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MEROMORPHIC SOLUTIONS OF SOME NON-LINEAR DIFFERENCE EQUATIONS WITH THREE EXPONENTIAL TERMS

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Abstract. In this paper, we study the existence of finite order meromorphic solutions of the following non-linear difference equation

$$f^n(z) + P_d(z, f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} + p_3 e^{\alpha_3 z},$$

where $n \geq 2$ is an integer, $P_d(z, f)$ is a difference polynomial in f of degree $d \leq n-2$ with small functions of f as its coefficients, p_j $(j = 1, 2, 3)$ are small meromorphic functions of f and α_j $(j = 1, 2, 3)$ are three distinct non-zero constants. We give the expressions of finite order meromorphic solutions of the above equation under some restrictions on α_j $(j = 1, 2, 3)$. Some examples are given to illustrate the accuracy of the conditions.

1. Introduction and main results

It is an important and difficult problem for complex functional equations to prove the existence of their solutions. In [15], Yang and Laine proved the following result.

Theorem A. *A non-linear difference equation*

$$f^3(z) + q(z) f(z+1) = c \sin bz, \tag{1.1}$$

where q is a non-constant polynomial and b, c are non-zero constants, does not admit entire solutions of finite order. If q is a non-zero constant, then equation (1.1) *possesses three district entire solutions of finite order, provided $b = 3\pi n$ and $q^3 = (-1)^{n+1} \frac{27}{4} c^2$ for a non-zero integer n.*

Recently, there has been a renewed interest (see [2–4, 10–12, 17]) in solvability and existence for entire or meromorphic solutions of non-linear difference

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equations or differential equations. In 2019, Chen, Gao and Zhang [3] improved and extended Theorem A and obtained the following result.

Theorem B. *Let $n \geq 2$ be an integer, q be a non-zero polynomial, c, λ, p_1, p_2 be non-zero constants. If there exists some entire solution f of finite order to equation*

$$f^n(z) + q(z)\Delta_c f(z) = p_1 e^{\lambda z} + p_2 e^{-\lambda z} \tag{1.2}$$

such that $\Delta_c f(z) = f(z+c) - f(z) \not\equiv 0$, then q is a constant, and $n = 2$ or $n = 3$. When $n = 2$, then $f(z) = q + c_1 e^{\frac{\lambda}{2}z} + c_2 e^{-\frac{\lambda}{2}z}$, where $q^4 = 4p_1p_2$, $c_1^2 = p_1$, $c_2^2 = p_2$, $\lambda c = 2k\pi i$, $k \in \mathbb{Z}$ and k is odd; When $n = 3$, then $f(z) = c_1 e^{\frac{\lambda}{3}z} + c_2 e^{-\frac{\lambda}{3}z}$, where $q^3 = \frac{27}{8}p_1p_2$, $c_1^3 = p_1$, $c_2^3 = p_2$, $\lambda c = 3k\pi i$, $k \in \mathbb{Z}$ and k is odd.

More recently, Liu and Mao showed in [12]:

Theorem C. *Let $n \geq 2$ be an integer, $P_d(z, f)$ be a difference polynomial in f of degree $d \leq n-1$ with polynomial coefficients, and let p_j, α_j $(j = 1, 2)$ be non-zero constants satisfying*

$$\frac{\alpha_1}{\alpha_2} \in \left\{-1, \frac{t}{n}, \frac{n}{t} : 1 \leq t \leq n-1\right\}.$$

If difference equation

$$f^n(z) + P_d(z, f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} \tag{1.3}$$

admits a finite order meromorphic solution $f(z)$, then one of the following holds:

(i) $f(z) = \gamma_0 + \gamma_1 e^{\frac{\alpha_1}{n}z} + \gamma_2 e^{\frac{\alpha_2}{n}z}$, and $\frac{\alpha_1}{\alpha_2} = -1$, where γ_1, γ_2 are constants satisfying $\gamma_j^n = p_j$, $j = 1, 2$, γ_0 is a polynomial.

(ii) $f(z) = \gamma_1 e^{\beta z} + \gamma_0$, and $\frac{\alpha_1}{\alpha_2} = \frac{n}{t}$ (or $\frac{t}{n}$), where $n\beta = \alpha_1$ (or α_2), γ_1 is a non-zero constant satisfying $\gamma_1^n = p_1$ (or $\gamma_1^n = p_2$), γ_0 is a polynomial. Moreover, if $P_d(z, 0) \not\equiv 0$, then $\gamma_0 \not\equiv 0$.

Clearly, there are only two exponential terms on the right side of difference equations studied above. It is natural to ask what we would obtain if the right side has three linearly independent exponential type functions. In this paper, we consider this question and obtain the following result.

Theorem 1.1. *Let $n \geq 3$ be an integer, $P_d(z, f)$ be a difference polynomial in f of degree $d \leq n-2$ with small functions of f as its coefficients, and let p_j, α_j $(j = 1, 2, 3)$ be non-zero constants. If*

$$\frac{\alpha_1}{\alpha_2} \in \left\{\frac{s}{n} : 1 \leq s \leq n-1\right\}, \ \frac{\alpha_3}{\alpha_2} \in \left\{\frac{t}{n} : 1 \leq t \leq n-1\right\}, \ s \neq t, \tag{1.4}$$

and if f is a finite order meromorphic solution of the difference equation

$$f^n(z) + P_d(z, f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} + p_3 e^{\alpha_3 z}, \tag{1.5}$$

then $f(z)=\gamma_1+\gamma_2 e^{\frac{\alpha_2}{n}z}$, γ_1 *is a constant,* γ_2 *is a non-zero constant satisfying* $\gamma_2^n=p_2$. *Moreover, if* $P_d(z,0)\not\equiv 0$, *then* $\gamma_1\not\equiv 0$.

Remark 1.2. In Theorem 1.1, the condition $P_d(z,0)\not\equiv 0$ is only a sufficient condition which guarantees $\gamma_1\neq 0$. See the following Example 1.3.

Example 1.3. $f(z)=1+e^z$ solves the difference equation

$$f^3(z)-f(z+\ln 2)=e^z+e^{3z}+3e^{2z}.$$

Here $n=3, d=1$, $\alpha_1=1$, $\alpha_2=3$, $\alpha_3=2$ satisfy $\frac{\alpha_1}{\alpha_2}=\frac{1}{3}$, $\frac{\alpha_3}{\alpha_2}=\frac{2}{3}$.

The conditions (1.4) and $d\leq n-2$ in Theorem 1.1 are necessary, which can be illustrated by the following two examples.

Example 1.4. Let $n=3$, $d=2$, and $\alpha_1=1$, $\alpha_2=3$, $\alpha_3=-3$ satisfy $\frac{\alpha_1}{\alpha_2}=\frac{1}{3}$, $\frac{\alpha_3}{\alpha_2}=-1$. Then the following difference equation

$$f^3(z)-3f^2(z)+2f(z)-f(z-\ln 2)+1=\frac{3}{2}e^z+e^{3z}+e^{-3z}$$

has a transcendental entire solution $f=1+e^z+e^{-z}$. But it does not satisfy the result of Theorem 1.1.

Example 1.5. Let $n=3$, $d=2$, and $\alpha_1=4$, $\alpha_2=6$, $\alpha_3=5$ satisfy $\frac{\alpha_1}{\alpha_2}=\frac{2}{3}$, $\frac{\alpha_3}{\alpha_2}=\frac{5}{6}$. Then the following difference equation

$$f^3(z)-\frac{1}{2}f^2(z)-\frac{1}{2}f(z)f(z+\pi i)=2e^{4z}+e^{6z}+3e^{5z}$$

has a transcendental entire solution $f=e^z(1+e^z)$. But it does not satisfy the result of Theorem 1.1.

