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A GORENSTEIN HOMOLOGICAL CHARACTERIZATION OF KRULL DOMAINS

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Dedicated to Professor Fanggui Wang for his 69th birthday

Abstract. In this note, we shed new light on Krull domains from the point view of Gorenstein homological algebra. By using the so-called w-operation, we show that an integral domain R is Krull if and only if for any nonzero proper w-ideal I, the Gorenstein global dimension of the w-factor ring $(R/I)_w$ is zero. Further, we obtain that an integral domain R is Dedekind if and only if for any nonzero proper ideal I, the Gorenstein global dimension of the factor ring R/I is zero.

1. Introduction

Throughout this paper, all the rings are commutative rings with 0 and 1 such that $0 \neq 1$. In order to avoid a trivial case, we assume that all integral domains are not field. It is well-known that Krull domains play an important role in the development of multiplicative ideal theory. By using star-operations, the Krull domains can be characterized those domains having nonzero ideal w-invertible (equivalently, t-invertible). So the Krull domains are also viewed as "Dedekind domains" in the sense of star-operations. In [3, Proposition 2.8], Bennis, Hu and Wang prove that an integral domain R is Dedekind if and only if every non-trivial factor ring of R is 2-SG-semisimple, where a ring R is called 2-*SG-semisimple* in [3] if every R-module is 2-SG-projective. Thus it is natural to ask whether we can characterize Krull domains from the point view of Gorenstein homological algebra.

Recall that an R-module M is called *Gorenstein projective* (G-projective) in [10] if M has a complete projective resolution

$$\mathbf{P} : \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$$

with $M \cong \ker(P^0 \to P^1)$. The *Gorenstein injective* (G-injective) module is defined dually. For an R-module M, the Gorenstein injective and projective

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dimensions of M are denoted by $\text{G-id}_R(M)$ and $\text{G-pd}_R(M)$, respectively. It is shown in [6, Theorem 1.1] that for a ring R,

$$\sup\{\text{G-pd}_R(M) \mid M \text{ is an } R\text{-module}\} = \sup\{\text{G-id}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

This common value is called the *Gorenstein global dimension* of R and denoted by G-gl.dim(R). As in [7], a ring R is called *Gorenstein semisimple* (G-semisimple) if every R-module is G-projective, i.e., G-gl.dim$(R) = 0$. It is shown in [7, Theorem 2.2] that a G-semisimple ring is precisely a QF-ring, where a ring R is called a *quasi-Frobenius ring* (QF-ring) if R is Noetherian and self-injective. Let n be a fixed positive integer. Recall from [5] that an R-module M is called *n-strongly Gorenstein projective* (n-SG-projective) if there exists an exact sequence of R-modules

$$0 \longrightarrow M \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

such that each P_i is projective and the functor $\text{Hom}_R(-, Q)$ leaves the sequence exact whenever Q is a projective R-module. The 1-SG-projective module is just the *strongly Gorenstein projective* module (SG-projective modules) in [4]. Accordingly, a ring R is called *n-SG-semisimple* in [7] if every R module is n-SG-projective. The 0-SG-semisimple ring is the so-called SG-semisimple ring. Both SG-semisimple rings and 2-SG-semisimple rings are QF-rings, which are investigated in [7] and [3], respectively. It is clear that an SG-semisimple ring is 2-SG-semisimple, and a 2-SG-semisimple ring is G-semisimple.

Recently, Chang and Kim use the w-operation to study the factor rings of Krull domains, and they prove in [8, Theorem 4.5] that an integral domain R is Krull if and only if for any nonzero proper w-ideal I of R, the w-factor ring $(R/I)_w$ of R modulo I is an Artinian PIR, where a ring R is called a *principal ideal ring* (PIR) if every ideal of R is principal. In this note, we shall further characterize Krull domains by their w-factor ring and extend the above Bennis-Hu-Wang's result to Krull domains. We prove in Theorem 9 that an integral domain R is Krull if and only if for any nonzero proper w-ideal I, the w-factor ring $(R/I)_w$ of R modulo I is 2-SG-semisimple. Equivalently, G-gl.dim$(R/I)_w = 0$ for any nonzero proper w-ideal I of R. Our result will be complementary for Chang-Kim Theorem ([8, Theorem 4.5]). It is also seen in Theorem 13 that an integral domain R is Dedekind if and only if R/I is a QF-ring for any nonzero proper ideal I of R.

