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ON WEAKLY (m, n)-PRIME IDEALS OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with identity and m, n be positive integers. In this paper, we introduce the class of weakly (m, n)-prime ideals generalizing (m, n)-prime and weakly (m, n)-closed ideals. A proper ideal I of R is called weakly (m, n)-prime if for $a, b \in R$, $0 \neq a^m b \in I$ implies either $a^n \in I$ or $b \in I$. We justify several properties and characterizations of weakly (m, n)-prime ideals with many supporting examples. Furthermore, we investigate weakly (m, n)-prime ideals under various contexts of constructions such as direct products, localizations and homomorphic images. Finally, we discuss the behaviour of this class of ideals in idealization and amalgamated rings.

1. Introduction

Let R be a commutative ring with identity. By $\dim(R)$, $J(R)$, $Nil(R)$, $reg(R)$ and $U(R)$, we denote the Krull dimension, Jacobson radical, nilpotent elements, regular elements and unit elements of R, respectively. For an ideal I of R and a positive integer n, we denote the set $\{x \in R : x^n \in I\}$ by $\sqrt[n]{I}$.

One of the most interesting and revolutionary concepts in commutative rings is the study of generalizations of prime ideals. Weakly prime ideals in a commutative ring with nonzero identity have been first introduced and studied by Anderson and Smith in [5]. Generalizing this concept, the weakly n-absorbing ideals are established in [21]. A proper ideal I of a ring R is called weakly n-absorbing if whenever $0 \neq a_1 \cdots a_{n+1} \in I$ for $a_1, \ldots, a_{n+1} \in R$, then there are n of the a_i's whose product is in I. Besides, in 2016, the notion of weakly semiprime ideals was presented. According to the definition in [7], a proper ideal I of a ring R is said to be semiprime (resp. weakly semiprime) if whenever $x^2 \in I$ (resp. $0 \neq x^2 \in I$) for some $x \in R$, then $x \in I$. Recall from [24] that a proper ideal I of R is said to be weakly 1-absorbing prime if for non-unit elements $a, b, c \in R$ with $0 \neq abc \in I$, either $ab \in I$ or $c \in I$. Trivially, any weakly

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prime ideal is weakly semiprime and weakly n-absorbing, but the converses of these implications do not hold. For a background and more examples about 1-absorbing ideal structures, we refer the reader to [11], [12] and [18].

Let m and n be positive integers. The concepts of prime and weakly prime ideals have been generalized in [2] and [3] by defining (m, n)-closed and weakly (m, n)-closed ideals. A proper ideal I of a ring R is called an (m, n)-closed (resp. a weakly (m, n)-closed) ideal of R if whenever $a^m \in I$ (resp. $0 \neq a^m \in I$) for some $a \in R$, then $a^n \in I$. In particular, I is said to be a semi-n-absorbing (resp. weakly semi-n-absorbing) ideal of R if for $x \in R$, $x^{n+1} \in I$ (resp. $0 \neq x^{n+1} \in I$) implies $x^n \in I$. More generalizations of prime ideals can be seen in [6, 8–10].

In a recent work [17], we introduced the class of (m, n)-prime ideals which lies properly between the classes of prime and (m, n)-closed ideals. A proper ideal I of a ring R is said to be (m, n)-prime if for $a, b \in R$ with $a^m b \in I$, then either $a^n \in I$ or $b \in I$. Motivated from this concept, the purpose of this paper is to define and study weakly (m, n)-prime ideals. We call a proper ideal I of R weakly (m, n)-prime if for $a, b \in R$, $0 \neq a^m b \in I$ implies either $a^n \in I$ or $b \in I$. Thus, an (m, n)-prime ideal is a weakly (m, n)-prime ideal, and the two concepts agree when R is reduced. However, this generalization is proper as we can see in Example 2.

Among many other results, we examine in Section 2 the relationship among the new class of ideals and the old ones in the literature. We illustrate the place of weakly (m, n)-prime ideals in a diagram and give many examples to verify that the arrows are not reversible (see Examples 1-3 and Remark 1). Then, we determine all weakly (m, n)-prime ideals that are not (m, n)-prime of $R = \mathbb{Z}_{p^k}$, where p is prime and $k > 0$ (Corollary 1). Moreover, we prove that if I is a weakly (m, n)-prime ideal of a ring R that is not (m, n)-prime, then aI, $bI \subseteq Nil(R)$ for some $a \notin \sqrt[n]{I}$ and $b \notin I$ (Corollary 5). Several characterizations of weakly (m, n)-prime ideals in different rings are given (see Theorems 3, 6).

