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RESIDUAL p-FINITENESS OF CERTAIN HNN EXTENSIONS OF FREE ABELIAN GROUPS OF FINITE RANK

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Abstract. Let p be a prime. A group G is said to be residually p-finite if for each non-trivial element x of G, there exists a normal subgroup N of index a power of p in G such that x is not in N. In this note we shall prove that certain HNN extensions of free abelian groups of finite rank are residually p-finite. In addition some of these HNN extensions are subgroup separable. Characterisations for certain one-relator groups and similar groups including the Baumslag-Solitar groups to be residually p-finite are proved.

1. Introduction

Let p be a prime. A group G is said to be residually p-finite if for each non-trivial element x of G, there exists a normal subgroup N of index a power of p in G such that $x \notin N$. However not many classes of groups are known to be residually p-finite. Free groups and finitely generated torsion-free nilpotent groups are residually p-finite for all primes p (see [6, 9]). Gruenberg in [6] had proved that residual nilpotence and being residually of prime-power order are equivalent properties for the class of finitely generated groups. In [8], Higman proved that a generalised free product of two finite p-groups amalgamating a cyclic subgroup, is residually p-finite. Kim and McCarron [11] then generalised Higman's result by proving that the generalised free product of residually p-groups amalgamating a finite cyclic subgroup, is residually p-finite. Other important results on the residual p-finiteness of generalised free products, tree products, polygonal products and certain one relator groups can be found in the papers [11–14, 16] by Kim, McCarron and Tang and [24, 25] by Wong and Tang.

On the other hand, the residual properties of HNN extensions of groups are difficult to obtain since one of the simplest type of an HNN extension of a cyclic group, the torsion-free Baumslag-Solitar group, $BS(2,3) = \langle t, a \mid t^{-1}a^2t = a^3\rangle$ is not even residually finite (see [4]). Another example is given by Kim and

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Tang [15] of a torsion-free HNN extension of a finitely generated free nilpotent group of class 2 that is not residually finite.

In this note we prove that certain HNN extensions of free abelian groups of finite rank are residually p-finite including proving a criterion. In addition we shall show some of these HNN extensions are subgroup separable. We shall apply our results to show characterisations for certain one-relator groups and similar groups, including the Baumslag-Solitar groups, to be residually p-finite. Some of these characterisations are based on those by Raptis and Varsos in [17] and Andredakis, Raptis and Varsos in [1] on residual finiteness.

Raptis and Varsos in the papers [17–19] proved important results on the residual nilpotence and residual p-finiteness of HNN extensions with base groups finitely generated abelian groups. In particular, Raptis and Varsos in [19], showed that the HNN extensions of a finite abelian p-group where the associated subgroups have trivial intersection are residually p-finite (see Theorem 3.3 in this paper). We shall use results from Andreadakis, Raptis and Varsos in the papers [2, 17, 19] directly. The motivation for this paper are the following two results by Raptis and Varsos in [19], [17], respectively (listed as Theorem 3.3 and Theorem 4.10 in this paper).

Theorem 3.3 ([19]). *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an HNN extension, where K is a finite abelian p-group. If $A \cap B = 1$, then G is residually p-finite.*

Theorem 4.10 ([17]). *Let K be a finitely generated abelian group, $A, B \leq K$ and $\varphi : A \to B$ an isomorphism, let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be the corresponding HNN extension. If $A \cap B$ is finite, then G is residually finite.*

Theorem 3.3 is not explicitly stated in [19] but it is a consequence of Corollary 5.1 and Corollary 1.2 in [19]. Theorem 3.3 will be used in the proof of Theorem 3.4.

In this paper, our main results are the following theorems:

Theorem 3.4. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an HNN extension, where K is a free abelian group of finite rank. If $A \cap B = 1$, then G is residually p-finite.*

Theorem 3.5. *Let $G = \langle t, K; t^{-1}At = A, \varphi\rangle$ be an HNN extension, where K is a free abelian group of finite rank. Then G is residually p-finite.*

Theorem 4.1. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an HNN extension, where K is a free abelian group of finite rank with $A \cap B = 1$. Then the following are equivalent*:

(i) *G is residually p-finite.*
(ii) *G is residually finite.*
(iii) *There exists $N \lhd_f K$ such that $(A \cap N)\varphi = B \cap N$ and $A \cap N, B \cap N$ are isolated in N.*
(iv) *There exists a free abelian group X of finite rank such that K is a subgroup of finite index in X and an automorphism $\bar{\varphi} \in \mathrm{Aut}X$ such that $\bar{\varphi}|_A = \varphi$.*

