $Q \subseteq T$. Since $Q \notin B$, there exists $S \in Ass(M)$ such that $S \subset Q$ and we have $PQ \subseteq S$. Thus $P \subseteq S$ and hence $P \subseteq S \subseteq Q \subseteq T$, which is a contradiction. So P and Q are not adjacent.

(ii) Let $P, Q \in Ass(M)$. Since $|B| \geq 3$, there exists $T \in B$ such that $T \neq P$ and $T \neq Q$. Assume that P and Q are adjacent. Thus $PQ \subseteq T$, hence $P \subseteq T$ or $Q \subseteq T$. So $P = T$ or $Q = T$, which is a contradiction. Therefore P and Q are not adjacent.

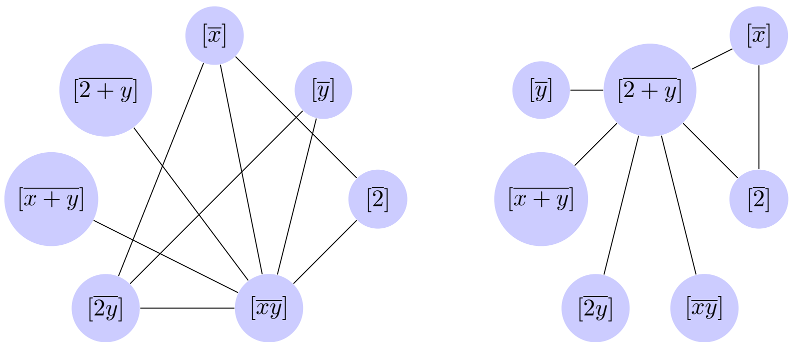
(iii) Suppose $S, T \in Ass(M)$ are adjacent. Then we have $TS \subseteq P$ and $TS \subseteq Q$. Hence $T \subseteq P$ or $S \subseteq P$ and $S \subseteq Q$ or $T \subseteq Q$. So $\{T, S\} = \{P, Q\}$.

(iv) Let $|Ass(M)| \geq 3$ and $P, Q, T \in Ass(M)$ be distinct three elements. Since $A_E(M)$ is a complete graph, P and Q are adjacent and hence by part (i), $P \in B$ or $Q \in B$. Let $P \in B$. Also Q and T are adjacent, hence $Q \in B$ or $T \in B$, which is a contradiction by parts (ii) and (iii). □

Proposition 2.6. *Let R be a Noetherian ring and $|V(A_E(R))| \geq 3$. Then $A_E(R)$ is not a complete graph. In particular, for every $[z]$ such that $ann(z)$ is a maximal element of $\Omega(R)$,* $\deg([z]) \leq 1$.

Proof. Suppose that $A_E(R)$ is a complete graph and $[x]$, $[y]$ and $[z]$ are three distinct vertices such that $ann(z)$ is a maximal element of $\Omega(R)$. We have $ann(x)ann(z) = 0$ and $ann(y)ann(z) = 0$. Then for every $r \in ann(x)$ and $s \in ann(z)$, $rs = 0$ and hence $ann(z) \subseteq ann(r)$. Now we have $ann(z) = ann(r)$ and so $[r] = [z]$. Since $rx = 0$, $[r][x] = [rx] = 0$ in $\Gamma_E(R)$. So $[z][x] = 0$ and hence $zx = 0$. Similarly, we have $yz = 0$. It follows that $x, y \in ann(z)$ and so $ann(x) \subseteq ann(y)$, because $ann(x)ann(z) = 0$. Similarly, $ann(y) \subseteq ann(x)$ and hence $ann(x) = ann(y)$. Therefore $[x] = [y]$, which is a contradiction, so $A_E(R)$ is not a complete graph. In particular, for every $[z]$ such that $ann(z)$ is a maximal element of $\Omega(R)$, $\deg([z]) \leq 1$. □

As the following example shows, for a ring R, we have always $V(A_E(R)) = V(\Gamma_E(R))$, but $A_E(R)$ and $\Gamma_E(R)$ are not necessary isomorphic.



$\Gamma_E(R)$ $A_E(R)$

Example 2.7. Let $R = \frac{\mathbb{Z}_8[x,y]}{\langle x^2,y^2,2x\rangle}$. Clearly, R is not a quotient of a Dedekind domain. Also $ann(\overline{x}) = \langle \overline{x}, \overline{2}\rangle$, $ann(\overline{y}) = \langle \overline{y}\rangle$, $ann(\overline{2}) = \langle \overline{x}, \overline{4}\rangle$, $ann(\overline{xy}) = \langle \overline{x}, \overline{y}, \overline{2}\rangle$, $ann(\overline{2y}) = \langle \overline{x}, \overline{y}, \overline{4}\rangle$, $ann(\overline{x+y}) = \langle \overline{x-y}\rangle$ and $ann(\overline{2+y}) = \langle \overline{4y}\rangle$. Clearly, $\Gamma_E(R)$ is not isomorphic with $A_E(R)$.

Proposition 2.8. *Let D be a Dedekind domain and I be a non-zero ideal of R. If $R = \frac{D}{I}$, then $A_E(R) \cong \Gamma_E(R)$.*