If $R = R_1 \times \cdots \times R_k$, where R_i's are commutative rings with identity, then a complete description of all weakly (m, n)-prime ideals of R is given in Theorem 7 and Corollaries 2, 8. For the particular case that $m \geq n$, we show that a rings for which every proper ideal is weakly (m, n)-prime must be zero dimensional (Theorem 5). Furthermore, a characterization for rings having only one weakly (m, n)-prime ideal disjoint with a multiplicatively closed set S is given in Proposition 9.

Let R be a ring and M an R-module. Recall that $R(+)M = \{(r, b) : r \in R,\ b \in M\}$ with coordinate-wise addition and multiplication defined as $(r_1, b_1)(r_2, b_2) = (r_1 r_2,\ r_1 b_2 + r_2 b_1)$ is a commutative ring with identity $(1, 0)$. This ring is called the idealization of M. For an ideal I of R and a submodule N of M, $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$, [22] and [23]. In the last section, we start by clarifying the relationships between the weakly (m,n)-prime ideals of a ring R and those of the idealization ring $R(+)M$ (Proposition 10). Next, for rings R and R', an ideal J of R' and a ring homomorphism $f : R \to R'$, we justify conditions under which some kinds of ideals in the amalgamated ring $R \bowtie^f J$ are weakly (m,n)-prime. The idealization and amalgamation extensions enables us to built more interesting examples of weakly (m,n)-prime ideals which are not (m,n)-prime.

2. Properties of weakly (m,n)-prime ideals

We begin this section with our main definition and several examples to show the place of the class of weakly (m,n)-prime ideals in the literature.

Definition 1. Let R be a ring and m, n be positive integers. A proper ideal I of R is called weakly (m,n)-prime in R if for $a,b \in R$, $0 \neq a^m b \in I$ implies either $a^n \in I$ or $b \in I$.

It is clear that any weakly (m,n)-prime ideal is weakly (m,n')-prime for all $n' \geq n$. By definition, the zero ideal of any ring is weakly (m,n)-prime for all positive integers m and n. On the other hand, for any prime integer p, the zero ideal in the ring $\mathbb{Z}_{p^{m+1}}$ is not (m,m)-prime since $p^m p = \bar{0}$ but, $p^m \neq \bar{0}$. If I is a weakly prime ideal of a ring R, then clearly, I is weakly (m,n)-prime in R for all positive integers m and n. Moreover, the classes of weakly $(1,1)$-prime ideals and weakly prime ideals in R are coincide. Unlike the case of weakly (m,n)-closed ideals, if $n > m$, then a proper ideal need not be weakly (m,n)-prime. Indeed, the ideal $I = p^4\mathbb{Z}$ of the ring of integers $\mathbb{Z}$ is not a weakly $(2,3)$-prime ideal of $\mathbb{Z}$ as $p^2 \cdot p^2 \in I$ but $p^3, p^2 \notin I$.

Example 1. Let (R,M) be a quasi local ring with $M^k = 0$ for some positive integer k. Then every proper ideal of R is weakly (m,n)-prime for all positive integers m and n such that $m \geq k$. Indeed, let I be a proper ideal of R. Suppose that $a^m b \in I$ and $b \notin I$ for some $a,b \in R$. Then a is non-unit and so $a \in M$. Therefore, $a^m b = 0$ and we are done.

The above general example gives plenty non-trivial examples of weakly (m,n)-prime ideals that are not (m,n)-prime.

Example 2. Consider the ideal $I = \langle \bar{4} \rangle$ of the ring $R = \mathbb{Z}_8$. Then Example 1 shows that I is a weakly $(3,1)$-prime ideal in R. However, I is neither $(3,1)$-prime nor weakly prime. Indeed, $\bar{2}^3 \in I$ and $\bar{0} \neq \bar{2} \cdot \bar{2} \in I$ but $\bar{2} \notin I$.

Note that unlike the (m,n)-prime case, we may find a weakly (m,n)-prime ideal that is not weakly (m',n) for $m' < m$. Indeed, the weakly $(3,1)$-prime ideal I in Example 2 is clearly not weakly $(2,1)$-prime.

Example 3. Consider the idealization ring $R = \mathbb{Z}_8(+)\langle \bar{4} \rangle$ and let $I = \bar{0}(+)\langle \bar{4} \rangle$. Let $(a,m_1),(b,m_2) \in R$ such that $(a^2 b, a^2 m_2 + 2abm_1) = (a,m_1)^2(b,m_2) \in I$ and $(a,m_1),(b,m_2) \notin I$. Then $a \neq \bar{0}$ and $b \neq \bar{0}$ and so clearly, $(a,m_1)^2(b,m_2) = (\bar{0},\bar{0})$. Therefore, I is a weakly $(2,1)$-prime ideal of R. On the other hand, I is not (m,n)-prime in R since for example, $(\bar{2},\bar{0})^2(\bar{2},\bar{0}) = (\bar{0},\bar{0}) \in I$ but, $(\bar{2},\bar{0}) \notin I$.