How to find the solutions of the equation (1.5) under the conditions $n=2$ and $d=0$? To this end, we shall prove the following result.

Theorem 1.6. *Let* p_l $(l=1,2,3)$ *be nonzero meromorphic functions,* α_l $(l=1,2,3)$ *be distinct nonzero constants. If* f *is a finite order meromrphic solution of equation*

$$f^2(z)=p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z}+p_3e^{\alpha_3 z}, \tag{1.6}$$

and satisfies $T(r,p_l)=S(r,f)$ $(l=1,2,3)$, *then* $f(z)=\gamma_i e^{\frac{\alpha_i}{2}z}+\gamma_j e^{\frac{\alpha_j}{2}z}$, *where* γ_i,γ_j *are meromorphic functions and satisfy* $\gamma_i^2=p_i$, $\gamma_j^2=p_j$, $2\gamma_i\gamma_j=p_k$, $\alpha_i+\alpha_j=2\alpha_k$, $\{i,j,k\}=\{1,2,3\}$.

We assume the reader is familiar with the basic results and standard notations of Nevanlinna theory (see [1,8,14]). Let f be a meromorphic function in the complex plane $\mathbb{C}$. We use $\sigma(f)$ to denote the order of growth of f. For simplicity, we denote by $S(r,f)$ any quantify satisfying $S(r,f)=o(T(r,f))$, as $r\to\infty$, outside of a possible exceptional set of finite logarithmic measure, we use $S(f)$ to denote the family of all small functions with respect to f. Let $N_{1)}\big(r,\frac{1}{f}\big)$ and $N_{(2}\big(r,\frac{1}{f}\big)$ denote the counting functions corresponding to simple and multiple zeros of f, respectively. In general, a difference polynomial in f is defined to be a polynomial in f and its shifts $f(z+c)$ with small functions as its coefficients, that is, a difference polynomial $P_d(z,f)$ in f is denoted by

$$P_d(z,f) = \sum_{\mu\in I} a_\mu(z) \prod_{j=1}^{t_\mu} f(z+\delta_{\mu j})^{l_{\mu j}}, \tag{1.7}$$

where I is a finite set of the index μ, a_μ ($\mu \in I$) are small meromorphic function of f, t_μ, $l_{\mu j}$ ($\mu \in I,\ j = 1,\dots,t_\mu$) are natural numbers, $\delta_{\mu j}$ ($\mu \in I,\ j = 1,\dots,t_\mu$) are distinct complex constants. The degree of $P_d(z,f)$ is defined by $d = \max\limits_{\mu\in I}\{l_\mu : l_\mu = \sum_{j=1}^{t_\mu} l_{\mu j}\}$.

2. Some lemmas

Lemma 2.1 (Clunie's Lemma [6]). *Let f be a transcendental meromorphic solution of $f^n(z)P(z,f) = Q(z,f)$, where $P(z,f)$ and $Q(z,f)$ are polynomials in f and its derivatives with meromorphic coefficients, say $\{a_\lambda \mid \lambda \in I\}$, such that $m(r,a_\lambda) = S(r,f)$ for all $\lambda \in I$, I is a finite set of the index λ. If the total degree of $Q(z,f)$ as a polynomial in f and its derivatives is $\le n$, then*

$$m(r, P(z,f)) = S(r,f).$$

Lemma 2.2 ([7, Corollary 3.3]). *Let f be a non-constant finite order meromorphic solution of $f^n(z)P(z,f) = Q(z,f)$, where $P(z,f)$ and $Q(z,f)$ are difference polynomials in f with small meromorphic coefficients, and let $c \in \mathbb{C}$, $\delta < 1$. If the total degree of $Q(z,f)$ as a polynomial in f and its shifts is $\le n$, then*

$$m(r, P(z,f)) = o\left(\frac{T(r+|c|,f)}{r^\delta}\right) + o(T(r,f))$$

for all r outside of a possible exceptional set with finite logarithmic measure.

Remark 2.3. Lemma 2.2 still holds for the case $P(z,f)$, $Q(z,f)$ being differential-difference polynomials in f with functions of small proximity related to f as its coefficients.

Lemma 2.4 ([5, Corollary 2.6]). *Let η_1, η_2 be two complex numbers such that $\eta_1 \ne \eta_2$ and let f be a finite order meromorphic function. Let σ be the order of f. Then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\sigma-1+\varepsilon}).$$

Lemma 2.5 ([12, Lemma 2.3]). *Let f be a transcendental meromorphic solution of the difference equation*

$$f^n(z) + P_d(z,f) = H(z),$$

where $n \ge 2$ is an integer, $P_d(z,f)$ is a difference polynomial in f of degree $d \le n-1$, and H is a meromorphic function satisfying $N(r,H) = S(r,f)$. If f is of finite order, then $N(r,f) = S(r,f)$ and $\sigma(f) = \sigma(H)$.

Lemma 2.6 ([12, Lemma 2.4])**.** *Let $n \geq 2$ be an integer, α_j $(j = 1, 2)$ be distinct non-zero constants, and let p_j $(j = 1, 2)$ be non-zero meromorphic functions. Then the equation*

$$f^n(z) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}$$

cannot admit a meromorphic solution f such that $T(r, p_j) = S(r, f)$ $(j = 1, 2)$.

Lemma 2.7 ([16, Theorem 1.51])**.** *Suppose that $f_1, f_2, \dots, f_n$ $(n \geq 2)$ are meromorphic functions and $g_1, g_2, \dots, g_n$ are entire functions satisfying the following conditions*:

(i) $\sum_{j=1}^{n} f_j e^{g_j} \equiv 0$.
(ii) *$g_j - g_k$ are not constants for $1 \leq j < k \leq n$.*
(iii) *For $1 \leq j \leq n, 1 \leq h < k \leq n$,*

$$T(r, f_j) = o(T(r, e^{g_h - g_k})) \ (r \to \infty, r \notin E),$$

where E is a set with finite linear measure. Then $f_j \equiv 0$ $(j = 1, \dots, n)$.

Lemma 2.8 ([9, Lemma 6])**.** *Suppose that $f(z)$ is a transcendental meromorphic function, $a, b, c, d \in S(f)$ such that $acd \not\equiv 0$. If*

$$af^2 + bff' + c(f')^2 = d,$$

then

$$c(b^2 - 4ac)\frac{d'}{d} + b(b^2 - 4ac) - c(b^2 - 4ac)' + (b^2 - 4ac)c' = 0.$$

Remark 2.9. The condition $acd \not\equiv 0$ in Lemma 2.8 is not necessary and it can be replaced by $cd \not\equiv 0$. Cf. the proof of Lemma 6 in [9].

Lemma 2.10 ([8, Theorem 3.9])**.** *Let $n \in \mathbb{N}$ and $f(z)$ be a non-constant meromorphic function. Suppose*

$$g(z) = f^n(z) + P_{n-1}(z, f),$$

where $P_{n-1}(z, f)$ is a differential polynomial in f of degree at most $n - 1$ with small functions of f as its coefficients and that

$$N(r, f) + N\left(r, \frac{1}{g}\right) = S(r, f).$$

Then $g(z) = (\gamma(z) + f(z))^n$, where $\gamma(z) \in S(f)$.