Next we recall for reader's convenience the following facts of w-modules. Let R be a ring. As in [19], a nonzero ideal J of R is called a *Glaz-Vasconcelos ideal* (*GV-ideal*) if J is finitely generated and the natural homomorphism $\varphi : J \to \text{Hom}_R(J, R)$ is an isomorphism. Denote the set of GV-ideals of R by $\text{GV}(R)$. Let M be an R-module. Then M is called *GV-torsion-free* if $Jx = 0$ with $J \in \text{GV}(R)$ and $x \in M$ implies $x = 0$, and M is called *GV-torsion* if for any $x \in M$, there exists $J \in \text{GV}(R)$ with $Jx = 0$. For a GV-torsion-free module M, set $M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \text{GV}(R)\}$ which is called the

w-envelope of M, where $E(M)$ is the injective hull of M. Let M be an GV-torsion-free module over R. Then M is called *a finite type module* if there exists submodule N of M such that $N_w = M_w$, and M is called a *w-module* over R if $M = M_w$. If further an ideal I of R is a w-module, then I is called a *w-ideal*. An ideal $\mathfrak{m}$ maximal among integral w-ideals is called a maximal w-ideal, and the set of maximal w-ideals is denoted by $w\text{-}\max(R)$. The *w-dimension* of a commutative ring R is defined to be $\sup\{\mathrm{ht}(\mathfrak{m}) \,|\, \mathfrak{m} \in w\text{-}\max(R)\}$ and it is denoted by $w\text{-}\dim(R)$. We say that two ideals I and J of R are *w-comaximal* if $(I+J)_w = R$, and a ring R is a *DW-ring* if every ideal of R is a w-ideal. As in [16], an integral domain R is called a *strong Mori domain* (an SM-domain) if R satisfies ACC on w-ideals. It is worth noting that a new description of w-envelope is recently given in [21]. Let M be a GV-torsion-free module over R and set $M[X] := R[X] \bigotimes_R M$. Then

$$S := \{\beta \in R[X] \,|\, \beta \text{ is regular and } \mathrm{Ann}_{M[X]}(\beta) = 0\}$$

is a multiplicative closed subset of $R[X]$. Thus, we can consider the localization $M[X]$ at S, denoted by $T'(M[X])$. Let $\Gamma_1(M) := \{u \in T'(M[X]) \mid$ there exists some $J \in \mathrm{GV}(R)$ such that $Ju \subseteq M\}$ and $\Gamma_2(M) := \{\sum_{i=1}^n a_i X^i / \sum_{i=1}^n b_i X^i \in T'(M[X]) \,|\, a_i \in M$ and $b_i \in R$ with $a_i b_j = b_i a_j$ for all $i,\ j$ and $(b_1, \ldots, b_n) \in \mathrm{GV}(R)\}$. By [21, Theorem 3.4], it is proved that $M_w = \Gamma_1(M) = \Gamma_2(M)$ for any GV-torsion-free R-module M. Further, if I is a nonzero proper w-ideal of R, then R/I is a GV-torsion-free R-module and its w-envelope has a natural ring structure by [21, Corollary 3.5], which is called the w-factor ring of R modulo I in [18]. We now proceed to state and prove our main results.