Remark 1. Let I be a proper ideal of a ring R.

(1) If I is a weakly 1-absorbing prime (resp. weakly prime) ideal of R, then I is a weakly (m,n)-prime ideal for $n \geq 2$ (resp. for all n). Indeed, let $a, b \in R$ with $0 \neq a^m b \in I$ and $b \notin I$. Then a is nonunit. If b is a unit, then $0 \neq a^m = a \cdot a^{m-2} \cdot a \in I$ and since I is 1-absorbing prime, we have $0 \neq a^{m-1} = a \cdot a^{m-2} \in I$ or $a \in I$. Continue this process to get $0 \neq a^2 \in I$ and so $a^n \in I$ for all $n \geq 2$. If I is weakly prime, then $a \in I$ and we are done.

(2) We may find a positive integer n such that I is weakly n-absorbing but not weakly (m,n)-prime in R for every positive integer m. For example, the ideal $I = 0(+)p\mathbb{Z}$ is a weakly 2-absorbing ideal in $\mathbb{Z}(+)\mathbb{Z}$ for any prime integer p, [1, Example 4.11]. But, I is not weakly $(m,2)$-prime for every positive integer m. Indeed, $(0,0) \neq (p,0)^m(0,1) \in I$ but $(p,0)^2, (0,1) \notin I$. Moreover, $I = \langle \bar{8} \rangle$ is a weakly $(m,2)$-prime ideal in $\mathbb{Z}_{16}$ for all $m \geq 4$ by Example 1. But, I is not weakly 2-absorbing since $0 \neq \bar{2} \cdot \bar{2} \cdot \bar{2} \in I$ with $\bar{2} \cdot \bar{2} \notin I$.

(3) For all positive integers m and n, it is proved in [17] that if I is an (m,n)-prime ideal of R, then I is semi n-absorbing in R. However, this is not true in the weakly case. For example, the weakly $(3,1)$-prime ideal of Example 2 is not weakly semi 1-absorbing.

(4) If I is a weakly (m,n)-prime ideal of a reduced ring R, then I is weakly primary in R. Indeed, if $0 \neq ab \in I$ for $a, b \in R$, then $0 \neq a^m b \in I$ since R is reduced. Thus, $a^n \in I$ or $b \in I$ and so $a \in \sqrt{I}$ or $b \in I$. Moreover, if I is weakly primary with $(\sqrt{I})^n \subseteq I$, then I is weakly (m,n)-prime in R for all positive integers m and n. Indeed, if $a, b \in R$ such that $0 \neq a^m b \in I$ and $b \notin I$, then $a \in \sqrt{I}$ and so $a^n \in (\sqrt{I})^n \subseteq I$.

(5) There are some weakly primary ideals that are not weakly (m,n)-prime. Consider the ideal $I = \langle \bar{4} \rangle (+)\mathbb{Z}_8$ in the ring $R = \mathbb{Z}_8(+)\mathbb{Z}_8$. Let $(a,b),(c,d) \in R$ with $(0,0) \neq (a,b)(c,d) \in \langle \bar{4} \rangle (+)\mathbb{Z}_8$ and $(a,b) \notin \langle \bar{4} \rangle (+)\mathbb{Z}_8$. Then $ac \in \langle \bar{4} \rangle$ and $a \notin \langle \bar{4} \rangle$ imply that $c \in \langle \bar{2} \rangle$ as $\langle \bar{4} \rangle$ is primary in R. Hence, $(c,d) \in \sqrt{I} = \langle \bar{2} \rangle (+)\mathbb{Z}_8$ and I is a (weakly) primary ideal of R. However, I is not weakly $(2,1)$-prime as $(0,0) \neq (2,1)^2(1,1) \in I$ but neither $(2,1) \in I$ nor $(1,1) \in I$.

(6) Suppose R is an integral domain and $I = \prod_{\alpha \in \Lambda} M_\alpha^{k_\alpha}$, where $\{M_\alpha : \alpha \in \Lambda\}$ is a family of distinct maximal ideals of R. If I is non-zero, then it is not weakly (m,n)-prime for all positive integers m and n. Indeed, for each $\beta \in \Lambda$, choose $x_\beta \in M_\beta$ such that $x_\beta \notin M_\alpha$ for all $\alpha \neq \beta$. Then clearly, $0 \neq (x_\beta^{k_\beta})^m (\prod_{\alpha \neq \beta} x_\alpha^{k_\alpha}) \in I$ but $(x_\beta^{k_\beta})^n \notin I$ and $\prod_{\alpha \neq \beta} x_\alpha^{k_\alpha} \notin I$.