By using the criterion Theorem 4.1, we have the following:

Corollary 4.12. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an HNN extension, where K is a free abelian group of finite rank. Suppose $A \cap B = 1$ and A and B have finite indices in K. Then G is residually p-finite and subgroup separable.*

Our last result, Theorem 5.4, which is a characterisation on the residually p-finiteness of the Baumslag-Solitar groups, can be derived from the Main Theorem of Kim and McCarron [12] but we shall provide a partial proof using Theorem 3.5.

Theorem 5.4. *Let $G = \langle t, a \mid t^{-1}a^r t = a^s\rangle$, where $r, s \in \mathbb{Z}$. Suppose the following conditions ($r = 1$, $s \equiv 1 \pmod p$) and ($s = 1$, $r \equiv 1 \pmod p$) do not occur. Then G is residually p-finite if and only if $|r| = |s|$.*

The notation used here is standard. In addition, the following will be used for any group G:

(i) p denotes a prime.
(ii) $\mathbb{N}$ denotes the set of natural numbers.
(iii) $\mathbb{Z}$ denotes the set of integers.
(iv) $H \leq G$ (resp. $H \leq_f G$) means H is a subgroup (resp. a subgroup of finite index) in G
(v) $N \lhd_f G$ (resp. $N \lhd_p G$) means N is a normal subgroup of finite index (resp. a normal subgroup of index a power of p) in G.
(vi) $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ denotes an HNN extension, where K is the base group, A, B are the associated subgroups and $\varphi : A \to B$ is the associated isomorphism from A to B.

2. Preliminaries

We now state the main definitions as well as some essential lemmas.

Definition 2.1. A group G is said to be residually p-finite if for each $1 \neq x \in G$, there exists $N \lhd_p G$ such that $x \notin N$.

Definition 2.2. A group G is said to be subgroup separable if for every finitely generated subgroup H of G and every $x \in G\backslash H$, there exists $N \lhd_f G$ such that $x \notin HN$.

We state the following well known theorem with a complete proof (see [22]).

Lemma 2.3. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an HNN extension, where K is a finite group. Then G is free-by-finite and hence G is subgroup separable (residually finite).*

Proof. The group G is free-by-finite (see [7, 10]). We note that free groups are subgroup separable (see [7]), and finite extensions of a subgroup separable group are again subgroup separable (see [20, 21]). Hence the group G is subgroup separable. □

Next we recall the definition of an isolated subgroup.

Definition 2.4. Let G be a group and $H < G$. Then the subgroup H is isolated in G if whenever $g^n \in H$ for $g \in G$ and $n \in \mathbb{N}$, we have $g \in H$.

Lemma 2.5. *Let G be a group and $H \lhd G$. Then H is isolated in G if and only if G/H is torsion free.*

3. The main results

In this section, we obtain two of our main results. Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$, where K is a free abelian group of finite rank. If $A \cap B = 1$ or $A = B$, then G is residually p-finite. We begin with three results of Raptis and Varsos.

Lemma 3.1 ([19, Corollary 5.1])**.** *Let K be a finite abelian p-group. Let $A, B \leq K$ and $\varphi : A \to B$ an isomorphism. If $A \cap B = 1$, then there exist a finite abelian p-group X and an automorphism θ of X such that $K \leq X$, $|\theta| = p^s$ for some $s \in \mathbb{N}$ and $\theta|_A = \varphi$.*

Lemma 3.2 ([19, Corollary 1.2])**.** *Let K be a finite p-group, $A, B \leq K$ and $\theta \in \mathrm{Aut}K$ such that $|\theta| = p^s$ for some $s \in \mathbb{N}$ and $A\theta = B$. If $\varphi = \theta|_A$, then the HNN extension $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ is residually p-finite.*

Theorem 3.3 is not explicitly stated in [19] but it is a consequence of Lemma 3.1 and Lemma 3.2.