2. The main results

We start by the following lemmas:

Lemma 1 (Krull Intersection Theorem for SM-domains [16, Theorem 1.8])**.** *Let R be an SM-domain and let M be a finite type w-module. If $B = \bigcap_{k=1}^{\infty}(I^k M)_w$ where I is an ideal of R, then $B = (IB)_w$. If in addition $I_w \neq R$, then $B = 0$.*

Lemma 2. *If I is an ideal of an SM-domain R and $I_w \neq R$, then*

$$\bigcap_{k=1}^{\infty}(I^k)_w = 0.$$

Proof. Since R is a projective R-module, R is a finite type w-module over R by [19, Corollary 2.4]. Take $M = R$ in Lemma 1. Then $\bigcap_{k=1}^{\infty}(I^k)_w = 0$. □

Lemma 3. *If P is a maximal w-ideal of an SM-domain R, then $P \neq (P^2)_w$.*

Proof. By Lemma 2, we have $\bigcap_{k=1}^{\infty}(P^k)_w = 0$. If $P = (P^2)_w$, then $P = (P^k)_w$ for any positive integer k. It means that $\bigcap_{k=1}^{\infty}(P^k)_w = P \neq 0$. Which is impossible. So $P \neq (P^2)_w$. □

Lemma 4 (w-theoretic version of Chinese Remainder Theorem for rings [21, Theorem 3.10]). *Let $\{I_i \mid i = 1, 2, \dots, n\}$ be a pairwise w-comaximal set of w-ideals in a ring R and let $I = I_1 \cap I_2 \cap \cdots \cap I_n$. Then the map*

$$\left(\frac{R}{I}\right)_w \cong \prod_{i=1}^{n} \left(\frac{R}{I_i}\right)_w$$

is a ring isomorphism.

For an Artinian local ring, we can use the following lemma to determinate when it is SG-semisimple (resp., 2-SG-semisimple, G-semisimple).

Lemma 5. *The following statements hold for an Artinian local $(R, \mathfrak{m})$.*

(1) *R is an SG-semisimple ring if and only if $\mathfrak{m}$ is SG-projective.*
(2) *R is a 2-SG-semisimple ring if and only if $\mathfrak{m}$ is principal.*
(3) *R is a G-semisimple ring if and only if $\mathfrak{m}$ is G-projective.*
(4) *If $\mathfrak{m}$ is SG-projective, then $\mathfrak{m}$ is principal.*
(5) *If $\mathfrak{m}$ is principal, then $\mathfrak{m}$ is G-projective.*

Proof. (1) This follows from [11, Corollary 3.4].
(2) This follows from [3, Lemma 2.2 and Theorem 2.6].
(3) This follows from [11, Corollary 2.6].
(4) and (5) are obvious. □

Let $\varphi : R \to T$ be a ring homomorphism. Then φ is called a *w-linked ring homomorphism* in [18] if T is a w-module over R. When I is a w-ideal of a ring R, R/I as an R-module is GV-torsion-free by [19, Theorem 2.7]. For a nonzero proper w-ideal I, we consider the natural composite map

$$\pi : R \twoheadrightarrow R/I \hookrightarrow (R/I)_w.$$

Then π is a w-linked ring homomorphism. Let A be an ideal of $(R/I)_w$. Define $A_{w_\pi} := A_w$, where A_w is the w-envelope of A as an R-module. Let us denote the w-envelope of A as an $(R/I)_w$-module by A_W, which is different from the w-envelope A_w ($= A_{w_R}$) of A as an R-module. Then $A_{w_R} = A_w \subseteq A_W$. Following [18], we say that A is a *w_π-ideal* of $(R/I)_w$ if $A_{w_\pi} = A$, and $(R/I)_w$ is a *DW_π-ring* if every ideal of $(R/I)_w$ is a w_π-ideal.

Lemma 6. *Let I be a nonzero proper w-ideal of an integral domain R. Denote by $\overline{0}$ the zero ideal of the w-factor ring $(R/I)_w$ of R modulo I. Then $\overline{0}_w = \overline{0}$.*

Proof. Since I is a w-ideal of R, $\pi : R \to (R/I)_w$ is a w-linked ring homomorphism. So $\overline{0} \subseteq \overline{0}_w \subseteq \overline{0}_W$ by [18, Theorem 3.3(1)]. But $\overline{0}_W = \overline{0}$ gives $\overline{0}_w = \overline{0}$. □

Now we start to prove our main results.