We illustrate the place of the class of weakly (m,n)-prime ideals for all positive integers m and n by the following diagram:

weakly semiprime	$\longrightarrow$	weakly semi n-absorbing	$\longleftarrow$	weakly n-absorbing
$\uparrow$	$\searrow$	$\uparrow$	$\swarrow$	
weakly prime	$\longrightarrow$	weakly (m,n)-closed		weakly 1-absorbing primary
$\downarrow$	$\searrow$	$\uparrow$		$\uparrow$
weakly 1-absorbing prime	$\longrightarrow$	weakly (m,n)-prime	$\longrightarrow$	weakly primary

Next, for a prime integer p and positive integers m, n and k, we justify weakly$(m.n)$-prime ideals in the ring $\mathbb{Z}_{p^k}$.

Theorem 1. *Let p be a prime integer and m, n, k positive integers. Let $I = \langle p^t \rangle$ be a proper ideal of $R = \mathbb{Z}_{p^k}$, where $1 \leq t \leq k$. Then I is a weakly (m, n)-prime ideal of R if and only if $t = k$ or $m \geq k$ or $n \geq t$.*

Proof. If $t = k$, then $I = 0$ is trivially a weakly (m, n)-prime ideal of R. Suppose $t \neq k$ and $m \geq k$ or $n \geq t$ and let $a, b \in R$ such that $0 \neq a^m b \in I$. If $b \notin I$, then $a^m \in \langle p \rangle$ and so $a \in \langle p \rangle$ as I is primary. If $m \geq k$, then $a^m \in \langle p^m \rangle = 0$, a contradiction. Hence, $b \in I$ and I is weakly (m, n)-prime in R. If $n \geq t$, then $a^n \in \langle p^n \rangle \subseteq \langle p^t \rangle$ and so again I is weakly (m, n)-prime in R. Conversely, suppose I is a weakly (m, n)-prime ideal of R but $t \neq k$, $m < k$ and $n < t$. We have two cases. Case 1: $m \leq t$. In this case, we have $0 \neq p^t = p^m p^{t-m} \in I$ but $p^n \notin I$ and $p^{t-m} \notin I$, a contradiction. Case 2: $m > t$. In this case, we have $0 \neq p^m \in I$ but $p^n \notin I$, a contradiction.

Therefore, we must have either $t = k$ or $m \geq k$ or $n \geq t$. □

By using Theorem 1 and [17, Theorem 3], we characterize weakly (m, n)-prime ideals of $\mathbb{Z}_{p^k}$ that is not (m, n)-prime.

Corollary 1. *Let p be a prime integer and m, n, k be positive integers. Let $I = \langle p^t \rangle$ be a proper ideal of $R = \mathbb{Z}_{p^k}$, where $1 \leq t \leq k$. Then I is a weakly (m, n)-prime ideal of R that is not (m, n)-prime if and only if $n < t$ and ($t = k$ or $m \geq k$).*

Theorem 2. *Let R be a ring such that every power of a prime ideal is primary. Let m, n and t be positive integers and $I = \langle p^t \rangle$, where p is a non-nilpotent prime element of R. The following are equivalent.*

(1) *I is an (m, n)-prime ideal of R.*
(2) *I is a weakly (m, n)-prime ideal of R.*
(3) *$n \geq t$.*

Proof. (1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) Suppose I is weakly (m, n)-prime in R and $n < t$. If $m \leq t$, then $0 \neq p^t = p^m p^{t-m} \in I$ but $p^n \notin I$ and $p^{t-m} \notin I$, a contradiction. Otherwise, if $m > t$, then $0 \neq p^m \in I$ but $p^n \notin I$ which is also a contradiction. Thus, $n \geq t$ as needed.

(3) $\Rightarrow$ (1) [17, Theorem 3]. □

Theorem 3. *Let m, n be positive integers and I be a proper ideal of a ring R. Then the following are equivalent.*

(1) *I is a weakly (m, n)-prime ideal of R.*
(2) *$(I : a^m) \subseteq I \cup (0 : a^m)$ for all $a \in R$ such that $a^n \notin I$.*
(3) *$(I : a^m) = I$ or $(I : a^m) = (0 : a^m)$ for all $a \in R$ such that $a^n \notin I$.*
(4) *Whenever $a \in R$ and J is an ideal of R with $0 \neq a^m J \subseteq I$, then $a^n \in I$ or $J \subseteq I$.*

Proof. (1)⇒(2) Let $a \in R$ such that $a^n \notin I$ and let $b \in (I : a^m)$. If $a^m b = 0$, then $b \in (0 : a^m)$. Suppose that $a^m b \neq 0$. Since I is weakly (m, n)-prime, we have $b \in I$. Thus, we get the required inclusion.