Proof. Since R is a quotient of a Dedekind domain, hence R is a PIR. If $x \in Z(R)^*$, there exists $a \in Z(R)^*$ such that $ann(x) = \langle a\rangle$. We define the map $f : V(A_E(R)) \longrightarrow V(\Gamma_E(R))$ by $f([x]) = [a]$. We shall show that f is one-to-one and onto. Let $[x], [y] \in V(A_E(R))$, $ann(x) = \langle a\rangle$ and $ann(y) = \langle b\rangle$. If $[x] = [y]$, then $ann(x) = ann(y)$ and hence $\langle a\rangle = \langle b\rangle$. So $[a] = [b]$. Therefore, f is well-defined. If $[a] = [b]$, then $ann(a) = ann(b)$. Since $ann(x) = \langle a\rangle$ and $ann(y) = \langle b\rangle$, $x \in ann(a) = ann(b)$ and hence $xb = 0$. Similarly, $ya = 0$, thus $a \in ann(y)$ and $b \in ann(x)$. However, $b \in \langle a\rangle$ and $a \in \langle b\rangle$ and hence $\langle a\rangle = \langle b\rangle$. Therefore, $[x] = [y]$. So f is one-to-one. Now we show that f is onto. Let $[a] \in V(\Gamma_E(R))$. So there exists $b \in D$ such that $a = b+I$. Let $I = P_1^{\alpha_1} \cdots P_k^{\alpha_k}$ be the decomposition of I into distinct prime (maximal) ideals of D. Then there exist $\{\beta_1, \dots, \beta_k\} \subseteq \mathbb{N} \cup \{0\}$ and an ideal J of D such that $\langle b\rangle = P_1^{\beta_1} \cdots P_k^{\beta_k} J$. For every i $(1 \leq i \leq k)$, let $T_i = P_i^2 \cup P_1 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_k$. If $P_i \subset T_i$, by the prime avoidance theorem, we have $P_i = P_i^2$ and hence $P_i R_{P_i} = (P_i R_{P_i})^2$. By Nakayama's Lemma $P_i R_{P_i} = 0$, hence $P_i = 0$, which is a contradiction. So $P_i \nsubseteq T_i$. Let $p_i \in P_i - T_i$ and $s = p_1^{\gamma_1} \cdots p_k^{\gamma_k}$, where

$$\gamma_i = \begin{cases} \alpha_i - \beta_i & \text{if } \beta_i < \alpha_i, \\ 0 & \text{if } \beta_i \geq \alpha_i. \end{cases}$$

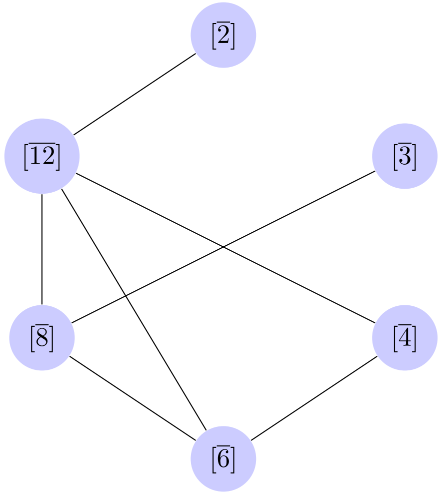
Let $x = s + I$. Now we have $ann(x) = ann(s + I) = P_1^{\alpha_1-\gamma_1} \cdots P_k^{\alpha_k-\gamma_k} + I$. Let $t = p_1^{\alpha_1-\gamma_1} \cdots p_k^{\alpha_k-\gamma_k} + I$. Then $ann(x) = \langle t\rangle$ and since $ann(a) = ann(t) = P_1^{\gamma_1} \cdots P_k^{\gamma_k} + I$, we have $[t] = [a]$. So $f([x]) = [a]$ and hence f is onto. Finally, $[x]$ and $[y]$ are adjacent in $A_E(R)$ if and only if $ann(x)ann(y) = 0$ if and only if $\langle a\rangle\langle b\rangle = 0$ if and only if $ab = 0$ if and only if $[a]$ and $[b]$ are adjacent in $\Gamma_E(R)$. □

The following example illustrate Proposition 2.8.

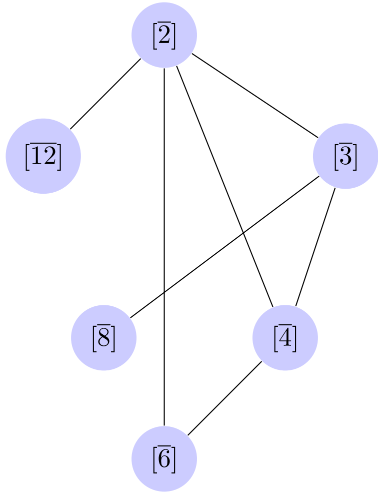
Example 2.9. Let $R = M = \mathbb{Z}_{24}$. Furthermore,

$$T(\mathbb{Z}_{24})^* = \{\overline{2}, \overline{3}, \overline{4}, \overline{6}, \overline{8}, \overline{9}, \overline{10}, \overline{12}, \overline{14}, \overline{15}, \overline{16}, \overline{18}, \overline{20}, \overline{21}, \overline{22}\}.$$

Now $V(\Gamma_E(\mathbb{Z}_{24})) = V(A_E(\mathbb{Z}_{24})) = \{[\overline{2}], [\overline{3}], [\overline{4}], [\overline{6}], [\overline{8}], [\overline{12}]\}$. Also, $ann(\overline{2}) = \langle \overline{12}\rangle$, $ann(\overline{3}) = \langle \overline{8}\rangle$, $ann(\overline{4}) = \langle \overline{6}\rangle$, $ann(\overline{6}) = \langle \overline{4}\rangle$, $ann(\overline{8}) = \langle \overline{3}\rangle$ and $ann(\overline{12}) = \langle \overline{2}\rangle$. Therefore by Proposition 2.8, $\Gamma_E(\mathbb{Z}_{24}) \cong A_E(\mathbb{Z}_{24})$.



$\Gamma_E(\mathbb{Z}_{24})$



$A_E(\mathbb{Z}_{24})$

3. Relationship between graph-theoretic properties and associated prime ideals of a module over a Dedekind domain

Let M be a torsion finitely generated module over a Dedekind domain R. In this section, we prove that, if $|V(A_E(M))| \geq 5$, then a vertex of $A_E(M)$ has degree two if and only if it is an associated prime of M.