Lemma 2.11. *Let $n \geq 3$ be an integer, $P_d(z, f)$ be a difference polynomial in f of degree $d \leq n - 2$ with small functions of f as its coefficients, p_j $(j = 1, 2, 3)$ are small functions of f and let α_j $(j = 1, 2, 3)$ be non-zero constants satisfying* (1.4)*. If f is a finite order meromorphic solution of difference equation*

$$f^n(z) + P_d(z, f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} + p_3 e^{\alpha_3 z}, \tag{2.1}$$

then $m\big(r, \frac{e^{\alpha_1 z}}{f^{n-2}}\big) = S(r, f)$ or $m\big(r, \frac{e^{\alpha_3 z}}{f^{n-2}}\big) = S(r, f)$.

Proof. Set

$$P_d(z,f)=\sum_{\mu\in I}b_\mu(z)\prod_{j=1}^{t_\mu}f(z+\delta_{\mu j})^{l_{\mu j}}, \tag{2.2}$$

where I is a finite set of the index μ, t_μ, $l_{\mu j}$ ($\mu\in I,\ j=1,\dots,t_\mu$) are natural numbers, $\delta_{\mu j}$ ($\mu\in I,\ j=1,\dots,t_\mu$) are distinct complex constants. Denote $g_{\mu j}(z):=\frac{f(z+\delta_{\mu j})}{f(z)}$ and substitute this equality into (2.2) yields

$$P_d(z,f)=\sum_{\mu\in I}\left(b_\mu(z)\prod_{j=1}^{t_\mu}g_{\mu j}^{l_{\mu j}}(z)\right)f^{l_\mu}(z)=\sum_{q=0}^{d}\beta_q(z)f^q(z), \tag{2.3}$$

where $l_\mu=\sum_{j=1}^{t_\mu}l_{\mu j}$, $d=\max_{\mu\in I}\{l_\mu\}$, $\beta_q(z)=\sum_{l_\mu=q}\left(b_\mu(z)\prod_{j=1}^{t_\mu}g_{\mu j}^{l_{\mu j}}(z)\right)$ ($q=0,\dots,d$). By applying Lemma 2.4, we have

$$m(r,\beta_q(z))=S(r,f)\quad (q=0,\dots,d). \tag{2.4}$$

Without loss of generality, we assume that $P_d(z,0)\not\equiv 0$. Otherwise, we make the transformation $g=f-c$ for a suitable constant c satisfying $c^n+P_d(z,c)\not\equiv 0$. Then (2.1) is changed to the form $g^n(z)+Q_d(z,g)=p_1e^{\alpha_1 z}+p_2e^{\alpha_2 z}+p_3e^{\alpha_3 z}$, where $Q_d(z,g)$ is a difference polynomial in g of degree at most $n-1$ with small functions of g as its coefficients, and $Q_d(z,0)=c^n+P_d(z,c)\not\equiv 0$. Noting that $P_d(z,0)\not\equiv 0$, it follows from (2.1) and (2.3) that

$$\frac{1}{\sum_{i=1}^{3}p_ie^{\alpha_i z}-P_d(z,0)}+\sum_{q=1}^{d}\frac{\beta_q}{\sum_{i=1}^{3}p_ie^{\alpha_i z}-P_d(z,0)}\left(\frac{1}{f}\right)^{n-q}=\left(\frac{1}{f}\right)^{n}. \tag{2.5}$$

It follows from (2.1) and Lemma 2.5 that $S(r,f)=S(r,e^z)$. On the other hand, by [13, Satz 2], then we have

$$m\left(r,\frac{1}{\sum_{i=1}^{3}p_ie^{\alpha_i z}-P_d(z,0)}\right)=S(r,f),$$
$$m\left(r,\frac{e^{\alpha_j z}}{\sum_{i=1}^{3}p_ie^{\alpha_i z}-P_d(z,0)}\right)=S(r,f),\ j=1,2,3.$$

By the above two equalities, (2.4) and (2.5), we obtain

$$m\left(r,\frac{1}{f}\right)=S(r,f),\quad m\left(r,\frac{e^{\alpha_j z}}{f^n}\right)=S(r,f),\ j=1,2,3. \tag{2.6}$$

By (1.4), we see that $s\neq t$, if $s<t$, then $1\le s\le n-2\le t\le n-1$, and the fact that

$$\left|\frac{e^{\alpha_1 z}}{f^{n-2}}\right|=\left|\frac{e^{\alpha_1 z}}{f^{s}}\right|\cdot\left|\frac{1}{f^{n-2-s}}\right|=\left|\frac{e^{\alpha_2 z}}{f^{n}}\right|^{\frac{s}{n}}\cdot\left|\frac{1}{f^{n-2-s}}\right|.$$

It follows from (2.6) that

$$m\left(r, \frac{e^{\alpha_1 z}}{f^{n-2}}\right) = S(r,f).$$

If $s > t$, similarly, we can prove that

$$m\left(r, \frac{e^{\alpha_3 z}}{f^{n-2}}\right) = S(r,f).$$

□

3. Proof of Theorem 1.1

Proof. Set $P_d(z,f) = P$. Then (1.5) can be rewritten as

$$f^n + P = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} + p_3 e^{\alpha_3 z}. \tag{3.1}$$

Differentiating (3.1) yields

$$nf^{n-1}f' + P' = \alpha_1 p_1 e^{\alpha_1 z} + \alpha_2 p_2 e^{\alpha_2 z} + \alpha_3 p_3 e^{\alpha_3 z}. \tag{3.2}$$

Eliminating $e^{\alpha_2 z}$ from (3.1) and (3.2), we have

$$\alpha_2 f^n - nf^{n-1}f' + \alpha_2 P - P' = (\alpha_2 - \alpha_1)p_1 e^{\alpha_1 z} + (\alpha_2 - \alpha_3)p_3 e^{\alpha_3 z}. \tag{3.3}$$

Differentiating (3.3) yields

$$\begin{aligned} &n\alpha_2 f^{n-1}f' - n(n-1)f^{n-2}(f')^2 - nf^{n-1}f'' + \alpha_2 P' - P'' \\ &= \alpha_1(\alpha_2 - \alpha_1)p_1 e^{\alpha_1 z} + \alpha_3(\alpha_2 - \alpha_3)p_3 e^{\alpha_3 z}. \end{aligned} \tag{3.4}$$

Eliminating $e^{\alpha_1 z}$ and $e^{\alpha_3 z}$ from (3.3) and (3.4), respectively, we have

$$\begin{aligned} &\alpha_1\alpha_2 f^n - n(\alpha_1 + \alpha_2)f^{n-1}f' + n(n-1)f^{n-2}(f')^2 + nf^{n-1}f'' \\ &+ \alpha_1\alpha_2 P - (\alpha_1 + \alpha_2)P' + P'' = (\alpha_2 - \alpha_3)(\alpha_1 - \alpha_3)p_3 e^{\alpha_3 z}, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &\alpha_2\alpha_3 f^n - n(\alpha_2 + \alpha_3)f^{n-1}f' + n(n-1)f^{n-2}(f')^2 + nf^{n-1}f'' \\ &+ \alpha_2\alpha_3 P - (\alpha_2 + \alpha_3)P' + P'' = (\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1)p_1 e^{\alpha_1 z}. \end{aligned} \tag{3.6}$$

Rewriting (3.6) as

$$f^{n-2}\varphi(z) = -[\alpha_2\alpha_3 P - (\alpha_2 + \alpha_3)P' + P''] + (\alpha_3 - \alpha_1)(\alpha_2 - \alpha_1)p_1 e^{\alpha_1 z}, \tag{3.7}$$

where

$$\varphi(z) = \alpha_2\alpha_3 f^2 - n(\alpha_2 + \alpha_3)ff' + n(n-1)(f')^2 + nff''. \tag{3.8}$$

By applying Lemma 2.11, we get $m\big(r, \frac{e^{\alpha_1 z}}{f^{n-2}}\big) = S(r,f)$ or $m\big(r, \frac{e^{\alpha_3 z}}{f^{n-2}}\big) = S(r,f)$.