(2)⇒(3) Clear.

(3)⇒(4) Let $a \in R$ and J be an ideal of R with $0 \neq a^m J \subseteq I$ and suppose $a^n \notin I$. Then $J \subseteq (I : a^m) \backslash (0 : a^m)$ and by our hypothesis, we have $J \subseteq (I : a^m) = I$.

(4)⇒(1) Suppose that $0 \neq a^m b \in I$ for some $a, b \in R$ and put $J = bR$. Then $0 \neq a^m J \subseteq I$ and by (4), we conclude that $a^n \in I$ or $b \in J \subseteq I$. Thus, I is a weakly (m, n)-prime ideal of R. □

For principal ideal rings, we have further characterizations for weakly $(m; n)$-prime ideals.

Corollary 2. *Let m, n be positive integers, R be a principal ideal ring and I be a proper ideal of R. Then the following are equivalent.*

(1) *I is a weakly (m, n)-prime ideal of R.*
(2) *$(I : a^m) \subseteq I \cup (0 : a^m)$ for all $a \in R$ such that $a^n \notin I$.*
(3) *$(I : a^m) = I$ or $(I : a^m) = (0 : a^m)$ for all $a \in R$ such that $a^n \notin I$.*
(4) *If $a \in R$ and J is an ideal of R with $0 \neq a^m J \subseteq I$, then $a^n \in I$ or $J \subseteq I$.*
(5) *If J and K are ideals of R with $0 \neq J^m K \subseteq I$, then $J^n \subseteq I$ or $K \subseteq I$.*
(6) *$(I : J^m) \subseteq I \cup (0 : J^m)$ for any ideal J of R such that nth power of which is not contained in I*
(7) *$(I : J^m) = I$ or $(I : J^m) = (0 : J^m)$ for any ideal J of R such that nth power of which is not contained in I*
(8) *If J is an ideal of R and $b \in R$ with $0 \neq J^m b \subseteq I$, then $J^n \subseteq I$ or $b \in I$.*

Proof. (1)⇒(2)⇒(3)⇒(4). Theorem 3.

(4)⇒(5) Since R is a principal ideal ring, we may put $J := \langle a \rangle$ for some $a \in R$ in (4).

(5)⇒(6) Let $b \in (I : J^m)$, where J is an ideal of R such that nth power of which is not contained in I. Then $J^m b$ is not contained in I as well. Put $K = \langle b \rangle$. Then, $J^m K \subseteq I$. If $J^m K = 0$, then $b \in K \subseteq (0 : J^m)$. Assume that $J^m K \neq 0$. Then (5) yields $K \subseteq I$. Thus, $(I : J^m) \subseteq I \cup (0 : J^m)$.

(6)⇒(7) Clear.

(7)⇒(8) Assume that $0 \neq J^m b \subseteq I$ and J^n is not contained in I. Then $(I : J^m) \neq (0 : J^m)$ and from (7), we conclude $b \in (I : J^m) = I$.

(8)⇒(1) Suppose that $0 \neq a^m b \in I$ and $a^n \notin I$. Put $J = \langle a \rangle$. Then $0 \neq J^m b$ is not contained in I and J^n is not contained I imply by (6) that $b \in I$ and we are done. □

Recall from [16] that an ideal of a ring is said to be quasi primary if its radical is prime.

Theorem 4. *Let R be a ring and I be a proper ideal of R. If I is a weakly (m,n)-prime ideal of R and the zero ideal of R is quasi primary, then I is quasi primary in R. Moreover, $a^n \in I$ for all $a \in \sqrt{I}\backslash\sqrt{0}$.*

Proof. Suppose that $ab \in \sqrt{I}$. Then $a^k b^k \in I$ for some positive integer k and so $a^{mk}b^k \in I$. If $a^{mk}b^k = 0$, then $\{0\}$ is quasi primary implies $a \in \sqrt{0} \subseteq \sqrt{I}$ or $b \in \sqrt{0} \subseteq \sqrt{I}$. Assume that $a^{mk}b^k \neq 0$. Then as I is weakly (m,n)-prime, we have $a^{nk} \in I$ or $b^k \in I$ and so again $a \in \sqrt{I}$ or $b \in \sqrt{I}$. Now, let $a \in \sqrt{I}\backslash\sqrt{0}$ and let t be the least positive integer such that $a^t \in I$. Since a is non-nilpotent, we have $0 \neq a^m a^{t-1} \in I$ and since $a^{t-1} \notin I$, we have $a^n \in I$. □

In general, if I is a quasi primary ideal in a ring R, then I need not be weakly (m,n)-prime in R. For example, consider the ring $R = \mathbb{Z}_2[\{X_n\}]_{n=1}^{\infty}$ and the ideal $I = \langle\{X_n^n\}_{n=1}^{\infty}\rangle$ of R. Then $\sqrt{I} = \langle\{X_n\}_{n=1}^{\infty}\rangle$ is a prime ideal of R, but I is not weakly (m,n)-prime, where $2n > m$. Indeed, $X_{2n}^m \cdot X_{2n}^{2n-m} \in I$ but neither $X_{2n}^n \in I$ nor $X_{2n}^{2n-m} \in I$.