Theorem 3.1. *Let M be a torsion finitely generated module over a Dedekind domain R. Suppose that $ann(M) = P_1^{\alpha_1} \cdots P_k^{\alpha_k}$ is the decomposition of $ann(M)$ into the distinct prime ideals of R. Then $|\Omega_R(M)| = (\prod_{i=1}^{k}(\alpha_i + 1)) - 1$ and $|Ass(M)| = k$.*

Proof. Since R is a Dedekind domain and M is a torsion finitely generated R-module, by [7, Theorem 10.15], there exist cyclic submodules of M, $\langle x_i \rangle$, $1 \leq i \leq n$, such that $M \cong \bigoplus_{i=1}^{n} \langle x_i \rangle$. If $x = x_1 + \cdots + x_n$, then $ann(x) = ann(M)$. For every i, $1 \leq i \leq k$, let $T_i = P_i^2 \cup P_1 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_k$. As in the proof of Proposition 2.8, we have $P_i \nsubseteq T_i$, $1 \leq i \leq k$. Then there exist $r_i \in P_i - T_i$, $1 \leq i \leq k$ such that $P_i R_{P_i} = (\frac{r_i}{1})$. Let $r = r_1^{\beta_1} \cdots r_k^{\beta_k} x$, where $0 \leq \beta_i \leq \alpha_i$. We have $ann(rx) = \{s \in R \mid sr \in ann(x)\}$ and so $P_1^{\gamma_1} \cdots P_k^{\gamma_k} \subseteq ann(rx)$, where $\gamma_i = \alpha_i - \beta_i$ $(1 \leq i \leq k)$. If for some i $(1 \leq i \leq k)$, $P_1^{\gamma_1} \cdots P_{i-1}^{\gamma_{i-1}} P_i^{\gamma_i - 1} P_{i+1}^{\gamma_{i+1}} \cdots P_k^{\gamma_k} \subseteq ann(rx)$, we have $t = r_1^{\alpha_1} \cdots r_{i-1}^{\alpha_{i-1}} r_i^{\alpha_i - 1} r_{i+1}^{\alpha_{i+1}} \cdots r_k^{\alpha_k} \in ann(x)$. So $\frac{t}{1} \in P_i^{\alpha_i} R_{P_i} = (\frac{r_i^{\alpha_i}}{1})$. We have $\frac{r_1^{\alpha_1} \cdots r_{i-1}^{\alpha_{i-1}} r_{i+1}^{\alpha_{i+1}} \cdots r_k^{\alpha_k}}{1} \in (\frac{r_i}{1}) = P_i R_{P_i}$ and hence there exists $s \in R - P_i$ such that $s r_1^{\alpha_1} \cdots r_{i-1}^{\alpha_{i-1}} r_{i+1}^{\alpha_{i+1}} \cdots r_k^{\alpha_k} \in P_i$, which is a contradiction. So $ann(rx) = P_1^{\gamma_1} \cdots P_k^{\gamma_k}$ and thus $T = \{P_1^{\gamma_1} \cdots P_k^{\gamma_k} \neq R \mid 0 \leq \gamma_i \leq \alpha_i\} \subseteq \Omega_R(M)$. Clearly $\Omega_R(M) \subseteq T$ and so $|\Omega_R(M)| = (\prod_{i=1}^{k}(\alpha_i + 1)) - 1$ and $|Ass(M)| = k$. □

Since R is a Dedekind domain, $S = \frac{R}{ann(M)}$ is an Artinian principal ideal ring. So $\Omega_R(M) = \Omega_R(S)$ and $A_E(M) = A_E(S)$, where S is an R-module. Also, $\Omega(S)$ is the set of all non-trivial ideals of S and hence $A_E(S) = \mathbb{AG}(S)$.

Let M be a torsion finitely generated module over a Dedekind domain R. Suppose that $ann(M) = P_1^{\alpha_1} \cdots P_k^{\alpha_k}$ is the decomposition of $ann(M)$ into prime ideals of R. In this section we find P_i $(1 \leq i \leq k)$, k and α_i $(1 \leq i \leq k)$, by the graph $A_E(M)$. Recall that there exists a natural bijective map from $\Omega_R(M)$ to the set of vertices of $A_E(M)$ given by $I \longrightarrow [x]$, where $I = ann(x)$. By Theorem 3.1, we have $|Ass(M)| = k$, $|V(A_E(M))| = (\prod_{i=1}^{k}(\alpha_i + 1)) - 1$ and $V(A_E(M)) = \{P_1^{\beta_1} \cdots P_k^{\beta_k} \neq R \mid 0 \leq \beta_i \leq \alpha_i\}$.

Lemma 3.2. *Let M be a torsion finitely generated module over a Dedekind domain R. Suppose that $ann(M) = P_1^{\alpha_1} \cdots P_k^{\alpha_k}$ is the decomposition of $ann(M)$ into the prime ideals of R. Then $V(A_E(M)) = \{P_1^{\beta_1} \cdots P_k^{\beta_k} \neq R \mid 0 \leq \beta_i \leq \alpha_i\}$ and*

$$\deg(P_1^{\beta_1} \cdots P_k^{\beta_k}) = \begin{cases} (\prod_{i=1}^{k}(\beta_i + 1)) - 2 & \textit{if } \beta_i = \alpha_i, \ \forall i, \\ (\prod_{i=1}^{k}(\beta_i + 1)) - 1 & \textit{if } \beta_i \geq \frac{\alpha_i}{2}, \ \forall i, \\ \prod_{i=1}^{k}(\beta_i + 1) & \textit{if } \exists i, \ \beta_i < \frac{\alpha_i}{2}. \end{cases}$$

Proof. By Definition, two distinct vertices $P_1^{\beta_1} \cdots P_k^{\beta_k}$ and $P_1^{\gamma_1} \cdots P_k^{\gamma_k}$ are adjacent if and only if $\beta_i + \gamma_i \geq \alpha_i$ $(1 \leq i \leq k)$. Hence the neighbourhood of $P_1^{\beta_1} \cdots P_k^{\beta_k} \in V(A_E(M))$, is the set $A = \{P_1^{\gamma_1} \cdots P_k^{\gamma_k} \neq R \mid \alpha_i - \beta_i \leq \gamma_i \leq \alpha_i, \forall i, 1 \leq i \leq k\}$.

Now we consider the following three cases:

(i) Let for every i $(1 \leq i \leq k)$, $\beta_i = \alpha_i$. Since $A_E(M)$ does not have any loop, $A = \{P_1^{\gamma_1} \cdots P_k^{\gamma_k} \mid 0 \leq \gamma_i \leq \alpha_i\} - \{P_1^{\alpha_1} \cdots P_k^{\alpha_k}, P_1^0 \cdots P_k^0\}$. So $\deg(P_1^{\alpha_1} \cdots P_k^{\alpha_k}) = (\prod_{i=1}^{k}(\alpha_i + 1)) - 2$.