Theorem 7. *Let R be an integral domain and let I be a nonzero proper w-ideal of R. If the w-factor ring $(R/I)_w$ of R modulo I is an Artinian ring, then $(R/I)_w$ is a DW_π-ring.*

Proof. Let $S = \{M_1 \cap M_2 \cap \cdots \cap M_k \mid k \geq 1$, each M_i is a maximal w_π-ideal of $(R/I)_w\}$. Set $\overline{R}_w = (R/I)_w$. Since $\overline{R}_w$ is an Artinian ring, $\overline{R}_w$ satisfies the minimal condition on S. Hence S has a minimal element. Say $M_1 \cap M_2 \cap \cdots \cap M_n$ where n is a positive integer. Let M be any maximal w_π-ideal of $\overline{R}_w$. Then $M \cap M_1 \cap M_2 \cap \cdots \cap M_n \in S$. Certainly

$$M \cap M_1 \cap M_2 \cap \cdots \cap M_n \subseteq M_1 \cap M_2 \cap \cdots \cap M_n.$$

But since $M_1 \cap M_2 \cap \cdots \cap M_n$ is a minimal element of S, we have

$$M \cap M_1 \cap M_2 \cap \cdots \cap M_n = M_1 \cap M_2 \cap \cdots \cap M_n.$$

So $M_1 M_2 \cdots M_n \subseteq M_1 \cap M_2 \cap \cdots \cap M_n \subseteq M$. It follows that $M = M_j$ for some $j \in \{1, 2, \ldots, n\}$. Hence $\overline{R}_w$ has only finite number of maximal w_π-ideals. Thus by [18, Theorem 3.11], $\overline{R}_w$ is a DW$_\pi$-ring. □

Theorem 8. *Let R be a Krull domain and let P be a maximal w-ideal of R. Then for any positive integer l, the w-factor ring $(R/(P^l)_w)_w$ of R modulo $(P^l)_w$ is a local 2-SG-semisimple ring, where $(P/(P^l)_w)_w$ is the only maximal ideal.*

Proof. Set $\overline{R}_w = (R/(P^l)_w)_w$ and $\overline{P}_w = (P/(P^l)_w)_w$. Since R is a Krull domain, R is an SM-domain with $w\text{-}\dim(R) = 1$. Hence $\overline{R}_w$ is an Artinian ring by [8, Corollary 2.7]. By Theorem 7, it follows that $\overline{R}_w$ is a DW$_\pi$-ring. Note that $\overline{P}_w$ is the only maximal w_π-ideal of $\overline{R}_w$ by [18, Proposition 4.4(3)]. Thus $\overline{P}_w$ is the only maximal ideal of $\overline{R}_w$. So $\overline{R}_w$ is an Artinian local ring. We next claim that $\overline{R}_w$ is a 2-SG-semisimple ring.

Case 1. $l = 1$. Then $\overline{R}_w = (R/P)_w$. Since P is a maximal w-ideal of R, $\overline{R}_w$ is a field by [18, Theorem 4.5(2)]. Certainly $\overline{R}_w$ is a 2-SG-semisimple ring.

Case 2. $l > 1$. Note that $(P^l)_w \neq 0$. Choose a nonzero element a in $(P^l)_w$. Then by [12, Corollary 1.5] there exists $b \in P$ such that $(a, b)_w = (a, b)_v = P_v = P_w = P$. So

$$\overline{P}_w = (b + (P^l)_w)_w = (b + (P^l)_w)_{w_\pi} = (b + (P^l)_w).$$

And hence $\overline{R}_w$ is a 2-SG-semisimple ring by Lemma 5(2).