We justified in Remark 1 that if I is a (weakly) primary ideal of a ring R such that $(\sqrt{I})^n \subseteq I$, then I is weakly (m,n)-prime in R for all positive integers m and n. However, even if $(\sqrt{I})^n \subseteq I$, I can be a quasi primary ideal that is not weakly (m,n)-prime in R. For example, consider the ring $R = \mathbb{Z}+pX\mathbb{Z}[X]$, where p is a prime integer and the ideal $P = pX\mathbb{Z}[X]$ of R. Then P is a prime ideal of R and so, $I = P^n$ is a quasi primary ideal in R for $n \leq m$. However, I is not weakly (m,n)-prime as p^m, $(pX^m) \in R$ with $0 \neq p^m(pX^m) \in I$ but neither $p^n \in I$ nor $pX^m \in I$.

A ring R is said to be a UN-ring if every non-unit element of R is a product a unit and a nilpotent element, [13]. It is verified in [13, Proposition 2(3)] that R is a UN-ring if and only if R has a unique prime ideal which is $\sqrt{0}$.

Corollary 3. *Let R be a UN-ring. If I is a weakly (m,n)-prime ideal of R, then $\sqrt{I}$ is a maximal ideal of R.*

Proof. Suppose I is a weakly (m,n)-prime ideal of R. Since $\sqrt{0}$ is the unique prime in R and $\sqrt{I}$ is also prime by Theorem 4, it follows that $\sqrt{I} = \sqrt{0}$ is the unique maximal ideal of R. □

Theorem 5. *Let m, n be positive integers, where $m \geq n$. If R is a ring in which every proper ideal is weakly (m,n)-prime, then* $\dim(R) = 0$.

Proof. Assume on the contrary that $\dim(R) \geq 1$ and let $P \subset Q$ be two prime ideals of R. Let $a \in Q\backslash P$ and $I = \langle a^{m+1}\rangle$. Then $0 \neq a^m a \in I$ and our assumption implies that $a^n \in I$ or $a \in I$. Hence $a^n = a^{m+1}r$ for some $r \in R$ and this implies that $a^n(1 - a^{m-n+1}r) = 0 \in P$. Since P is prime and $a \notin P$, we conclude that $1 - a^{m-n+1}r \in P \subset Q$. Thus, we have $1 \in Q$, a contradiction. Therefore, $\dim(R) = 0$. □

However, the converses of Corollary 3 and Theorem 5 do not hold in general. Let $k > m > n$ be positive integers. Then the ideal $I = \langle\overline{p^m}\rangle$ of the zero dimensional UN-ring $R = \mathbb{Z}_{p^k}$ is not weakly (m,n)-prime by Theorem 1. Note that $\sqrt{I} = \langle p \rangle$ is the unique maximal ideal of R.

Proposition 1. *Let R be a ring, $a, b \in J(R)$ and m, n be positive integers. Then $I = \langle a^n b \rangle$ is a weakly (m,n)-prime ideal of R if and only if $a^n b = 0$.*

Proof. Suppose $I = \langle a^n b \rangle$ is weakly (m,n)-prime in R but $a^n b \neq 0$. We have two cases. Case I: If $m \geq n$, then $a^m b \in I$ and so $a^n \in I$ or $b \in I$ as I is weakly (m,n)-prime. If $a^n \in I$, then there exists some $r \in R$ such that $a^n = a^n br$, and so $a^n(1 - br) = 0$. Therefore, $1 - br \in U(R)$ as $b \in J(R)$ and so $a^n = 0$, a contradiction. If $b \in I$, then $b = a^n br'$ for some $r' \in R$ and hence $b(1 - a^n r') = 0$. Thus, $(1 - a^n r') \in U(R)$ as $a \in J(R)$ and so $b = 0$, a contradiction. Case II: If $m < n$, then $a^m a^{n-m} b = a^n b \in I$ implies $a^n \in I$ or $a^{n-m} b \in I$. If $a^n \in I$, then similar to the above argument, we get a contradiction. If $a^{n-m} b \in I$, then $a^{n-m} b(1 - sa^m) = 0$ for some $s \in R$. Hence, $a \in J(R)$ implies $1 - sa^m \in U(R)$ and so $a^{n-m} b = 0$, a contradiction. Therefore, $a^n b = 0$. The converse part is immediate since the zero ideal is always weakly (m,n)-prime. □

Let I be a proper ideal of a ring R. Then the ideal $\langle a^n : a \in I \rangle$ of R generated by nth powers of elements of I is denoted by I_n. Note that $I_n \subseteq I^n \subseteq I$ and the equality holds when $n = 1$. Moreover, it is verified that if $n!$ is a unit of R, then $I_n = I^n$ [4]. In view of Proposition 1, we have the following corollary.