If $m\big(r, \frac{e^{\alpha_1 z}}{f^{n-2}}\big) = S(r,f)$, by (3.7), (3.8) and the same proof of Lemma 2.2, we have

$$m(r,\varphi) = S(r,f). \tag{3.9}$$

It follows from (1.5) and Lemma 2.5 that $N(r,f) = S(r,f)$. Combining (3.8), we get $N(r,\varphi) = S(r,f)$. By (3.9), we get $T(r,\varphi) = S(r,f)$. We consider two cases below.

Case 1. If $\varphi \equiv 0$, it follows from (3.7) that $\alpha_2\alpha_3 P - (\alpha_2+\alpha_3)P' + P'' = (\alpha_3-\alpha_1)(\alpha_2-\alpha_1)p_1 e^{\alpha_1 z}$. Then the general solution of above equation is of the form $P = p_1e^{\alpha_1 z} + c_2e^{\alpha_2 z} + c_3e^{\alpha_3 z}$, $c_2,\ c_3 \in \mathbb{C}$. It follows from (1.5) that $f^n = (p_2-c_2)e^{\alpha_2 z} + (p_3-c_3)e^{\alpha_3 z}$. By Lemma 2.6, we get $p_2 \neq c_2$ and $p_3 = c_3$ or $p_2 = c_2$ and $p_3 \neq c_3$. If $p_2 \neq c_2$ and $p_3 = c_3$, then $f^n = (p_2-c_2)e^{\alpha_2 z}$, and $f = \gamma_2 e^{\frac{\alpha_2}{n}z}$, $\gamma_2^n = p_2 - c_2$. Substituting the expression of f into (2.3), we have

$$\begin{aligned} P &= \beta_d(\gamma_2 e^{\frac{\alpha_2}{n}z})^d + \beta_{d-1}(\gamma_2 e^{\frac{\alpha_2}{n}z})^{d-1} + \cdots + \beta_0 \\ &= p_1e^{\alpha_1 z} + c_2e^{\alpha_2 z} + p_3e^{\alpha_3 z}. \end{aligned}$$

Noting that $d \leq n-2$ and by applying Lemma 2.7, we obtain $c_2 = 0$. Then $f = \gamma_2 e^{\frac{\alpha_2}{n}z}$, $\gamma_2^n = p_2$, $P = p_1e^{\alpha_1 z} + p_3e^{\alpha_3 z}$, $\frac{\alpha_1}{\alpha_2} = \frac{s}{n}$, $\frac{\alpha_3}{\alpha_2} = \frac{t}{n}$, $1 \leq s < t \leq d$. If $p_2 = c_2$ and $p_3 \neq c_3$, similarly, we conclude that $f^n = p_3e^{\alpha_3 z}$, $\gamma_3^n = p_3$, $P = p_1e^{\alpha_1 z} + p_2e^{\alpha_2 z}$, $\frac{\alpha_1}{\alpha_3} = \frac{s}{n}$, $\frac{\alpha_2}{\alpha_3} = \frac{t}{n}$, $1 \leq s < t \leq d$, which contradicts with (1.4).

Case 2. If $\varphi \not\equiv 0$, it follows from (3.8) that

$$\frac{1}{f^2} = \frac{1}{\varphi}\left[\alpha_2\alpha_3 - n(\alpha_2+\alpha_3)\frac{f'}{f} + n(n-1)\left(\frac{f'}{f}\right)^2 + n\frac{f''}{f}\right].$$

From the above equality, we obtain

$$2m\left(r, \frac{1}{f}\right) = m\left(r, \frac{1}{f^2}\right) \leq S(r,f).$$

If z_0 is a multiple zero of f, then z_0 must be a zero of φ, and $N_{(2}\big(r, \frac{1}{f}\big) \leq N\big(r, \frac{1}{\varphi}\big) \leq T(r,\varphi) = S(r,f)$, which implies

$$T\left(r, \frac{1}{f}\right) = N_{1)}\left(r, \frac{1}{f}\right) + S(r,f).$$

Differentiating (3.8) yields

$$\varphi' = 2\alpha_2\alpha_3 ff' - n(\alpha_2+\alpha_3)ff'' - n(\alpha_2+\alpha_3)(f')^2 + n(2n-1)f'f'' + nff'''. \tag{3.10}$$

Combining (3.8) and (3.10), we get

$$\begin{aligned} &\alpha_2\alpha_3\varphi' f^2 - [n(\alpha_2+\alpha_3)\varphi' + 2\alpha_2\alpha_3\varphi]ff' \\ &+ [n(n-1)\varphi' + n(\alpha_2+\alpha_3)\varphi](f')^2 \\ &+ [n\varphi' + n(\alpha_2+\alpha_3)\varphi]ff'' - n(2n-1)\varphi f'f'' - n\varphi ff''' = 0. \end{aligned} \tag{3.11}$$

Suppose z_1 is a simple zero of f which is not the zero of coefficients of (3.11). It follows from (3.11) that z_1 is a zero of $(2n-1)\varphi f'' - [(n-1)\varphi' + (\alpha_2+\alpha_3)\varphi]f'$. Denote

$$\alpha = \frac{(2n-1)\varphi f'' - [(n-1)\varphi' + (\alpha_2+\alpha_3)\varphi]f'}{f}. \tag{3.12}$$

Then we have $T(r,\alpha)=S(r,f)$. It follows from (3.12) that

$$f''=\frac{1}{2n-1}\left[(n-1)\frac{\varphi'}{\varphi}+(\alpha_2+\alpha_3)\right]f'+\frac{\alpha}{(2n-1)\varphi}f. \tag{3.13}$$

Substituting (3.13) into (3.8) yields

$$q_1f^2+q_2ff'+q_3(f')^2=\varphi, \tag{3.14}$$

where

$$\begin{cases} q_1=\alpha_2\alpha_3+\frac{n\alpha}{(2n-1)\varphi}, \\ q_2=\frac{n(n-1)}{2n-1}\left[\frac{\varphi'}{\varphi}-2(\alpha_2+\alpha_3)\right], \\ q_3=n(n-1). \end{cases} \tag{3.15}$$

By (3.14) and Lemma 2.8, we have

$$q_3(q_2^2-4q_1q_3)\frac{\varphi'}{\varphi}=q_3(q_2^2-4q_1q_3)'-q_2(q_2^2-4q_1q_3). \tag{3.16}$$

We distinguish two subcases as follows.

Subcase 2.1. Suppose that $q_2^2-4q_1q_3\not\equiv 0$. It follows from (3.15) and (3.16) that

$$2(\alpha_2+\alpha_3)=2n\frac{\varphi'}{\varphi}-(2n-1)\frac{(q_2^2-4q_1q_3)'}{q_2^2-4q_1q_3}. \tag{3.17}$$

By integration, there exists a $c_4\in\mathbb{C}\setminus\{0\}$ such that

$$e^{2(\alpha_2+\alpha_3)z}=c_4\varphi^{2n}(q_2^2-4q_1q_3)^{-(2n-1)}, \tag{3.18}$$

which implies $e^{2(\alpha_2+\alpha_3)z}\in S(f)$, then $\alpha_2+\alpha_3=0$, a contraction.