(ii) Let there exist i $(1 \leq i \leq k)$ such that $\beta_i \neq \alpha_i$ and for every i $(1 \leq i \leq k)$, $\beta_i \geq \frac{\alpha_i}{2}$. So $\alpha_i \leq 2\beta_i$ and hence $P_1^{\beta_1} \cdots P_k^{\beta_k} \in A$. Since $A_E(M)$ does not have any loop, $A = \{P_1^{\gamma_1} \cdots P_k^{\gamma_k} \mid \alpha_i - \beta_i \leq \gamma_i \leq \alpha_i\} - \{P_1^{\beta_1} \cdots P_k^{\beta_k}\}$. So $\deg(P_1^{\beta_1} \cdots P_k^{\beta_k}) = (\prod_{i=1}^{k}(\beta_i + 1)) - 1$.

(iii) Let there exist i $(1 \leq i \leq k)$ such that $\beta_i < \frac{\alpha_i}{2}$. So $A = \{P_1^{\gamma_1} \cdots P_k^{\gamma_k} \mid \alpha_i - \beta_i \leq \gamma_i \leq \alpha_i\}$, and hence $\deg(P_1^{\beta_1} \cdots P_k^{\beta_k}) = \prod_{i=1}^{k}(\beta_i + 1)$. □

Theorem 3.3. *Let M be a torsion finitely generated module over a Dedekind domain R. If $|V(A_E(M))| \geq 5$, then a vertex of $A_E(M)$ has degree two if and only if it is an associated prime of M.*

Proof. As in Lemma 3.2, we have $ann(M) = P_1^{\alpha_1} \cdots P_k^{\alpha_k}$. In the proof of Lemma 3.2, we showed that $V(A_E(M)) = \{P_1^{\beta_1} \cdots P_k^{\beta_k} \neq R \mid 0 \leq \beta_i \leq \alpha_i\}$

and

$$\deg(P_1^{\beta_1}\cdots P_k^{\beta_k}) = \begin{cases} (\prod_{i=1}^{k}(\beta_i+1)) - 2 & \text{if } \beta_i = \alpha_i,\ \forall i, \\ (\prod_{i=1}^{k}(\beta_i+1)) - 1 & \text{if } \beta_i \geq \frac{\alpha_i}{2},\ \forall i, \\ \prod_{i=1}^{k}(\beta_i+1) & \text{if } \exists i,\ \beta_i < \frac{\alpha_i}{2}. \end{cases}$$

By Theorem 3.1, we have $Ass(M) = \{P_1, \ldots, P_k\}$ and by Lemma 3.2, $\deg P_i = 1+1 = 2$. Conversely, let $\deg(P_1^{\beta_1}\cdots P_k^{\beta_k}) = 2$. If $(\prod_{i=1}^{k}(\beta_i+1)) - 2 = 2$, then $(\prod_{i=1}^{k}(\beta_i+1)) = 4$. We have the following two cases:

(i) $\alpha_1 = 3$ and for every $i \neq 1$, $\alpha_i = 0$.

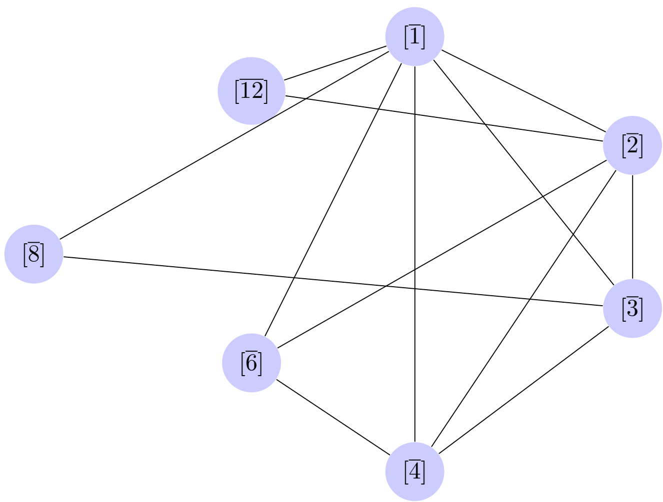
(ii) $\alpha_1 = \alpha_2 = 1$ and for every $i \neq 1, 2$, $\alpha_i = 0$.

In both above cases, $|V(A_E(M))| \leq 4$, which is a contradiction.

If $(\prod_{i=1}^{k}(\beta_i+1)) - 1 = 2$, then $(\prod_{i=1}^{k}(\beta_i+1)) = 3$. So $\beta_1 = 2$ and for every $i \neq 1$, $\beta_i = 0$. Hence $\alpha_1 \leq 4$ and for every $i \neq 1$, $\alpha_i = 0$. Therefore $|V(A_E(M))| \leq 4$, which is a contradiction. Now let $\prod_{i=1}^{k}(\beta_i+1) = 2$. So there exists i $(1 \leq i \leq k)$ such that $\beta_i + 1 = 2$ and for every $j \neq i$ $(1 \leq j \leq k)$, $\beta_j + 1 = 1$. So $\beta_i = 1$ and for every $j \neq i$ $(1 \leq j \leq k)$, $\beta_j = 0$. Therefore, $P_1^{\beta_1}\cdots P_k^{\beta_k} = P_i \in Ass(M)$. □

If $|V(A_E(M)| = 1, 2, 3$ or 4, then Theorem 3.3 is not necessary true. For example for $\mathbb{Z}$-modules $\mathbb{Z}_2$, $\mathbb{Z}_4$, $\mathbb{Z}_8$ and $\mathbb{Z}_{16}$ we have $A_E(\mathbb{Z}_2) \cong K_1$, $A_E(\mathbb{Z}_4) \cong K_2$, $A_E(\mathbb{Z}_8) \cong K_3$ and $A_E(\mathbb{Z}_{16}) \cong \Theta_{2,2,1}$, where $\Theta_{2,2,1}$ is the graph K_4 with one edge deleted. But $Ass(\mathbb{Z}_2) = Ass(\mathbb{Z}_4) = Ass(\mathbb{Z}_8) = Ass(\mathbb{Z}_{16}) = \{2\mathbb{Z}\}$.

The following examples illustrate Theorem 3.3, when R is a PID and R is a Dedekind domain but it is not a PID, respectively.