Consequently, $\overline{R}_w$ is a local 2-SG-semisimple ring. □

Theorem 9. *The following statements are equivalent for an integral domain R.*

(1) *R is a Krull domain.*
(2) *$(R/I)_w$ is 2-SG-semisimple for any nonzero proper w-ideal I of R.*
(3) *$(R/I)_w$ is G-semisimple for any nonzero proper w-ideal I of R.*
(4) *G-gl.dim$(R/I)_w = 0$ for any nonzero proper w-ideal I of R.*
(5) *$(R/I)_w$ is a QF-ring for any nonzero proper w-ideal I of R.*

Proof. (1) ⇒ (2) Let I be a nonzero proper w-ideal of R. Then by [1, Corollary 3.2], I is a t-product of prime t-ideals. Since R is a Krull domain, $t = w$ over R. Hence I is a w-product of prime w-ideals. Write $I = (P_1^{l_1} P_2^{l_2} \cdots P_n^{l_n})_w =$

$((P_1^{l_1})_w(P_2^{l_2})_w\cdots(P_n^{l_n})_w)_w$. Since R is a Krull domain, w-$\dim(R)=1$. Hence $\{P_i\,|\,i=1,2,\ldots,n\}$ is a pairwise w-comaximal set of w-ideals. It follows from [21, Lemma 3.8] that $\{(P_i^{l_i})_w\,|\,i=1,2,\ldots,n\}$ is likewise a pairwise w-comaximal set of w-ideals. So $I=((P_1^{l_1})_w(P_2^{l_2})_w\cdots(P_n^{l_n})_w)_w=(P_1^{l_1})_w\cap(P_2^{l_2})_w\cap\cdots\cap(P_n^{l_n})_w$ by [21, Lemma 3.9]. Applying Lemma 4, we have the following ring isomorphism:

$$\left(\frac{R}{I}\right)_w=\left(\frac{R}{(P_1^{l_1})_w\bigcap(P_2^{l_2})_w\bigcap\cdots\bigcap(P_n^{l_n})_w}\right)_w\cong\prod_{i=1}^{n}\left(\frac{R}{(P_i^{l_i})_w}\right)_w.$$

Since the w-fractor ring $(R/(P_i^{l_i})_w)_w$ is a local 2-SG-simple ring by Theorem 8, $\prod_{i=1}^n(R/(P_i^{l_i})_w)_w$ is a 2-SG-simple ring by [3, Lemma 2.1]. Hence $(R/I)_w$ is a 2-SG-simple ring.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are clear.

(4) $\Rightarrow$ (5) follows from [6, Proposition 2.6].

(5) $\Rightarrow$ (1) By (5) and [8, Theorem 3.5], it follows that R is an SM-domain. Let P be a nonzero prime w-ideal of R. By Lemma 3, we have $P\neq(P^2)_w$. Take $a\in P\backslash(P^2)_w$. Set $\overline{a}=a+(P^2)_w$, $\overline{P}_w=(P/(P^2)_w)_w$ and $\overline{R}_w=(R/(P^2)_w)_w$. Then $\overline{R}_w$ is a QF-ring by (5). And hence $\overline{R}_w$ is an Artinian ring. By Proposition 7, $\overline{R}_w$ is a DW$_\pi$-ring. Since P is a maximal w-ideal of R, $\overline{P}_w$ is the only maximal w_π-ideal of $\overline{R}_w$ by [18, Proposition 4.4]. Hence $\overline{R}_w$ is an Artinian local ring and its maximal ideal is $\overline{P}_w$. Note that $\overline{P}_w\cdot\overline{P}_w\subseteq(\overline{P}\cdot\overline{P})_w=\overline{0}_w$. But since $\overline{0}_w=\overline{0}$ by Lemma 6, we have $\overline{P}_w\cdot\overline{P}_w=0$. So $\overline{P}_w\subseteq\mathrm{Ann}_{\overline{R}_w}(\overline{a})_w$ and $\overline{P}_w\subseteq\mathrm{Ann}_{\overline{R}_w}\overline{P}_w$. But as $a\notin(P^2)_w$, $\mathrm{Ann}_{\overline{R}_w}(\overline{a})_w\neq\overline{R}_w$. Hence $\overline{P}_w=\mathrm{Ann}_{\overline{R}_w}(\overline{a})_w$. Also since $\overline{P}_w$ is the maximal ideal of $\overline{R}_w$, either $\overline{P}_w=\mathrm{Ann}_{\overline{R}_w}\overline{P}_w$ or $\mathrm{Ann}_{\overline{R}_w}\overline{P}_w=\overline{R}_w$. We claim that $\mathrm{Ann}_{\overline{R}_w}\overline{P}_w\subset\overline{R}_w$. Assume on the contrary that $\mathrm{Ann}_{\overline{R}_w}\overline{P}_w=\overline{R}_w$. Then by Lemma 6, $\overline{P}_w=\overline{0}=\overline{0}_w$. It follows from [18, Proposition 4.4(2)] that $P=(P^2)_w$, which is a contradiction. Thus $\mathrm{Ann}_{\overline{R}_w}(\overline{a})_w=\overline{P}_w=\mathrm{Ann}_{\overline{R}_w}\overline{P}_w$. Since $\overline{R}_w$ is a QF-ring, it follows from [14, Theorem 4.6.11] that