Subcase 2.2. Suppose that $q_2^2-4q_1q_3\equiv 0$. Differentiating (3.14) yields

$$\varphi'=q_1'f^2+(2q_1+q_2')ff'+q_2(f')^2+q_2ff''+2q_3f'f''. \tag{3.19}$$

Suppose z_2 is a simple zero of f which is not the zero of $q_1,\ q_2$. Then it follows from (3.14) and (3.19) that z_2 is a zero of $2\varphi f''-\left(\varphi'-\frac{q_2}{q_3}\varphi\right)f'$. Denote

$$\beta=\frac{2\varphi f''-\left(\varphi'-\frac{q_2}{q_3}\varphi\right)f'}{f}. \tag{3.20}$$

Then we have $T(r,\beta)=S(r,f)$ and deduce that

$$f''=\left(\frac{1}{2}\frac{\varphi'}{\varphi}-\frac{q_2}{2q_3}\right)f'+\frac{\beta}{2\varphi}f. \tag{3.21}$$

Substituting (3.21) into (3.19) yields

$$\varphi'=q_4f^2+q_5ff'+q_3\frac{\varphi'}{\varphi}(f')^2, \tag{3.22}$$

where $q_4=q_1'+\frac{q_2\beta}{2\varphi}$, $q_5=2q_1+q_2'+\frac{q_2}{2}\frac{\varphi'}{\varphi}-\frac{q_2^2}{2q_3}+\frac{q_3\beta}{\varphi}=q_2'+\frac{q_2}{2}\frac{\varphi'}{\varphi}+\frac{q_3\beta}{\varphi}$. Eliminating $(f')^2$ from (3.14) and (3.22), we have

$$A_1f+A_2f'\equiv 0, \tag{3.23}$$

where

$$A_1 = q_4 - q_1\frac{\varphi'}{\varphi} = q_1' - q_1\frac{\varphi'}{\varphi} + \frac{q_2\beta}{2\varphi},$$

$$A_2 = q_5 - q_2\frac{\varphi'}{\varphi} = q_2' - \frac{q_2}{2}\frac{\varphi'}{\varphi} + \frac{q_3\beta}{\varphi},$$

and A_1, A_2 are small functions of f. Suppose z_3 is a simple zero of f which is not the zero of A_1, A_2. Then it follows from (3.23) that $A_1 = A_2 \equiv 0$. By (3.21), we have

$$f'' = \left(\frac{1}{2}\frac{\varphi'}{\varphi} - \frac{q_2}{2q_3}\right) f' - \frac{1}{q_2}\left(q_1' - q_1\frac{\varphi'}{\varphi}\right) f, \tag{3.24}$$

where $q_2 = \frac{n(n-1)}{2n-1}\left[\frac{\varphi'}{\varphi} - 2(\alpha_2+\alpha_3)\right] \not\equiv 0$. If $q_2 \equiv 0$, then $\frac{\varphi'}{\varphi} = 2(\alpha_2+\alpha_3)$. By integration, we get $\varphi = c_5 e^{(\alpha_2+\alpha_3)z} \in S(f)$, $c_5 \in \mathbb{C}\setminus\{0\}$, which implies $\alpha_2+\alpha_3 = 0$, a contradiction. Substituting $q_2^2 = 4q_1q_3$ and (3.15) into (3.24) yields

$$\begin{aligned} f'' = {} & \frac{1}{2n-1}\left[(n-1)\frac{\varphi'}{\varphi} + (\alpha_2+\alpha_3)\right] f' \\ & - \frac{1}{2(2n-1)}\left[\left(\frac{\varphi'}{\varphi}\right)' - \frac{1}{2}\left(\frac{\varphi'}{\varphi}\right)^2 + (\alpha_2+\alpha_3)\frac{\varphi'}{\varphi}\right] f. \end{aligned} \tag{3.25}$$

It follows from (3.13) and (3.25) that

$$\frac{\alpha}{\varphi} = -\frac{1}{2}\left[\left(\frac{\varphi'}{\varphi}\right)' - \frac{1}{2}\left(\frac{\varphi'}{\varphi}\right)^2 + (\alpha_2+\alpha_3)\frac{\varphi'}{\varphi}\right]. \tag{3.26}$$

If $\varphi' \not\equiv 0$, differentiating (3.26) gives

$$\left(\frac{\alpha}{\varphi}\right)' = -\frac{1}{2}\left[\left(\frac{\varphi'}{\varphi}\right)'' - \left(\frac{\varphi'}{\varphi}\right)'\frac{\varphi'}{\varphi} + (\alpha_2+\alpha_3)\left(\frac{\varphi'}{\varphi}\right)'\right]. \tag{3.27}$$

It follows from (3.15) and $q_2^2 = 4q_1q_3$ that $q_2q_2' = 2q_1'q_3$, namely,

$$\left(\frac{\alpha}{\varphi}\right)' = \frac{n-1}{2(2n-1)}\left(\frac{\varphi'}{\varphi}\right)'\left[\frac{\varphi'}{\varphi} - 2(\alpha_2+\alpha_3)\right]. \tag{3.28}$$

Denote $\gamma := \frac{\varphi'}{\varphi}$. By (3.27) and (3.28), we have

$$(\alpha_2+\alpha_3)\gamma' = n\gamma\gamma' - (2n-1)\gamma''. \tag{3.29}$$

If $\gamma' \equiv 0$, then $\varphi = c_6 e^{c_7 z}$, $c_6, c_7 \in \mathbb{C}$. It follows from $\varphi' \not\equiv 0$ that $c_6c_7 \neq 0$, which implies that $\varphi \notin S(f)$, a contradiction. If $\gamma' \not\equiv 0$, it follows from (3.29) that

$$e^{(\alpha_2+\alpha_3)z} = c_8\varphi^n\left(\left(\frac{\varphi'}{\varphi}\right)'\right)^{-(2n-1)}, \quad c_8 \in \mathbb{C}\setminus\{0\},$$

which implies that $e^{(\alpha_2+\alpha_3)z} \in S(f)$, then $\alpha_2 + \alpha_3 = 0$, a contradiction. If $\varphi' \equiv 0$, by (3.26), we have $\frac{\alpha}{\varphi} \equiv 0$. Combining (3.15) and substituting these equalities into $q_2^2 = 4q_1q_3$ yield

$$n(n-1)\left(\frac{\alpha_3}{\alpha_2}\right)^2 - [n^2 + (n-1)^2]\frac{\alpha_3}{\alpha_2} + n(n-1) = 0. \tag{3.30}$$

Solving the equation (3.30) shows that $\frac{\alpha_3}{\alpha_2} = \frac{n-1}{n}$ or $\frac{\alpha_3}{\alpha_2} = \frac{n}{n-1}$. Noting that (1.4), we have $\frac{\alpha_3}{\alpha_2} = \frac{n-1}{n}$. Substituting $\varphi' \equiv 0$ into (3.25), we get

$$f'' = \frac{1}{2n-1}(\alpha_2 + \alpha_3)f'. \tag{3.31}$$

Note that $T\big(r, \frac{1}{f}\big) = N_{1)}\big(r, \frac{1}{f}\big) + S(r, f)$. Then we obtain the solution of (3.31) of the form

$$f = \frac{(2n-1)c_9}{\alpha_2 + \alpha_3} e^{\frac{\alpha_2+\alpha_3}{2n-1}z} + c_{10},\ \ c_9, c_{10} \in \mathbb{C} \setminus \{0\}.$$

By $\frac{\alpha_3}{\alpha_2} = \frac{n-1}{n}$, we get $f = \frac{nc_9}{\alpha_2} e^{\frac{\alpha_2}{n}z} + c_{10}$. Substituting this formula into (1.5) gives $\big(\frac{nc_9}{\alpha_2}\big)^n = p_2$, $\frac{\alpha_1}{\alpha_2} = \frac{s}{n}$, $1 \le s \le n-1$.