$A_E(\mathbb{Z}_{24})$

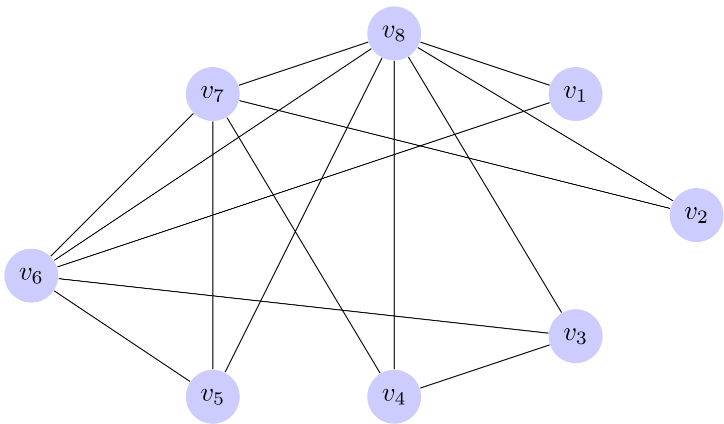
Example 3.4. Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_{24}$. We have

$$V(A_E(\mathbb{Z}_{24})) = \{[\overline{1}], [\overline{2}], [\overline{3}], [\overline{4}], [\overline{6}], [\overline{8}], [\overline{12}]\}.$$

Then $Ass(\mathbb{Z}_{24}) = \{ann(\overline{12}) = 2\mathbb{Z}, ann(\overline{8}) = 3\mathbb{Z}\}$.

Example 3.5. Let $R = \mathbb{Z}[\sqrt{10}]$, $I = \langle 10, 10\sqrt{10}\rangle$ and $M = \frac{R}{I}$. We know that R is a Dedekind domain, but it is not a PID. We have $ann(5\sqrt{10}+I) = \langle 2, \sqrt{10}\rangle$, $ann(2\sqrt{10} + I) = \langle 5, \sqrt{10}\rangle$, $ann(5 + I) = \langle 2, 2\sqrt{10}\rangle$, $ann(2 + I) = \langle 5, 5\sqrt{10}\rangle$, $ann(\sqrt{10}+I) = \langle 10, \sqrt{10}\rangle$, $ann(2+5\sqrt{10}+I) = \langle 10, 5\sqrt{10}\rangle$, $ann(5+2\sqrt{10}+I) = \langle 10, 2\sqrt{10}\rangle$ and $ann(1 + I) = \langle 10, 10\sqrt{10}\rangle$.

Put $v_1 = [5\sqrt{10}+I]$, $v_2 = [2\sqrt{10}+I]$, $v_3 = [5+I]$, $v_4 = [2+I]$, $v_5 = [\sqrt{10}+I]$, $v_6 = [2 + 5\sqrt{10} + I]$, $v_7 = [5 + 2\sqrt{10} + I]$ and $v_8 = [1 + I]$.



$A_E(M)$

Then $Ass(M) = \{P_1 = \langle 2, \sqrt{10}\rangle, P_2 = \langle 5, \sqrt{10}\rangle\}$.

Corollary 3.6. *Let R be a Dedekind domain and $0 \neq I$ be an ideal of R and $S = \frac{R}{I}$. If $|V(A_E(S))| \geq 4$, then a vertex of $A_E(S)$ has degree one if and only if it is an associated prime ideal of S.*

Proof. The proof follows from Theorem 3.3 and the observation after Theorem 3.3. □

In Example 2.9, $A_E(\mathbb{Z}_{24})$ is an example for Corollary 3.6.

Corollary 3.7. *Let R be a Dedekind domain and M be a torsion finitely generated R-module. If $A_E(M) = C_n$, then $n = 3$.*

Proof. By the proof of Theorem 3.1, there exists $x \in T(M)^*$ such that $ann(M) = ann(x)$ and hence $[x]$ is adjacent to any vertex. So $\deg([x]) = n - 1$. Since the degree of any vertex in C_n is two, hence $n - 1 = 2$ and we have $n = 3$. □

Proposition 3.8. *Let R be a Dedekind domain and M be a torsion finitely generated R-module. If $A_E(M) = K_n$, then $1 \leq n \leq 3$.*

Proof. Since $A_E(M)$ is complete, by part (iv) of Proposition 2.5, we have $|Ass(M)| \leq 2$. Suppose that $Ass(M) = \{P, Q\}$. Since $A_E(M)$ is complete, $PQ \subseteq ann(M)$. But $ann(M) \subseteq P$ and $ann(M) \subseteq Q$, hence $ann(M) \subseteq PQ$ and thus $ann(M) = PQ$. So by Theorem 3.1, $V(A_E(M)) = \{P, Q, PQ\}$ and hence $A_E(M) = K_3$. Now suppose that $|Ass(M)| = 1$ and $Ass(M) = \{P\}$. By Theorem 3.1, $ann(M) = P^\alpha$ for some $\alpha \in \mathbb{N}$. Thus $V(A_E(M)) = \{P, \ldots, P^\alpha\}$. If $\alpha \geq 4$, then P and P^2 are not adjacent, which is a contradiction. Therefore, $1 \leq \alpha \leq 3$ and we have $1 \leq n \leq 3$. □

Theorem 3.9. *Let M_1 and M_2 be torsion finitely generated modules over a Dedekind domain R such that $A_E(M_1) \cong A_E(M_2)$ and $|V(A_E(M_1))| = |V(A_E(M_2))| \geq 5$. If $ann(M_1) = P_1^{\alpha_1} \cdots P_k^{\alpha_k}$ and $ann(M_2) = Q_1^{\beta_1} \cdots Q_s^{\beta_s}$ are the decompositions of $ann(M_i)$, $i = 1, 2$, into prime ideals of R such that $\alpha_1 \geq \cdots \geq \alpha_k$ and $\beta_1 \geq \cdots \geq \beta_s$, then $k = s$ and $|Ass(M_1)| = |Ass(M_2)| = k$. Furthermore, for every i, $1 \leq i \leq k$, $\alpha_i = \beta_i$.*