Theorem 3.3 ([19])**.** *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an HNN extension, where K is a finite abelian p-group. If $A \cap B = 1$, then G is residually p-finite.*

Proof. By Lemma 3.1, there exist a finite abelian p-group X and an automorphism θ of X such that $K \leq X$, $|\theta| = p^s$ for some $s \in \mathbb{N}$ and $\theta|_A = \varphi$. Let $G^* = \langle t, X; t^{-1}At = B, \varphi\rangle$. Now by Lemma 3.2, G^* is residually p-finite. Since $G < G^*$, G is residually p-finite. □

First we consider the HNN extension $G = \langle t, K; t^{-1}At = B, \varphi\rangle$, where K is a free abelian group of finite rank and $A \cap B = 1$.

Theorem 3.4. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an HNN extension, where K is a free abelian group of finite rank. If $A \cap B = 1$, then G is residually p-finite.*

Proof. Since K is free abelian, then by Proposition 2.2 of Baumslag [3], $\bigcap K^{p^n} = 1$ for almost all primes p. Since $K \neq A$, $K \neq B$, then again by Proposition 2.2 of Baumslag [3], for a prime p, we have $\bigcap AK^{p^n} = A$, $\bigcap BK^{p^n} = B$ for every $n \in \mathbb{N}$ and also $A \cap K^{p^n} = A^{p^n}$, $B \cap K^{p^n} = B^{p^n}$ for every $n \in \mathbb{N}$. Furthermore $AK^{p^n} \cap BK^{p^n} = K^{p^n}$ since $AK^{p^n} \cap BK^{p^n} = (A \cap B)K^{p^n} = 1K^{p^n} = K^{p^n}$.

Let $1 \neq x \in G$ be a reduced element in G. We prove the theorem by constructing a residually p-finite image group $\bar{G}$ of G such that $\bar{x} \neq \bar{1}$. Then there exists $\bar{P} \lhd_f \bar{G}$ such that $\tilde{x} \neq \tilde{1}$ in $\tilde{G} = \bar{G}/\bar{P}$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $\bar{x} \neq \bar{1}$ in $\bar{G} = G/P$ and we are done.

Case 1. $\|x\| = 0$, that is, $x \in K$. Since $\bigcap K^{p^n} = 1$, there exists $r \in \mathbb{N}$ such that $x \notin K^{p^r}$. Furthermore $(A \cap K^{p^r})\varphi = (A^{p^r})\varphi = B^{p^r} = B \cap K^{p^r}$. Hence we can form $\bar{G} = \langle t, \bar{K}; t^{-1}\bar{A}t = \bar{B}, \bar{\varphi}\rangle$, where $\bar{K} = K/K^{p^r}$, $\bar{A} = AK^{p^r}/K^{p^r}$, $\bar{B} = BK^{p^r}/K^{p^r}$ and $\bar{\varphi} : \bar{A} \to \bar{B}$ is the isomorphism induced by φ. Clearly $\bar{G}$ is a homomorphic image of G. Furthermore from above, we have $\bar{A} \cap \bar{B} = (AK^{p^r}/K^{p^r}) \cap (BK^{p^r}/K^{p^r}) = (AK^{p^r} \cap BK^{p^r})/K^{p^r} = K^{p^r}/K^{p^r} = \bar{1}$. Let $\bar{x}$ denote the image of x in $\bar{G}$. Then $\bar{x}$ has order p^s in $\bar{G}$ for some integer s and so $\bar{x} \neq \bar{1}$. Since $\bar{K}$ is a finite abelian p-group and $\bar{A} \cap \bar{B} = \bar{1}$, then by Theorem 3.3, $\bar{G}$ is residually p-finite and our result now follows.

Case 2. $\|x\| \geq 1$. Without loss of generality, we let $x = t^{e_1}x_1t^{e_2}x_2 \cdots t^{e_n}x_n$, where $x_i \in K$ and $e_i = \pm 1$, $1 \leq i \leq n$, $n \geq 1$. Let u_i denote those x_i in $K \backslash A$, v_i denote those x_i in $K \backslash B$ and w_i those x_i in $A \cup B \backslash 1$. Since $\bigcap K^{p^n} = 1$, $\bigcap AK^{p^n} = A$ and $\bigcap BK^{p^n} = B$ for all $n \in \mathbb{N}$, we can find $r \in \mathbb{N}$ such that $u_i \notin AK^{p^r}$, $v_i \notin BK^{p^r}$ and $w_i \notin K^{p^r}$ for all i. We proceed as in Case 1 to form $\bar{G}$. Then $\bar{x}$ is reduced in $\bar{G}$ and $\|\bar{x}\| = \|x\| \geq 1$. It follows that $\bar{x} \neq \bar{1}$. Since $\bar{G}$ is residually p-finite, we are done. □

Next we consider the HNN extension $G = \langle t, K; t^{-1}At = A, \varphi\rangle$, where K is a free abelian group of finite rank and φ is an automorphism of A, that is, Theorem 3.5 below.