Corollary 4. *Let m, n be positive integers. If R is a ring in which all proper ideals are weakly (m,n)-prime, then $J(R)_n J(R) = 0$.*

Following [20], a non-zero ideal I of a ring R is called secondary if for each $a \in R$, either $aI = I$ or $a^k I = 0$ for some positive integer k. In this case, $P = \sqrt{(0 :_R I)}$ is clearly a prime ideal of R. More general, we have the following definition.

Definition 2. Let I be a non-zero ideal of a ring R and let m, n be positive integers. Then I is called (m,n)-secondary if for each $a \in R$, n is the smallest positive integer such that either $a^m I = I$ or $a^n I = 0$.

The following result is an analogues to [21, Theorem 2.8].

Proposition 2. *Let I and J be ideals of a ring R and m, n be positive integers. If I is (m,n)-secondary and J is weakly (m,n)-prime in R, then $I \cap J$ is (m,n)-secondary.*

Proof. Let $a \in R$. If $a^n I = 0$, then $a^n(I \cap J) = 0$. Suppose $a^n I \neq 0$. Then $a^m I = I$ as I is (m,n)-secondary. We prove that $a^m(I \cap J) = I \cap J$. Let $0 \neq x \in I \cap J$. Then $x = a^m b \in J$ for some $b \in I$. By assumption, either $a^n \in J$ or $b \in J$. If $b \in J$, then $x = a^m b \in a^m(I \cap J)$ and $a^m(I \cap J) = I \cap J$. Suppose $a^n \in J$. If $n > m$, then $a^{n-m} I = a^{n-m}(a^m I) = a^n I = 0$ which is a contradiction. If $n \leq m$, then $I = a^m I \subseteq a^n I \subseteq J$ and so $a^m(I \cap J) = a^m I = I = I \cap J$. It follows that $I \cap J$ is (m,n)-secondary in R. □

An ideal I of a ring R is said to be divided if $I \subseteq \langle x \rangle$ for every $x \in R \backslash I$. Next, we determine a condition under which a weakly (m,n)-prime ideal in a ring is weakly primary.

Proposition 3. *Let I be a weakly (m,n)-prime ideal of a ring R. If $\sqrt{I}$ is a divided weakly prime ideal of R, then I is weakly primary in R.*

Proof. Let $0 \neq ab \in I \subseteq \sqrt{I}$ and $b \notin \sqrt{I}$ for $a, b \in R$. Then $a \in \sqrt{I}$ as $\sqrt{I}$ is weakly prime. Note that $b^{m-1} \notin \sqrt{I}$. Since $\sqrt{I}$ is divided, then $\sqrt{I} \subseteq \langle b^{m-1} \rangle$ and so $a = b^{m-1}r$ for some $r \in R$. Now, $0 \neq b^m r = ba \in I$ and $b^n \notin I$ imply $r \in I$ as I is weakly (m,n)-prime. Thus, $a = b^{m-1}r \in I$ as needed. □

Definition 3. Let I be a weakly (m,n)-prime ideal of a ring R and $a, b \in R$. Then (a,b) is said to be an (m,n)-zero of I provided that $a^m b = 0$ and $a^n, b \notin I$.

It is clear that a weakly (m,n)-prime ideal I of R is not (m,n)-prime if and only if I has an (m,n)-zero.

Lemma 1. *Let m and n be positive integers and I be a weakly (m,n)-prime ideal of R. If (a,b) is an (m,n)-zero of I, then*

(1) *$(a+x)^m b = 0$ for every $x \in I$. In particular, if $char(R) = m$ is prime, then $x^m b = 0$ for every $x \in I$.*
(2) *$a^m(b+x) = 0$ for every $x \in I$.*
(3) *$a^m I = 0$.*

Proof. (1) Suppose (a,b) is an (m,n)-zero of I. Assume on the contrary that $(a+x)^m b \neq 0$ for some $x \in I$. Then

$$0 \neq (a+x)^m b = \underbrace{a^m b}_{0} + \sum_{k=1}^{m} \begin{pmatrix} m \\ k \end{pmatrix} a^{m-k} x^k b \in I$$

and $b \notin I$ imply that $(a+x)^n \in I$. Also, since (a,b) is an (m,n)-zero of I, $a^n \notin I$ and so we get $(a+x)^n \notin I$, a contradiction. Therefore, $(a+x)^m b = 0$ for every $x \in I$. The "in particular" statement is clear since whenever $char(R) = m$ is prime, $0 = (a+x)^m b = a^m b + x^m b = x^m b$ for every $x \in I$.