$$\begin{aligned}(\overline{a})_w&=\mathrm{Ann}_{\overline{R}_w}(\mathrm{Ann}_{\overline{R}_w}(\overline{a})_w)\\&=\mathrm{Ann}_{\overline{R}_w}(\mathrm{Ann}_{\overline{R}_w}\overline{P}_w)=\overline{P}_w.\end{aligned}$$

So $P=(aR+P^2)_w$ by [18, Proposition 4.4(2)]. Thus $PR_P=aR_P+P^2R_P$. Note that R is an SM-domain of w-dimension one by [8, Corollary 2.5]. So P is a finite type w-ideal, and hence PR_P is a finitely generated ideal of R_P. By Nakayama lemma, we have $PR_P=aR_P$. Hence P is a w-locally principal ideal of finite type. So P is w-invertible. Therefore R is a Krull domain by [16, Theorem 2.8]. □

Corollary 10 (cf. [8, Corollary 4.9])**.** *An SM-domain R is Krull if and only if for any nonzero proper w-ideal I of R, $(R/I)_w$ is self-injective.*

Proof. This follows from Theorem 9 and [8, Theorem 3.5]. □

Corollary 11. *An SM-domain R of w-dimension one is Krull if and only if for any nonzero proper w-ideal I of R, every prime ideal of $(R/I)_w$ is G-projective.*

Proof. [11, Corollary 2.5] gives that an Artinian ring is a QF-ring if and only if every prime ideal is G-projective. Applying Theorem 8 and [8, Corollary 3.7], the statement is immediate. □

1960s, Bass in [2] defined the some finitistic dimensions. Let R be a ring. The *finitistic dimension* of R, denoted by $\mathrm{FPD}(R)$, is defined to be the supremum of the projective dimensions of R-modules with finite projective dimensions. The *finitistic weak dimension* of R, denoted by $\mathrm{FFD}(R)$, is defined to be the supremum of the flat dimensions of R-modules with finite flat dimensions. For an R-module M, M is said to be have finite projective resolution, denoted by $M \in \mathcal{FPR}$, if there exist a positive integer n and an exact sequence

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \to 0$$

with each P_i is finitely generated and projective. The *small finitistic dimension* of R, denoted by $\mathrm{fPD}(R)$, is the supremum of the projective dimensions of R-modules in $\mathcal{FPR}$. It is clear that for a ring R, $\mathrm{fPD}(R) \leq \mathrm{FFD}(R) \leq \mathrm{FPD}(R)$ and if $w.\mathrm{gl.dim}(R) < \infty$, then $\mathrm{FFD}(R) = w.\mathrm{gl.dim}(R)$. It is worth noting in [9, Theorem 3.2] that for any ring R, $\mathrm{FFD}(R) \leq \mathrm{IFD}(R)$, where $\mathrm{IFD}(R)$ is defined to be $\sup\{\mathrm{fd}_R E \mid E \text{ is an injective } R\text{-module}\}$.

Using those finitistic dimensions, some important ring-theoretic properties can be characterized. For example, it is well-known in [13] that the finitistic dimension $\mathrm{FPD}(R)$ over a Noetherian ring R coincides with the Krull dimension $\dim(R)$. In [15, Theorem 4.1], it was proved that the finitistic weak dimension of a pseudo-valuation domain but not a valuation domain is 1 or 2. Recently, Zhang and Wang in [20, Corollary 3.7] gives an important relationship between DW-rings and their small finitistic dimensions, and they prove that a ring R is a DW-ring if and only if $\mathrm{fPD}(R) \leq 1$. Using this result, we can further characterize Dedekind domains from the point view of factor rings, and we need to establish the following proposition.