If $m\big(r, \frac{e^{\alpha_3 z}}{f^{n-2}}\big) = S(r, f)$, similarly, we also obtain $f(z) = \gamma_1 + \gamma_2 e^{\frac{\alpha_2}{n}z}$, γ_1 is a constant, γ_2 is a non-zero constant satisfying $\gamma_2^n = p_2$. □

4. Proof of Theorem 1.6

Proof. Suppose that (1.6) has a meromorphic solution f such that $T(r, p_l) = S(r, f)$ $(l = 1, 2, 3)$. It follows from (1.6) that

$$\begin{aligned} N(r,f) = \frac{1}{2}N(r,f^2) &\le \frac{1}{2}(N(r,p_1e^{\alpha_1 z}) + N(r,p_2e^{\alpha_2 z}) + N(r,p_3e^{\alpha_3 z})) + O(1) \\ &= S(r,f), \end{aligned}$$

$$T(r,f) = \frac{1}{2}T(r,f^2) \le O(T(r,e^z)) + S(r,f).$$

Then $N(r, f) \le S(r, f)$, $S(r, f) \subset S(r, e^z)$. Rewrite (1.6) as

$$(fe^{-\frac{\alpha_3}{2}z})^2 - p_3 = p_1e^{(\alpha_1-\alpha_3)z} + p_2e^{(\alpha_2-\alpha_3)z}. \tag{4.1}$$

Set $g = fe^{-\frac{\alpha_3}{2}z}$, $\beta_1 = \alpha_1 - \alpha_3$, $\beta_2 = \alpha_2 - \alpha_3$, which implies $\beta_1 \ne \beta_2$. Rewrite (4.1) in the form

$$g^2 - p_3 = p_1e^{\beta_1 z} + p_2e^{\beta_2 z}. \tag{4.2}$$

By $g = fe^{-\frac{\alpha_3}{2}z}$, we have

$$N(r,g) \le N(r,f) + N(r,e^{-\frac{\alpha_3}{2}z}) \le S(r,f),$$

$$T(r,g) \le T(r,f) + T(r,e^{-\frac{\alpha_3}{2}z}) \le O(T(r,e^z)) + S(r,f) \le O(T(r,e^z)) + S(r,e^z),$$

and $N(r, g) \le S(r, f)$, $S(r, g) \subset S(r, e^z)$. Differentiating (4.2) yields

$$2gg' - p_3' = (p_1' + \beta_1p_1)e^{\beta_1 z} + (p_2' + \beta_2p_2)e^{\beta_2 z}. \tag{4.3}$$

Eliminating $e^{\beta_2 z}$ and $e^{\beta_1 z}$ from (4.2) and (4.3), respectively, we have

$$g[2p_2g' - (p_2' + \beta_2 p_2)g] + P = Ae^{\beta_1 z}, \tag{4.4}$$

$$g[2p_1g' - (p_1' + \beta_1 p_1)g] + Q = -Ae^{\beta_2 z}, \tag{4.5}$$

where $A = p_2(p_1' + \beta_1 p_1) - p_1(p_2' + \beta_2 p_2)$, $P = -p_2p_3' + p_3(p_2' + \beta_2 p_2)$, $Q = -p_1p_3' + p_3(p_1' + \beta_1 p_1)$. We claim that $A \not\equiv 0$. Otherwise $A \equiv 0$, and $\beta_1 - \beta_2 = \frac{p_2'}{p_2} - \frac{p_1'}{p_1}$. By integration, we get $\frac{p_2}{p_1} = c_1 e^{(\beta_1-\beta_2)z} \in S(f)$, $c_1 \in \mathbb{C} \setminus \{0\}$, which implies $\beta_1 = \beta_2$, a contradiction. By $S(r,f) \subset S(r,e^z)$, $S(r,g) \subset S(r,e^z)$ and (4.4), we have

$$\begin{aligned}
T(r, e^{\beta_1 z}) &\le m(r, Ae^{\beta_1 z}) + m\left(r, \frac{1}{A}\right) \\
&\le m\left(r, g^2\left(2p_2\frac{g'}{g} - (p_2' + \beta_2 p_2)\right) + P\right) + T(r,A) \\
&\le m(r, g^2) + S(r,g) + S(r,f) \\
&\le 2T(r,g) + S(r,e^z),
\end{aligned}$$

and $S(r,e^z) \subset S(r,g)$. This shows that $S(r,e^z) = S(r,g)$. Differentiating (4.4) yields

$$-(p_2' + \beta_2 p_2)'g^2 - 2\beta_2 p_2 gg' + 2p_2(g')^2 + 2p_2 gg'' + P' = (A' + A\beta_1)e^{\beta_1 z}. \tag{4.6}$$

Eliminating $e^{\beta_1 z}$ from (4.4) and (4.6), we have

$$d_1 g^2 + d_2 gg' + d_3 (g')^2 + d_4 gg'' = R, \tag{4.7}$$

where

$$\begin{cases}
d_1 = (p_2' + \beta_2 p_2)(A' + A\beta_1) - (p_2' + \beta_2 p_2)'A, \\
d_2 = -2p_2(\beta_1 + \beta_2)A - 2p_2 A', \\
d_3 = 2p_2 A, \\
d_4 = 2p_2 A, \\
R = (A' + A\beta_1)P - AP'.
\end{cases} \tag{4.8}$$

If $d_1 \equiv 0$, then $\frac{(p_2'+\beta_2 p_2)'}{p_2'+\beta_2 p_2} - \frac{A'}{A} = \beta_1$. By integration, we have $c_2 e^{\beta_1 z} = \frac{p_2'+\beta_2 p_2}{A} \in S(f)$, $c_2 \in \mathbb{C} \setminus \{0\}$, which implies $\beta_1 = 0$, a contradiction. Hence $d_1 \not\equiv 0$. If $R \equiv 0$, that is $(A' + A\beta_1)P - AP' \equiv 0$. If $P \equiv 0$, then $\frac{p_3'}{p_3} - \frac{p_2'}{p_2} = \beta_2$. By integration, we have $c_3 e^{\beta_2 z} = \frac{p_3}{p_2} \in S(f)$, $c_3 \in \mathbb{C} \setminus \{0\}$, which implies $\beta_2 = 0$, a contradiction. If $P \not\equiv 0$, then $\frac{A'}{A} - \frac{P'}{P} = -\beta_1$, by integration, we get $c_4 e^{-\beta_1 z} = \frac{A}{P} \in S(f)$, $c_4 \in \mathbb{C} \setminus \{0\}$, which implies $\beta_1 = 0$, a contradiction. So $R \not\equiv 0$. From (4.8), we have $T(r,R) = S(r,g)$. By (4.7), we deduce that $2m\left(r, \frac{1}{g}\right) = m\left(r, \frac{1}{R}\left(d_1 + d_2\frac{g'}{g} + d_3\left(\frac{g'}{g}\right)^2 + d_4\frac{g''}{g}\right)\right) \le S(r,g)$. Therefore, $T(r,g) = N\left(r, \frac{1}{g}\right) + S(r,g)$. Suppose z_0 is a multiple zero of g which is not the zero of d_j $(j = 1,2,3,4)$. By (4.7), we conclude that z_0 must be a zero of R, which implies $N_{(2}\big(r,\frac{1}{g}\big) \le T(r,R) = S(r,g)$. Then $T(r,g) = N_{1)}\big(r,\frac{1}{g}\big)+S(r,g)$. Differentiating (4.7) yields

$$R' = d_1'g^2 + (2d_1 + d_2')gg' + (d_2 + d_3')(g')^2 + (d_2 + d_4')gg'' + (2d_3 + d_4)g'g'' + d_4gg'''. \tag{4.9}$$