Proof. By Theorem 3.1, $k = |Ass(M_1)| = |Ass(M_2)| = s$. If $\alpha_1 = 1$, then it is clear that $\alpha_i = \beta_i = 1$ $(1 \leq i \leq k)$. So let $\alpha_1 > 1$. By Lemma 3.2, $\deg(P_1^{\alpha_1 - 1} P_2^{\alpha_2} \cdots P_k^{\alpha_k}) = \alpha_1(\alpha_2 + 1) \cdots (\alpha_k + 1) - 1$ is the second maximum degree of $A_E(M_1)$. Then $\alpha_1(\alpha_2+1) \cdots (\alpha_k+1) - 1 = \deg(P_1^{\alpha_1 - 1} P_2^{\alpha_2} \cdots P_k^{\alpha_k}) = \deg(Q_1^{\beta_1 - 1} Q_2^{\beta_2} \cdots Q_k^{\beta_k}) = \beta_1(\beta_2 + 1) \cdots (\beta_k + 1) - 1$ and we have $\prod_{i=1}^{k}(\alpha_i + 1) = \prod_{i=1}^{k}(\beta_i + 1)$. Thus $\alpha_1 = \beta_1$. Now for every $0 \leq s \leq \alpha_1$, we have $\deg(P_1^{\alpha_1 - s} P_2^{\alpha_2} \cdots P_k^{\alpha_k}) = \deg(Q_1^{\alpha_1 - s} Q_2^{\beta_2} \cdots Q_k^{\beta_k})$ and there exists s such that

$$\begin{aligned} \deg(P_1^{\alpha_1 - s - 1} P_2^{\alpha_2} \cdots P_k^{\alpha_k}) &< \deg(P_1^{\alpha_1} P_2^{\alpha_2 - 1} P_3^{\alpha_3} \cdots P_k^{\alpha_k}) \\ &\leq \deg(P_1^{\alpha_1 - s} P_2^{\alpha_2} \cdots P_k^{\alpha_k}). \end{aligned}$$

Therefore, $\deg(P_1^{\alpha_1} P_2^{\alpha_2 - 1} P_3^{\alpha_3} \cdots p_k^{\alpha_k}) = \deg(Q_1^{\alpha_1} Q_2^{\beta_2 - 1} Q_3^{\beta_3} \cdots Q_k^{\beta_k})$. So $\alpha_2 = \beta_2$. Let $\alpha_i = \beta_i$, for every i, $1 \leq i \leq t-1$. Then there exist s_i, $0 \leq s_i \leq \alpha_i$ $(1 \leq i \leq t-1)$, such that $\deg(P_1^{\alpha_1} \cdots P_{t-1}^{\alpha_{t-1}} P_t^{\alpha_t - 1} P_{t+1}^{\alpha_{t+1}} \cdots P_k^{\alpha_k}) \leq \deg(P_1^{\alpha_1 - s_1} \cdots P_{t-1}^{\alpha_{t-1} - s_{t-1}} P_t^{\alpha_t} \cdots P_k^{\alpha_k})$. Also for every i $(1 \leq i \leq t-1)$, we have

$$\begin{aligned} &\deg(P_1^{\alpha_1 - s_1} \cdots P_{i-1}^{\alpha_{i-1} - s_{i-1}} P_i^{\alpha_i - s_i - 1} P_{i+1}^{\alpha_{i+1} - s_{i+1}} \cdots P_{t-1}^{\alpha_{t-1} - s_{t-1}} P_t^{\alpha_t} \cdots P_k^{\alpha_k}) \\ < \ &\deg(P_1^{\alpha_1} \cdots P_{t-1}^{\alpha_{t-1}} P_t^{\alpha_t - 1} P_{t+1}^{\alpha_{t+1}} \cdots P_k^{\alpha_k}). \end{aligned}$$

Hence

$$\deg(P_1^{\alpha_1} \cdots P_{t-1}^{\alpha_{t-1}} P_t^{\alpha_t - 1} P_{t+1}^{\alpha_{t+1}} \cdots P_k^{\alpha_k}) = \deg(Q_1^{\alpha_1} \cdots Q_{t-1}^{\alpha_{t-1}} Q_t^{\beta_t - 1} Q_{t+1}^{\beta_{t+1}} \cdots Q_k^{\beta_k})$$

and it follows that $\alpha_t = \beta_t$. Therefore for every i $(1 \leq i \leq k)$, $\alpha_i = \beta_i$. □

Note that Theorem 3.9 is true, when $|V(A_E(M_1))| = |V(A_E(M_2))| = 1, 2$ or 4. Also Theorem 3.9 is not necessarily true, when $|V(A_E(M_1))| = |V(A_E(M_2))| = 3$. For example for $\mathbb{Z}$-modules $\mathbb{Z}_6$ and $\mathbb{Z}_8$, we have $A_E(\mathbb{Z}_6) \cong A_E(\mathbb{Z}_8) \cong K_3$. But $Ass(\mathbb{Z}_6) = \{2\mathbb{Z}, 3\mathbb{Z}\}$, $Ass(\mathbb{Z}_8) = \{2\mathbb{Z}\}$, $ann(\mathbb{Z}_8) = (2\mathbb{Z})^3$ and $ann(\mathbb{Z}_6) = (2\mathbb{Z})(3\mathbb{Z})$.

Let $n \in \mathbb{N}$ and p be a prime number. Then by Theorem 3.1, we have $|\Omega_{\mathbb{Z}}(\mathbb{Z}_{p^n})| = |V(A_E(\mathbb{Z}_{p^n}))| = n$. Now for every $n \in \mathbb{N}$, we shall obtain the number of graphs Γ (up to isomorphism) such that there exist a Dedekind domain R and a torsion finitely generated R-module M with $|V(A_E(M))| = n$ and $A_E(M) \cong \Gamma$. Let $k \in \mathbb{N}$ and $\{\alpha_1, \ldots, \alpha_k\} \subseteq \mathbb{N}$ be such that $n+1 = \prod_{i=1}^{k}(\alpha_i + 1)$. Also, $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where p_i $(1 \leq i \leq k)$ are prime numbers and for $i \neq j$ $(1 \leq i, j \leq k)$, $p_i \neq p_j$. Then for $\mathbb{Z}$-module $\mathbb{Z}_m$, we have $ann(\mathbb{Z}_m) = (p_1\mathbb{Z})^{\alpha_1} \cdots (p_k\mathbb{Z})^{\alpha_k}$ and $|V(A_E(\mathbb{Z}_m))| = (\prod_{i=1}^{k}(\alpha_i + 1)) - 1 = n$. Conversely, let the graph Γ be such that $|V(\Gamma)| = n$ and suppose that there exist a Dedekind domain R and a torsion finitely generated R-module M such that $A_E(M) \cong \Gamma$. Let $ann(M) = P_1^{\alpha_1} \cdots P_k^{\alpha_k}$ be the decomposition of $ann(M)$ to prime ideals of R. Let $m = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, where p_i $(1 \leq i \leq k)$ are prime numbers and for $i \neq j$ $(1 \leq i, j \leq k)$, $p_i \neq p_j$. So for $\mathbb{Z}$-module $\mathbb{Z}_m$, we have $A_E(M) \cong A_E(\mathbb{Z}_m)$. Then the number of graphs Γ such that there exist a Dedekind domain R and a torsion finitely generated R-module M with $A_E(M) \cong \Gamma$ and $|V(A_E(M))| = n$, is equal the number of products $n + 1 = \prod_{i=1}^{k} a_i$, where $k \in \mathbb{N}$ and $a_i \geq 2$ $(1 \leq i \leq k)$.