Theorem 3.5. *Let $G = \langle t, K; t^{-1}At = A, \varphi\rangle$ be an HNN extension, where K is a free abelian group of finite rank. Then G is residually p-finite.*

Proof. If $K = A$, then $G = \langle t, K; t^{-1}Kt = K, \varphi\rangle$ is free abelian and hence residually p-finite. Now let $K \neq A$. Since K is free abelian, then by Proposition 2.2 of Baumslag [3], $\bigcap K^{p^n} = 1$ for almost all primes p. Since $K \neq A$, then again by Proposition 2.2 of Baumslag [3], for a prime p, we have $\bigcap AK^{p^n} = A$ for all $n \in \mathbb{N}$.

Let $1 \neq x \in G$ be a reduced element in G. As in the proof of Theorem 3.4, we shall proceed by constructing a residually p-finite image group $\bar{G}$ of G such that $\bar{x} \neq \bar{1}$. The result will then follows.

Case 1. $\|x\| = 0$, that is, $x \in K$. Since $\bigcap K^{p^n} = 1$, there exists $r \in \mathbb{N}$ such that $x \notin K^{p^r}$. Note that since K^{p^r} is characteristic in K, then $(A \cap K^{p^r})$ is characteristic in A. Since φ is an automorphism of A, we have $(A \cap K^{p^r})\varphi = A \cap K^{p^r}$. Hence φ induces an isomorphism from $\bar{A} = AK^{p^r}/K^{p^r}$ onto itself which we denote by $\bar{\varphi}$. We can form $\bar{G} = \langle t, \bar{K}; t^{-1}\bar{A}t = \bar{A}, \bar{\varphi}\rangle$, where $\bar{K} = K/K^{p^r}$, $\bar{A} = AK^{p^r}/K^{p^r}$ and $\bar{\varphi}$ is the isomorphism induced by φ. Clearly $\bar{G}$ is a homomorphic image of G. Let $\bar{x}$ denote the image of x in $\bar{G}$. Then $\bar{x}$ has order p^s in $\bar{G}$ for some integer s and so $\bar{x} \neq \bar{1}$. Let θ be the homomorphism of $\bar{G}$ onto the finite p-group $\bar{K}$ and J be the kernel of θ. Then $J \cap \bar{K} = \bar{1}$. Therefore J is a finitely generated free group and hence J is a residually p-finite group. So $\bar{G}$ is an extension of residually p-finite group by a finite p-group. It follows that $\bar{G}$ is residually p-finite and we are done.

Case 2. $\|x\| \geq 1$. With loss of generality, we let $x = t^{e_1}x_1t^{e_2}x_2 \cdots t^{e_n}x_n$, where $x_i \in K$ and $e_i = \pm 1$, $1 \leq i \leq n$, $n \geq 1$. Let u_i denote those x_i in $K\backslash A$ and v_i denote those x_i in $A\backslash\{1\}$. Since $\bigcap K^{p^n} = 1$ and $\bigcap AK^{p^n} = A$ for all $n \in \mathbb{N}$, there exists $r \in \mathbb{Z}$ such that $u_i \notin AK^{p^r}$ and $v_i \notin K^{p^r}$ for all i. We form $\bar{G} = \langle t, \bar{K}; t^{-1}\bar{A}t = \bar{A}, \bar{\varphi}\rangle$, where $\bar{K} = K/K^{p^r}$, $\bar{A} = AK^{p^r}/K^{p^r}$ and $\bar{\varphi}$ is the isomorphism induced by φ. Clearly $\bar{G}$ is a homomorphic image of G. Let $\bar{x}$ denote the image of x in $\bar{G}$. Then $\bar{x}$ is reduced in $\bar{G}$ and $\|\bar{x}\| = \|x\|$. This implies that $\bar{x} \neq \bar{1}$ in $\bar{G}$. Arguing as in Case 1 above, our result follows. □

4. A criterion

In this section, we prove a criterion, Theorem 4.1, for certain HNN extensions of free abelian groups of finite rank to be residually p-finite. From this criterion we shall give another proof of Theorem 3.4 and also show that certain of these HNN extensions are residually p-finite if and only if they are subgroup separable.