Combining (4.7) and (4.9), we have

$$(d_1'R - d_1R')g^2 + [(2d_1 + d_2')R - d_2R']gg' + [(d_2 + d_3')R - d_3R'](g')^2 + [(d_2 + d_4')R - d_4R']gg'' + (2d_3 + d_4)Rg'g'' + d_4Rgg''' = 0. \tag{4.10}$$

Suppose z_1 is a simple zero of g, which is not the zero of coefficients of (4.10). It follows from (4.10) that z_1 is a zero of $(2d_3+d_4)Rg''+[(d_2+d_3')R-d_3R']g'$. Set

$$\alpha = \frac{(2d_3 + d_4)Rg'' + [(d_2 + d_3')R - d_3R']g'}{g}. \tag{4.11}$$

Then we have $T(r,\alpha) = S(r,g)$. It follows from (4.11) that

$$g'' = \frac{d_3R' - (d_2 + d_3')R}{(2d_3 + d_4)R}g' + \frac{\alpha}{(2d_3 + d_4)R}g. \tag{4.12}$$

Substituting (4.12) into (4.7) yields

$$q_1g^2 + q_2gg' + q_3(g')^2 = R, \tag{4.13}$$

where

$$\begin{cases} q_1 = d_1 + \frac{d_4\alpha}{(2d_3+d_4)R}, \\ q_2 = d_2 + \frac{d_4[d_3R'-(d_2+d_3')R]}{(2d_3+d_4)R}, \\ q_3 = d_3, \end{cases}$$

are small functions of g. It follows from (4.8) that

$$\frac{q_2}{q_3} = \frac{1}{3}\frac{R'}{R} - \frac{1}{3}\frac{p_2'}{p_2} - \frac{2}{3}(\beta_1 + \beta_2) - \frac{A'}{A}. \tag{4.14}$$

From (4.13) and Lemma 2.8, we get

$$q_3(q_2^2-4q_1q_3)\frac{R'}{R} + q_2(q_2^2-4q_1q_3) - q_3(q_2^2-4q_1q_3)' + (q_2^2-4q_1q_3)q_3' = 0. \tag{4.15}$$

Now we consider the following two cases.

Case 1. Suppose that $q_2^2 - 4q_1q_3 \equiv 0$. It follows from (4.13) that

$$q_3\left(g' + \frac{q_2}{2q_3}g\right)^2 = R,$$

and $g'+\frac{q_2}{2q_3}g$ must be a small function of g. Set $\gamma = g'+\frac{q_2}{2q_3}g$. Note that $R \not\equiv 0$. Then $\gamma \not\equiv 0$. Substituting $g' = \gamma - \frac{q_2}{2q_3}g$ into (4.4) and (4.5), respectively, we have

$$g^2R_1 - 2p_2\gamma g - P = -Ae^{\beta_1 z}, \tag{4.16}$$

and

$$g^2R_2 - 2p_1\gamma g - Q = Ae^{\beta_2 z}, \tag{4.17}$$

where $R_1 = p_2' + \beta_2 p_2 + p_2\frac{q_2}{q_3}$, $R_2 = p_1' + \beta_1 p_1 + p_1\frac{q_2}{q_3}$. Now we discuss the following four subcases.

Subcase 1.1. Suppose that $R_1 \equiv 0$ and $R_2 \equiv 0$. By $R_1 \equiv 0$ and (4.14), we get

$$\frac{p_2'}{p_2} + \beta_2 = -\frac{1}{3}\frac{R'}{R} + \frac{1}{3}\frac{p_2'}{p_2} + \frac{2}{3}(\beta_1+\beta_2) + \frac{A'}{A}.$$

Then by integration, we get

$$(p_2 e^{\beta_2 z})^3 = c_5\frac{p_2 A^3}{R}e^{2(\beta_1+\beta_2)z},\ c_5 \in \mathbb{C}\setminus\{0\},$$

which implies $e^{[3\beta_2 - 2(\beta_1+\beta_2)]z} \in S(g)$, and thus $3\beta_2 - 2(\beta_1+\beta_2) = 0$, that is $\frac{\beta_1}{\beta_2} = \frac{1}{2}$. By $R_2 \equiv 0$ and (4.14), we get

$$\frac{p_1'}{p_1} + \beta_1 = -\frac{1}{3}\frac{R'}{R} + \frac{1}{3}\frac{p_2'}{p_2} + \frac{2}{3}(\beta_1+\beta_2) + \frac{A'}{A}.$$

Then by integration, we have

$$(p_1 e^{\beta_1 z})^3 = c_6\frac{p_2 A^3}{R}e^{2(\beta_1+\beta_2)z},\ c_6 \in \mathbb{C}\setminus\{0\},$$

which implies $e^{[3\beta_1 - 2(\beta_1+\beta_2)]z} \in S(g)$, and thus $3\beta_1 - 2(\beta_1+\beta_2) = 0$, that is $\frac{\beta_1}{\beta_2} = 2$, which is impossible.

Subcase 1.2. Suppose that $R_1 \equiv 0$ and $R_2 \not\equiv 0$. By $R_1 \equiv 0$ and the Subcase 1.1, we get $\frac{\beta_1}{\beta_2} = \frac{1}{2}$, and

$$e^{\beta_1 z} = e^{\frac{1}{2}\beta_2 z}. \tag{4.18}$$

By (4.17) and Lemma 2.10, we have

$$(g+\nu_1)^2 = \frac{A}{R_2}e^{\beta_2 z}, \tag{4.19}$$

that is $g = \mu_1 e^{\frac{\beta_2}{2}z} - \nu_1$, μ_1, ν_1 are small functions of f. Note that $T(r,g) = N_{1)}\big(r, \frac{1}{g}\big) + S(r,g)$, so $\mu_1\nu_1 \not\equiv 0$. It follows from (4.18), (4.19) and (4.2) that

$$(\mu_1 e^{\frac{\beta_2}{2}z} - \nu_1)^2 - p_3 = p_1 e^{\frac{1}{2}\beta_2 z} + p_2 e^{\beta_2 z},$$

that is,

$$(\mu_1^2 - p_2)e^{\beta_2 z} + (-2\mu_1\nu_1 - p_1)e^{\frac{1}{2}\beta_2 z} + \nu_1^2 - p_3 = 0.$$

By applying Lemma 2.7, we get $\mu_1^2 = p_2$, $-2\mu_1\nu_1 = p_1$, $\nu_1^2 = p_3$. Note that $g = fe^{-\frac{\alpha_3}{2}z}$, $\beta_1 = \alpha_1 - \alpha_3$, $\beta_2 = \alpha_2 - \alpha_3$, we conclude that $f = \mu_1 e^{\frac{\alpha_2}{2}z} - \nu_1 e^{\frac{\alpha_3}{2}z}$ and $2\alpha_1 = \alpha_2 + \alpha_3$.

Subcase 1.3. Suppose that $R_1 \not\equiv 0$ and $R_2 \equiv 0$. By $R_2 \equiv 0$ and the Subcase 1.1, we get $\frac{\beta_1}{\beta_2} = 2$. Similar to the Subcase 1.2, we obtain $f = \mu_2 e^{\frac{\alpha_1}{2}z} - \nu_2 e^{\frac{\alpha_3}{2}z}$, $\mu_2^2 = p_1$, $-2\mu_2\nu_2 = p_2$, $\nu_2^2 = p_3$ and $2\alpha_2 = \alpha_1 + \alpha_3$.