Example 3.10. Since $7 + 1 = 8$ and $8 = 8$, $8 = 4 \times 2$, $8 = 2 \times 2 \times 2$, there exist three graphs Γ such that there exist a Dedekind domain R and a torsion finitely generated R-module M with $A_E(M) \cong \Gamma$ and $|V(A_E(M))| = 7$. For example, $\mathbb{Z}_{128}$, $\mathbb{Z}_{30}$, $\mathbb{Z}_{24}$.

The number 3 is an exception, because $3+1 = 4$ and $4 = 4$, $4 = 2 \times 2$. Hence we must have two graphs such that there exist a Dedekind domain R and a torsion finitely generated R-module M with $|V(A_E(M))| = 3$, and $A_E(M) \cong \Gamma$. But $A_E(\mathbb{Z}_{p^3}) \cong A_E(\mathbb{Z}_{pq}) \cong K_3$, where p, q are prime numbers.

Then for every torsion finitely generated module M over a Dedekind domain R, there exists $m \in \mathbb{N}$ such that $A_E(M) \cong A_E(\mathbb{Z}_m)$. Now let $ann(M) = P_1^{\alpha_1} \cdots P_k^{\alpha_k}$ be the decomposition of $ann(M)$ into prime ideals of R and $m = p_1^{\beta_1} \cdots p_s^{\beta_s}$ be the decomposition of m into prime numbers of $\mathbb{Z}$ such that $\alpha_1 \geq \cdots \geq \alpha_k$ and $\beta_1 \geq \cdots \geq \beta_s$. By Theorem 3.9, we have $k = s$ and for every i $(1 \leq i \leq k)$, $\alpha_i = \beta_i$.

Theorem 3.11. *Let M be a finitely generated module over a Dedekind domain R with $ann(M) \notin Spec(R)$. Suppose that $ann(M) = P_1^{\alpha_1} \cdots P_k^{\alpha_k} Q_1^{\beta_1} \cdots Q_t^{\beta_t}$ is the decomposition of $ann(M)$ into prime ideals of R such that for every i $(1 \leq i \leq k)$, α_i is even and for every j $(1 \leq j \leq t)$, β_j is odd. Then $\nu(A_E(M)) = (\prod_{i=1}^{k}(\frac{\alpha_i}{2} + 1)\prod_{j=1}^{t}(\frac{\beta_j+1}{2})) + t$.*

Proof. Let $ann(M) = T_1^{\gamma_1} \cdots T_s^{\gamma_s}$, where $\{T_1, \ldots, T_s\} = \{P_1, \ldots, P_k, Q_1, \ldots, Q_t\}$ and $\{\gamma_1, \ldots, \gamma_s\} = \{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_t\}$. We define the function $f : \mathbb{N} \longrightarrow \mathbb{N}$ by

$$f(\gamma) = \begin{cases} \frac{\gamma}{2} + 1 & \text{if } \gamma \text{ is even,} \\ \frac{\gamma+1}{2} & \text{if } \gamma \text{ is odd.} \end{cases}$$

Now we consider $\{(i_1, \ldots, i_s) \mid 0 \leq i_j \leq \gamma_j \text{ and } 1 \leq j \leq s\}$ with the following ordering:

$$\begin{aligned}(0,\ldots,0) &< (1,0,\ldots,0) < \cdots < (\gamma_1,0,\ldots,0) < (0,1,0,\ldots,) \\ &< (1,1,0,\ldots,0) < (\gamma_1,1,0,\ldots,0) < (0,2,\ldots,0) \\ &< (1,2,\ldots,0) < \cdots < (0,\gamma_2,0\cdots,0) < (1,\gamma_2,0,\ldots,0) \\ &< \cdots < (0,\ldots,0,\gamma_s) < (1,0,\ldots,0,\gamma_s) < \cdots < (\gamma_1,\ldots,\gamma_s).\end{aligned}$$

For every $(i_1, \ldots, i_s)$, we consider the subsets $V_{(i_1,\ldots,i_s)}$ of $V(A_E(M))$ that satisfy the following conditions:

(i) $T_1^{\gamma_1-i_1}\cdots T_s^{\gamma_s-i_s} \in V_{(i_1,\ldots,i_s)}$;

(ii) for every $(l_1,\ldots,l_s) < (i_1,\ldots,i_s)$, $V_{(i_1,\ldots,i_s)} \bigcap V_{(l_1,\ldots,l_s)} = \emptyset$;

(iii) for every $v \in V(A_E(M))$ such that $v \notin \bigcup_{(l_1,\ldots,l_s)<(i_1,\ldots,i_s)} V_{(l_1,\ldots,l_s)}$ and v and $T_1^{\gamma_1-i_1}\cdots T_s^{\gamma_s-i_s}$ are not adjacent, then $v \in V_{(i_1,\ldots,i_s)}$.