Theorem 4.1. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an HNN extension, where K is a free abelian group of finite rank and $A \cap B = 1$. Then the following are equivalent*:

(i) *G is residually p-finite.*
(ii) *G is residually finite.*
(iii) *There exists $N \lhd_f K$ such that $(A \cap N)\varphi = B \cap N$ and $A \cap N$, $B \cap N$ are isolated in N.*
(iv) *There exists a free abelian group X of finite rank such that K is a subgroup of finite index in X and an automorphism $\bar{\varphi} \in \mathrm{Aut}X$ such that $\bar{\varphi}|_A = \varphi$.*

We shall prove Theorem 4.1 in this order: (i)⇒(ii)⇒(iii)⇒(iv)⇒(i).

Trivially (i) implies (ii). We now prove (ii)⇒(iii). This will be done in Lemma 4.3. First we show Lemma 4.2.

Lemma 4.2. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an HNN extension, where K is a free abelian group of finite rank and $K \neq A$, $K \neq B$. Suppose that G is residually finite. Then $\cap AN = A$ and $\cap BN = B$, where $N \in \Delta = \{N \mid N \lhd_f K$ and $(A \cap N)\varphi = B \cap N\}$.*

Proof. Let $x \in \cap AN\backslash A$, where $N \in \Delta$ and $y \in K\backslash B$. Then $z = [t^{-1}xt, y] \neq 1$. Let $x = an$, where $a \in A$, $n \in N$ and suppose $t^{-1}at = b$, where $b \in B$. Then $zN = [t^{-1}ant, y]N = [t^{-1}at, y]N = [b, y]N = N$ since K is abelian. This implies that $z \in \cap N$. Since G is residually finite, then by Theorem 2.3 of Choon and Bin [5] we have $\cap N = 1$. Thus we have a contradiction since $z \neq 1$. Therefore $\cap AN = A$ and in a similar way we prove $\cap BN = B$. □

We now show (ii)⇒(iii).

Lemma 4.3. *Let $G = \langle t, K; t^{-1}At = B, \varphi \rangle$ be an HNN extension, where K is a free abelian group of finite rank and $K \neq A$, $K \neq B$. Suppose that G is residually finite. Then there exists $N \lhd_f K$ such that $(A \cap N)\varphi = B \cap N$ and $A \cap N, B \cap N$ are isolated in N.*

Proof. By Lemma 4.2, $\cap AN = A$ and $\cap BN = A$, where $N \in \Delta = \{N \mid N \lhd_f K$ and $(A \cap N)\varphi = B \cap N\}$. Let S/A and T/B be the torsion parts of K/A and K/B, respectively. Since K is finitely generated, S/A and T/B are finite. For each non-trivial element $xA \in S/A$, there exists $N_x \in \Delta$ such that $N_x \cap xA = \emptyset$ since $\cap AN = A$. Similarly, for each non-trivial element $yB \in T/B$, there exists $N_y \in \Delta$ such that $N_y \cap yB = \emptyset$ since $\cap BN = B$. Let $N = (\cap N_x) \cap (\cap N_y)$ where the intersection extends over all the finitely many elements of S/A and T/B. Clearly $N \lhd_f K$ and $(A \cap N)\varphi = B \cap N$. By the construction of N, AN/A and BN/B are torsion free. Since $N/A \cap N \simeq AN/A$ and $N/B \cap N \simeq BN/B$, then $N/A \cap N$ and $N/B \cap N$ are torsion free. Thus $A \cap N$ and $B \cap N$ are isolated in N by Lemma 2.5. Hence N is the required normal subgroup. □

Next we prove (iii)⇒(iv). This will be done in Lemma 4.7. First we shall need the following three lemmas in Andreadakis, Raptis and Varsos [2].