Subcase 1.4. Suppose that $R_1 \not\equiv 0$ and $R_2 \not\equiv 0$. By (4.16), (4.17) and Lemma 2.10, we get

$$(g+\nu_3)^2 = -\frac{A}{R_1}e^{\beta_1 z}, \quad (g+\nu_4)^2 = \frac{A}{R_2}e^{\beta_2 z}.$$

Then

$$g = \mu_3 e^{\frac{\beta_1}{2}z} - \nu_3, \quad g = \mu_4 e^{\frac{\beta_2}{2}z} - \nu_4, \tag{4.20}$$

where $\mu_3^2 = -\frac{A}{R_1} \not\equiv 0$, $\mu_4^2 = \frac{A}{R_2} \not\equiv 0$. It follows from (4.20) that

$$\mu_3 e^{\frac{\beta_1}{2}z} - \mu_4 e^{\frac{\beta_2}{2}z} - (\nu_3 - \nu_4) = 0.$$

Noting that $\beta_1 \neq \beta_2$ and by applying Lemma 2.7, we get $\mu_3 = \mu_4 \equiv 0$, a contradiction.

Case 2. Suppose that $q_2^2 - 4q_1q_3 \not\equiv 0$. It follows from (4.15) that

$$\frac{q_2}{q_3} = \frac{(q_2^2-4q_1q_3)'}{q_2^2-4q_1q_3} - \frac{q_3'}{q_3} - \frac{R'}{R}.$$

By (4.14) and the above equality, we get

$$2(\beta_1+\beta_2) = 4\frac{R'}{R} + 3\frac{q_3'}{q_3} - \frac{p_2'}{p_2} - 3\frac{A'}{A} - 3\frac{(q_2^2-4q_1q_3)'}{q_2^2-4q_1q_3}.$$

By integration, we have

$$e^{2(\beta_1+\beta_2)z} = c_7\frac{R^4q_3^3}{p_2A^3(q_2^2-4q_1q_3)^3}, \ c_7 \in \mathbb{C}\setminus\{0\},$$

which implies $e^{2(\beta_1+\beta_2)z} \in S(g)$, and thus $\beta_1+\beta_2 = 0$. Multiplying (4.4) and (4.5) deduces that

$$g^2\psi + T = -A^2, \tag{4.21}$$

where

$$\psi = [2p_1g' - (p_1' + \beta_1p_1)g][2p_2g' - (p_2'+\beta_2p_2)g],$$
$$T = g[2p_1g' - (p_1'+\beta_1p_1)g]P + g[2p_2g' - (p_2'+\beta_2p_2)g]Q + PQ,$$

are differential polynomials in g of degree at most 2, with small functions of g as its coefficients. By (4.21) and Lemma 2.1, we obtain $m(r,\psi) = S(r,g)$. Note that $N(r,g) = S(r,g)$. Then $T(r,\psi) = S(r,g)$. If $\psi \equiv 0$, then $2p_1g' - (p_1' + \beta_1p_1)g \equiv 0$ or $2p_2g' - (p_2'+\beta_2p_2)g \equiv 0$, so we deduce that $\overline{N}(r,\frac{1}{g}) = S(r,g)$, a contradiction. Hence $\psi \not\equiv 0$. Denote $\psi = \psi_1\psi_2$, where

$$\psi_j = (p_j' + \beta_jp_j)g - 2p_jg', \ j = 1,2. \tag{4.22}$$

Then

$$N\left(r,\frac{1}{\psi_j}\right) + N(r,\psi_j) \le N(r,\psi_1) + N(r,\psi_2) + T(r,\psi) + O(1) = S(r,g).$$

It follows from (4.22) that

$$g=-\frac{p_2}{B}\psi_1+\frac{p_1}{B}\psi_2,\quad g'=-\frac{p_2'+\beta_2p_2}{2B}\psi_1+\frac{p_1'+\beta_1p_1}{2B}\psi_2, \tag{4.23}$$

where $B=p_1(p_2'+\beta_2p_2)-p_2(p_1'+\beta_1p_1)\not\equiv 0$. Otherwise $\frac{p_2'}{p_2}-\frac{p_1'}{p_1}=\beta_1-\beta_2$, and then $c_8e^{(\beta_1-\beta_2)z}=\frac{p_2}{p_1}\in S(g)$, $c_8\in\mathbb{C}\setminus\{0\}$, which implies $\beta_1=\beta_2$, a contradiction. Differentiating the first equality of (4.23), we get

$$g'=-\left(\left(\frac{p_2}{B}\right)'+\frac{p_2}{B}\frac{\psi_1'}{\psi_1}\right)\psi_1+\left(\left(\frac{p_1}{B}\right)'+\frac{p_1}{B}\frac{\psi_2'}{\psi_2}\right)\psi_2. \tag{4.24}$$

Substituting (4.24) into the second equality of (4.23), we have

$$D_1\psi_1-D_2\psi_2=0, \tag{4.25}$$

where

$$\begin{aligned} D_1&=\frac{p_2'+\beta_2p_2}{2B}-\left(\frac{p_2}{B}\right)'-\frac{p_2}{B}\frac{\psi_1'}{\psi_1},\\ D_2&=\frac{p_1'+\beta_1p_1}{2B}-\left(\frac{p_1}{B}\right)'-\frac{p_1}{B}\frac{\psi_2'}{\psi_2}, \end{aligned} \tag{4.26}$$

and $T(r,D_j)=S(r,g),\ j=1,2$. If $D_1\not\equiv 0$, it follows from (4.25) that

$$T(r,\psi_1)=\frac{1}{2}T(r,\psi_1^2)=\frac{1}{2}T\left(r,\frac{D_2\psi}{D_1}\right)\le S(r,g).$$

It follows from the first equality of (4.23) that

$$T(r,g)=T\left(r,\frac{g\psi_1}{\psi_1}\right)\le T\left(r,-\frac{p_2}{B}\psi_1^2+\frac{p_1}{B}\psi\right)+T(r,\psi_1)+O(1)=S(r,g),$$

which is impossible. So $D_1\equiv 0$, then $D_2\equiv 0$. Combining with (4.26), we obtain

$$\psi_1=c_9Bp_2^{-\frac{1}{2}}e^{\frac{\beta_2}{2}z},\quad \psi_2=c_{10}Bp_1^{-\frac{1}{2}}e^{\frac{\beta_1}{2}z},\ c_9,\ c_{10}\in\mathbb{C}\setminus\{0\}. \tag{4.27}$$

Substituting (4.27) into the first equality of (4.23) yields

$$g=\gamma_1e^{\frac{\beta_1}{2}z}+\gamma_2e^{\frac{\beta_2}{2}z}, \tag{4.28}$$

where $\gamma_1=c_{10}p_1^{\frac{1}{2}}\not\equiv 0$, $\gamma_2=-c_9p_2^{\frac{1}{2}}\not\equiv 0$, $\beta_1+\beta_2=0$. Substituting (4.28) into (4.2) yields

$$(\gamma_1^2-p_1)e^{\beta_1z}+(\gamma_2^2-p_2)e^{\beta_2z}+2\gamma_1\gamma_2-p_3=0.$$

By applying Lemma 2.7, we have $\gamma_1^2=p_1$, $\gamma_2^2=p_2$, $2\gamma_1\gamma_2=p_3$. Note that $g=fe^{-\frac{\alpha_3}{2}z}$, $\beta_1=\alpha_1-\alpha_3$, $\beta_2=\alpha_2-\alpha_3$. By (4.28), we conclude that $f=\gamma_1e^{\frac{\alpha_1}{2}z}+\gamma_2e^{\frac{\alpha_2}{2}z}$ and $2\alpha_3=\alpha_1+\alpha_2$. □

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