Now we have $V_{(i_1,0,\ldots,0)} \neq \emptyset$. If γ_1 is even, then $0 \leq i_1 \leq f(\gamma_1) - 1$ and if γ_1 is odd, then $0 \leq i_1 \leq f(\gamma_1)$. Also, $V_{(i_1,i_2,0,\ldots,0)} \neq \emptyset$, when $0 \leq i_1 \leq f(\gamma_1) - 1$ and $0 \leq i_2 \leq f(\gamma_2) - 1$. Moreover, if γ_1 is odd (or γ_2 is odd), then $V_{(\frac{\gamma_1+1}{2},0,\ldots,0)} \neq \emptyset$ (or $V_{(0,\frac{\gamma_2+1}{2},0,\ldots,0)} \neq \emptyset$). Similarly, we have $V_{(i_1,\ldots,i_s)} \neq \emptyset$, when $0 \leq i_j \leq f(\gamma_j) - 1$ and if γ_j is odd, we have $V_{(0,\ldots,0,\frac{\gamma_j+1}{2},0,\ldots,0)} \neq \emptyset$. Let $I = \{\gamma_j \mid \gamma_j \text{ is odd}, 1 \leq j \leq s\}$, $|I| = t$, $A = \{V_{(i_1,\ldots,i_s)} \mid V_{(i_1,\ldots,i_s)} \neq \emptyset\}$ and $|A| = a$. Then $a = \prod_{i=1}^{s} f(\gamma_i) + t$. Since $\bigcup_{(i_1,\ldots,i_s)} V_{(i_1,\ldots,i_s)} = V(A_E(M))$ and for every $V_{(i_1,\ldots,i_s)}, V_{(l_1,\ldots,l_s)} \in A$, $V_{(i_1,\ldots,i_s)} \cap V_{(l_1,\ldots,l_s)} = \emptyset$ and the vertices of no $V_{(i_1,\ldots,i_s)}$ are adjacent, the set A is a colour partition of $A_E(M)$. Since $|A| = a$, hence $\nu(A_E(M)) \leq a$. Now let $T_1^{\gamma_1-i_1}\cdots T_s^{\gamma_s-i_s} \in V_{(i_1,\ldots,i_s)}$ and $T_1^{\gamma_1-i_1'}\cdots T_s^{\gamma_s-i_s'} \in V_{(i_1',\ldots,i_s')}$. We consider j, $1 \leq j \leq s$. Now we have the following two cases:

(i) γ_j is even. If $0 \leq i_j \leq \frac{\gamma_j}{2}$ and $0 \leq i_j' \leq \frac{\gamma_j}{2}$, then $i_j + i_j' \leq \gamma_j$. So $2\gamma_j - (i_j + i_j') \geq \gamma_j$.

(ii) γ_j is odd. If $0 \leq i_j \leq \frac{\gamma_j-1}{2}$ and $0 \leq i_j' \leq \frac{\gamma_j-1}{2}$, then $i_j + i_j' \leq \gamma_j - 1$. So $2\gamma_j - (i_j + i_j') \geq \gamma_j + 1$. But if $i_j = \frac{\gamma_j+1}{2}$ and $0 \leq i_j' \leq \frac{\gamma_j-1}{2}$, then $i_j + i_j' \leq \gamma_j$ and hence $2\gamma_j - (i_j + i_j') \geq \gamma_j$. Therefore, the induced subgraph generated by $\{T_1^{\gamma_1-i_1}\cdots T_s^{\gamma_s-i_s} \mid T_1^{\gamma_1-i_1}\cdots T_s^{\gamma_s-i_s} \in V_{(i_1,\ldots,i_s)}\}$ is the complete graph K_a. So $\nu(A_E(M)) \geq a$ and hence

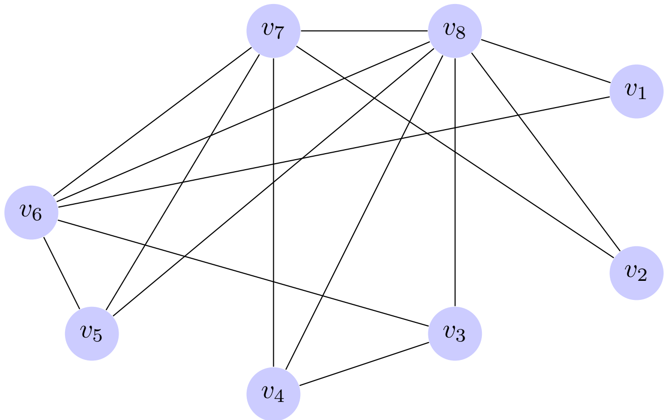
$$\nu(A_E(M)) = a = (\prod_{i=1}^{s} f(\gamma_i)) + t = (\prod_{i=1}^{k}(\frac{\alpha_i}{2}+1)\prod_{j=1}^{t}(\frac{\beta_j+1}{2})) + t.$$

□

Now suppose $ann(M) \in Spec(R)$. Since $|V(A_E(M))| = 1$, hence $\nu(A_E(M)) = 1$. But by Theorem 3.11, we have $\nu(A_E(M)) = \frac{1+1}{2} + 1 = 2$. Therefore, Theorem 3.11 is not necessarily valid in the case $ann(M) \in Spec(R)$.

In Example 3.5, we show that $|V(A_E(M))| = 8$ and $|Ass(M)| = 2$. Then we have $(\alpha_1 + 1)(\alpha_2 + 1) = 8 + 1 = 9$, hence $\alpha_1 = \alpha_2 = 2$. So $ann(M) = P_1^2 P_2^2$. In the following example, we obtain $\nu(A_E(M))$ for Example 3.5.

Example 3.12. Let $R = \mathbb{Z}[\sqrt{10}]$, $I = \langle 10, 10\sqrt{10} \rangle$ and $M = \frac{R}{I}$. Since $ann(M) = P_1^2 P_2^2$, we have $\nu(A_E(M)) = (\frac{2}{2} + 1) \times (\frac{2}{2} + 1) = 4$.



$A_M(R)$

Corollary 3.13. *Let M be a torsion finitely generated module over a Dedekind domain R. Then the clique number and the chromatic number of $A_E(M)$ are equal.*

Proof. In the notation of Theorem 3.11, we have $\nu(A_E(M)) = a$, where K_a is a subgraph of $A_E(M)$. So $a \leq \chi(A_E(M))$. Let $a \neq \chi(A_E(M))$. Then there exists $b > a$ such that K_b is a complete subgraph of $A_E(M)$ and hence $\nu(A_E(M)) \geq b > a$, which is a contradiction. So $\nu(A_E(M)) = \chi(A_E(M)) = a$. □

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