Proposition 12. *An integral domain R is a DW-domain if and only if*

$$\mathrm{fPD}(R/(a)) = 0$$

for any nonzero nonunit a of R.

Proof. [20, Corollary 3.7] gives that R is a DW-domain if and only if $\mathrm{fPD}(R) \leq 1$. Thus the result follows immediately from [17, Theorem 4.13]. □

Recall that a ring R is called an *IF-ring* if every injective R-module is flat. If R is a QF-ring, then every injective R-module is projective. So every QF-ring is an IF-ring.

Theorem 13. *The following statements are equivalent for an integral domain R.*

(1) *R is Dedekind.*
(2) *R/I is a QF-ring for any nonzero proper ideal I of R.*
(3) *G-gl.dim$(R/I) = 0$ for any nonzero proper ideal I of R.*

Proof. (1) $\Rightarrow$ (2) Since R is a Dedekind domain, R is a DW-domain and a Krull domain. Let I be any nonzero proper ideal of R. Then $R/I = (R/I)_w$. By Theorem 8, it follows that R/I is a QF-ring.

(2) $\Rightarrow$ (1) Let a be any nonzero nonunit of R and let $T = R/(a)$. Since T is a QF-ring, T an IF-ring. Hence every injective T-module is flat. And so IFD$(T) = 0$. By [9, Theorem 3.2], we have FFD$(T) \leq$ IFD(T). It follows that FFD$(T) = 0$. Also since fPD$(T) \leq$ FFD(T), we have fPD$(T) = 0$. Since a is arbitrary nonzero nonunit in R, we conclude from Proposition 12 that R is a DW-domain. Thus $(R/I)_w = R/I$ is a QF-ring for any nonzero proper ideal I of R. So R is also a Kull domain by Theorem 9. It means that R is a DW-domain and a Krull domain. So R is a Dedekind domain.

(2) $\Leftrightarrow$ (3) This follows from [6, Proposition 2.6]. □

Now we give an example to show that the w-factor ring is different from the factor ring.

Example 14. Let $\mathbb{Z}$ be the ring of integer numbers. Then the polynomial ring $\mathbb{Z}[X]$ is a Krull domain but not a DW-domain. Note that (X) is a maximal w-ideal of $\mathbb{Z}[X]$. So $(\mathbb{Z}[X]/(X))_W$ is a field by [18, Theorem 4.5(2)], where W is the w-operation over $\mathbb{Z}[X]$. However, $\mathbb{Z}[X]/(X)$ is a PID. Thus $\mathbb{Z}[X]/(X) \subset (\mathbb{Z}[X]/(X))_W$. Let $\mathbb{Q}$ be the field of the rational numbers. We remain denoting by $\mathbb{Q}$ the quotient field of $\mathbb{Z}[X]/(X)$. Then $\mathbb{Q} \subseteq (\mathbb{Z}[X]/(X))_W$. But since $\mathbb{Q}$ is the injective hull of $\mathbb{Z}[X]/(X)$, we have $\mathbb{Q} = (\mathbb{Z}[X]/(X))_W$.

Applying Theorem 9, we can provide a new approach to construct lots of non-trivial QF-rings.

Example 15. As Example 14, let $R = \mathbb{Z}[X]$ and $P = (X)$. Then for any positive integer $k > 1$, the w-factor ring $(R/(P^k)_w)_w$ of R modulo $(P^k)_w$ is a QF-ring by Theorem 9. Since R is an SM-domain, it follows from Lemma 3 that $(P^k)_w \subset P$. Hence $0 \neq (P^k)_w$ and $(P^k)_w$ is not a maximal w-ideal of R. So $(R/(P^k)_w)_w$ is not a field by [18, Theorem 4.5(2)]. Thus $(R/(P^k)_w)_w$ is a QF-ring but not a field.

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