Lemma 4.4 ([2, Lemma 1])**.** *Let K be a free abelian group of finite rank, A, B subgroups of K which are direct factors of K and $\varphi : A \to B$ an isomorphism. Then there exists an automorphism $\theta \in \mathrm{Aut}K$ such that $\theta|_A = \varphi$.*

Lemma 4.5 ([2, Lemma 2])**.** *Let A, K be free abelian groups and $\theta_1, \theta_2 : A \to K$ monomorphisms such that $\theta_1|_H = \theta_2|_H$ for some subgroup H of finite index in A. Then $\theta_1 = \theta_2$.*

Lemma 4.6 ([2, Proposition 1])**.** *Let K be a free abelian group of finite rank $r(K) = n$. Let A, B be subgroups of K of finite index in K and $\varphi : A \to B$ an isomorphism. Suppose that there exists a subgroup $H \leq_f K$ with $H \leq A \cap B$ and $H\varphi = H$. Then there exists a free abelian group X with finite rank $r(X) = r(K) = n$ such that K is a subgroup of finite index in X and an automorphism $\theta \in \mathrm{Aut}X$ such that $\theta|_A = \varphi$.*

We now show (iii)⇒(iv).

Lemma 4.7. *Let K be a free abelian group of finite rank. Let A, B be subgroups of K and $\varphi : A \to B$ an isomorphism. Suppose that there exists $N \lhd_f K$ such that $(A \cap N)\varphi = B \cap N$ and $A \cap N$, $B \cap N$ are isolated in N. Then there exists a free abelian group X with finite rank such that K is a subgroup of finite index in X and an automorphism $\bar{\varphi} \in \mathrm{Aut}X$ such that $\bar{\varphi}|_A = \varphi$.*

Proof. Since $A \cap N, B \cap N$ are isolated in N, then $N/A \cap N$ and $N/B \cap N$ are torsion free by Lemma 2.5. Thus $N/A \cap N$ is free abelian and there exists $L < N$ such that $N = (A \cap N) \times L$. Similarly, $N = (B \cap N) \times M$ for some $M < N$. Since $(A \cap N)\varphi = B \cap N$, then by Lemma 4.4, there exists an automorphism τ of N with $\tau|_{A \cap N} = \varphi|_{A \cap N}$.

Next we put $K = K$, $A = B = H = N$ with $\tau : N \to N$ an isomorphism in Lemma 4.6. Note that all the conditions of Lemma 4.6 are satisfied. Hence there exists a free abelian group X with finite rank such that $K \leq_f X$ and an automorphism $\bar{\varphi} \in \mathrm{Aut}X$ such that $\bar{\varphi}|_N = \tau$. Since $\bar{\varphi} \in \mathrm{Aut}X$, we have $\bar{\varphi}|_A : A \to X$. So there are two monomorphisms $\bar{\varphi}|_A : A \to X$ and $\varphi : A \longrightarrow X$ such that $\bar{\varphi}|_{A\cap N} = \tau|_{A\cap N} = \varphi|_{A\cap N}$ for the subgroup $A \cap N \lhd_f A$. Therefore by Lemma 4.5, $\bar{\varphi}|_A = \varphi$ and our result follows. □

Theorem 4.8 below follows from Theorem 3.4 but we shall provide an independent proof using the fact that φ is an automorphism of K.

Lemma 4.8. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an HNN extension, where K is a free abelian group of finite rank and $A \cap B = 1$. If φ is an automorphism of K, then G is residually p-finite.*

Proof. Since K is free abelian, then by Proposition 2.2 of Baumslag [3], $\bigcap K^{p^n} = 1$ for almost all primes p. Since $K \neq A$, $K \neq B$, then again by Proposition 2.2 of Baumslag [3], for a prime p, we have $\bigcap AK^{p^n} = A$ and $\bigcap BK^{p^n} = B$ for all $n \in \mathbb{N}$. Furthermore $AK^{p^n} \cap BK^{p^n} = K^{p^n}$ since $AK^{p^n} \cap BK^{p^n} = (A \cap B)K^{p^n} = 1K^{p^n} = K^{p^n}$.

Let $1 \neq x \in G$ be a reduced element in G. We prove the lemma by constructing a residually p-finite image group $\bar{G}$ of G such that $\bar{x} \neq \bar{1}$. Then there exists $\bar{P} \lhd_f \bar{G}$ such that $\tilde{x} \neq \tilde{1}$ in $\tilde{G} = \bar{G}/\bar{P}$. Let P be the preimage of $\bar{P}$ in G. Then $P \lhd_f G$ such that $\bar{x} \neq \bar{1}$ in $\bar{G} = G/P$ and we are done.

Case 1. $\|x\| = 0$, that is, $x \in K$. Since $\bigcap K^{p^n} = 1$, there exists $r \in \mathbb{N}$ such that $x \notin K^{p^r}$. Note that K^{p^r} is characteristic in K. Since φ is an automorphism of K such that $A\varphi = B$ and K^{p^r} is characteristic in K, we have $(A \cap K^{p^r})\varphi = B \cap K^{p^r}$. This implies that φ induces an isomorphism from $\bar{A} = AK^{p^r}/K^{p^r}$ onto $\bar{B} = BK^{p^r}/K^{p^r}$ which we denote by $\bar{\varphi}$. So we can form the HNN extension $\bar{G} = \langle t, \bar{K}; t^{-1}\bar{A}t = \bar{B}, \bar{\varphi}\rangle$, where $\bar{K} = K/K^{p^r}$. We now proceed as in the proof of Case 1 in Theorem 3.4 and our result follows.

Case 2. $\|x\| \geq 1$. The proof of this case is similar to the proof of Case 2 in Theorem 3.4. □

We now show (iv)⇒(i).

Lemma 4.9. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an HNN extension, where K is a free abelian group of finite rank and $A \cap B = 1$. If there exists a free abelian group X with finite rank such that K is a subgroup of finite index in X and an automorphism $\bar{\varphi} \in \mathrm{Aut}X$ such that $\bar{\varphi}|_A = \varphi$, then G is residually p-finite.*

Proof. Let $G^* = \langle t, X; t^{-1}At = B, \varphi\rangle$. Now φ comes from the automorphism $\bar{\varphi}$ of X and hence by Theorem 4.8, G^* is residually p-finite. Since $G < G^*$, G is residually p-finite. □

Remark. Theorem 4.1 now follows from Lemmas 4.3, 4.7 and 4.8.

Next we give another proof of Theorem 3.4 by using Theorem 4.1 and the following theorem from Raptis & Varsos [17].

Theorem 4.10 ([17, Proposition 1]). *Let K be a finitely generated abelian group, $A, B \leq K$ and $\varphi : A \to B$ an isomorphism, let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be the corresponding HNN extension. If $A \cap B$ is finite, then G is residually finite.*

Remark. Another proof of Theorem 3.4:
Proof of Theorem 3.4. By Theorem 4.10, G is residually finite. The result now follows from Theorem 4.1. □

Now suppose $A \cap B = 1$ and A and B have finite indices in K. Then we can show that $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ is residually p-finite if and only if G is subgroup separable.

Corollary 4.11. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an HNN extension, where K is a free abelian group of finite rank. Suppose $A \cap B = 1$ and A and B have finite indices in K. Then G is residually p-finite if and only if G is subgroup separable.*

Proof. Suppose G is subgroup separable. Then G is residually finite and by Theorem 4.1, G is residually p-finite.

Suppose G is residually p-finite. Then from Theorem 4.1, there exists a free abelian group X of finite rank such that K is a subgroup of finite index in X and an automorphism $\bar{\varphi} \in \mathrm{Aut}X$ such that $\bar{\varphi}|_A = \varphi$. Hence by Theorem 1 of Wong [23], G is subgroup separable. □

Corollary 4.12. *Let $G = \langle t, K; t^{-1}At = B, \varphi\rangle$ be an HNN extension, where K is a free abelian group of finite rank. Suppose $A \cap B = 1$ and A and B have finite indices in K. Then G is residually p-finite and subgroup separable.*

Proof. It follows from Theorem 3.4 and Corollary 4.11. □

5. Some applications

In this section we show first characterisations for certain one-relator groups and similar groups, to be residually p-finite. Characterisations on the residual finiteness of these groups by Raptis and Varsos in [17] and Andredakis, Raptis and Varsos in [1] are used.

Theorem 5.1. *Let $G = \langle t, K; t^{-1}ut = w\rangle$ be an HNN extension, where K is a free abelian group of finite rank and $u, w \in K$. Then G is residually p-finite if and only if $\langle u\rangle \cap \langle w\rangle = 1$ or there exists a primitive element s of K such that $u = w^{\pm 1} = s^r$, where $r \in \mathbb{Z}$.*

Proof. The group G can be written as $G = \langle t, K; t^{-1}At = B, \varphi\rangle$, where K is a free abelian group of finite rank, $A = \langle u\rangle$, $B = \langle w\rangle$ and $\varphi : A \to B$ with $\varphi(u) = w$ is an isomorphism. If $\langle u\rangle \cap \langle w\rangle = 1$, then $A \cap B = 1$ and hence

G is residually p-finite by Theorem 3.4. Suppose $u = w^{\pm 1} = s^r$, where s is a primitive element of K. Then $A = B$ and hence G is residually p-finite by Theorem 3.5.

Suppose G is residually p-finite and $\langle u \rangle \cap \langle w \rangle \neq 1$. Since G is residually finite and $\langle u \rangle$ and $\langle v \rangle$ are not finite, then by Proposition 2 of Raptis and Varsos [17], there exists a primitive element s of K such that $u = w^{\pm 1} = s^r$, where $r \in \mathbb{Z}$. □

Theorem 5.2. *Let $G = \langle t, a, b; t^{-1}a^m t = a^n b^k, [a, b] \rangle$, where $m, n, k \in \mathbb{Z}$. Then G is residually p-finite if and only if $k \neq 0$ or $m = |n|$.*

Proof. The group G can be written as an HNN extension $G = \langle t, K; t^{-1}At = B, \varphi \rangle$, where $K = \langle a, b \rangle$ is a free abelian group of rank 2, $A = \langle a^m \rangle$, $B = \langle a^n b^k \rangle$ and $\varphi : A \to B$ with $\varphi(a^m) = a^n b^k$ is an isomorphism. If $k \neq 0$, then $A \cap B = \langle a^m \rangle \cap \langle a^n b^k \rangle = 1$ and hence G is residually p-finite by Theorem 3.4. If $k = 0$ and $m = |n|$, then $A = B$ and hence G is residually p-finite by Theorem 3.5.

If $k = 0$ and $m \neq |n|$, then G is not residually finite by Theorem 2 of Andredakis, Raptis and Varsos [1]. □

Theorem 5.3. *Let $G = \langle t, a_1, a_2, \ldots, a_n; t^{-1}a_i^{h_i} t = a_i^{k_i}, [a_i, a_j], i, j = 1, 2, \ldots, n \rangle$, where not all $h_i = 1$ for $i = 1, 2, \ldots, n$ and not all $k_j = 1$ for $j = 1, 2, \ldots, n$. Then G is residually p-finite if and only if $|h_i| = |k_i|$, $i = 1, 2, \ldots, n$.*

Proof. The group G can be written as an HNN extension $G = \langle t, K; t^{-1}At = B, \varphi \rangle$, where $K = \langle a_1, a_2, \ldots, a_n; [a_i, a_j] \rangle$ is a free abelian group of rank n, $A = \langle a_1^{h_1}, a_2^{h_2}, \ldots, a_n^{h_n} \rangle$, $B = \langle a_1^{k_1}, a_2^{k_2}, \ldots, a_n^{k_n} \rangle$ and $\varphi : A \to B$ with $\varphi(a_i^{h_i}) = a_i^{k_i}$, $i = 1, 2, \ldots, n$, is an isomorphism.

Suppose $|h_i| = |k_i|$, $i = 1, 2, \ldots, n$. Then we have $A = B$ and hence G is residually p-finite by Theorem 3.5.

If $|h_i| \neq |k_i|$ for some $i = 1, 2, \ldots, n$, then G is not residually finite by Corollary 3 of Andredakis, Raptis and Varsos [1]. □

Remark. We note that if $|h_i| = |k_i|$, $i = 1, 2, \ldots, n$, then G is subgroup separable by Corollary 2 of Wong [23].

A characterisation for the Baumslag-Solitar groups to be residually p-finite is shown in the Main Theorem of Kim and McCarron [12]. We shall provide a partial proof using Theorem 3.5.

Theorem 5.4. *Let $G = \langle t, a \mid t^{-1}a^r t = a^s \rangle$, where $r, s \in \mathbb{Z}$. Suppose the following conditions ($r = 1$, $s \equiv 1 \pmod{p}$) and ($s = 1$, $r \equiv 1 \pmod{p}$) do not occur. Then G is residually p-finite if and only if $|r| = |s|$.*

Proof. The group G can be written as an HNN extension $G = \langle t, K; t^{-1}At = B, \varphi \rangle$, where $K = \langle a \rangle$ is infinite cyclic, $A = \langle a^r \rangle$, $B = \langle a^s \rangle$ and $\varphi : A \to B$ with $a^r \to a^s$ is an isomorphism.

Suppose $|r| = |s|$. Then $A = B$ and hence G is residually p-finite by Theorem 3.5.

If $|r| \neq |s|$, then G is not residually p-finite by Main Theorem of Kim and McCarron [12]. $\square$

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