(2) Assume that $a^m(b+x) \neq 0$ for some $x \in I$. Then

$$0 \neq a^m(b+x) = \underbrace{a^m b}_{0} + a^m x \in I$$

and since $a^n \notin I$, we have $(b+x) \in I$. Hence, we get $b \in I$, a contradiction. Thus, $a^m(b+x) = 0$.

(3) Suppose that $a^m x \neq 0$ for some $x \in I$. From (2), we have

$$a^m(b+x) = \underbrace{a^m b}_{0} + \underbrace{a^m x}_{\neq 0} = 0,$$

a contradiction. Thus, $a^m I = 0$. □

Proposition 4. *Let m and n be positive integers, I be a weakly (m,n)-prime ideal of a ring R and (a,b) be an (m,n)-zero of I. Then aI, $bI \subseteq Nil(R)$.*

Proof. By Lemma 1(3), $a^m I = 0$, and thus, $aI \subseteq Nil(R)$. Now, let $x \in I$. By Lemma 1(1), we have $(a+x)b \in Nil(R)$ and note that $ab \in Nil(R)$ as $a^m b = 0$. Thus, $bx = (a+x)b - ab \in Nil(R)$ and so $bI \subseteq Nil(R)$. □

Corollary 5. *Let m and n be positive integers and I be a weakly (m,n)-prime ideal of a ring R that is not (m,n)-prime. Then $aI, bI \subseteq Nil(R)$ for some $a \notin \sqrt[n]{I}$ and $b \notin I$.*

Proposition 5. *Let m and n be positive integers and I be an ideal of a ring R. Then I is a weakly (m,n)-prime ideal of R if and only if $(I : x)$ is a weakly (m,n)-prime ideal in R for all $x \in reg(R)\backslash I$.*

Proof. Note that for $x \in reg(R)\backslash I$, $(I : x)$ is proper in R. Let $a, b \in R$ and $x \in reg(R)\backslash I$ such that $0 \neq a^m b \in (I : x)$. Since x is regular, we conclude $0 \neq a^m bx \in I$ which implies either $a^n \in I$ or $bx \in I$. Thus, $a^n \in (I : x)$ or $b \in (I : x)$ as needed. The converse part follows directly since $1 \in reg(R)\backslash I$. □

If $(I : x)$ is a weakly (m,n)-prime ideal in a ring R for some $x \in reg(R) \setminus I$, then I may not be a weakly (m,n)-prime ideal of a ring R. For example, the ideal $I = 0(+)\langle 2\rangle$ is not a weakly $(1,2)$-prime ideal of the ring $R = \mathbb{Z}(+)\mathbb{Z}$ since $(0,0) \neq (2,0)(0,1) \in I$ but $(2,0)^2, (0,1) \notin I$. However, for $x = (2,0) \in reg(R)\backslash I$, we have $(I : x) = 0(+)\mathbb{Z}$ is clearly weakly $(1,2)$-prime in R.

Proposition 6. *Let m and n be positive integers and $\{I_\alpha\}_{\alpha\in\Lambda}$ be a family of weakly (m,n)-prime ideals of a ring R, where $\sqrt[n]{I_\alpha} = \sqrt[n]{I_\beta}$ for all $\alpha, \beta \in \Lambda$. Then $\bigcap_{\alpha\in\Lambda} I_\alpha$ is a weakly (m,n)-prime ideal of R.*

Proof. Let $0 \neq a^m b \in \bigcap_{\alpha\in\Lambda} I_\alpha$ and $b \notin \bigcap_{\alpha\in\Lambda} I_\alpha$ for $a, b \in R$. Then $b \notin I_\beta$ for some $\beta \in \Lambda$. Since $0 \neq a^m b \in I_\beta$, then by assumption, $a^n \in I_\beta$ and so $a \in \sqrt[n]{I_\beta}$. Thus, $a \in \sqrt[n]{I_\alpha}$ for all $\alpha \in \Lambda$ and $a^n \in \bigcap_{\alpha\in\Lambda} I_\alpha$. Thus, $\bigcap_{\alpha\in\Lambda} I_\alpha$ is a weakly (m,n)-prime ideal of R. □

In general, if I and J are two weakly (m,n)-prime ideals with distinct n^{th}-radicals, then $I \cap J$ need not be weakly (m,n)-prime. For example, the ideals $\langle\bar{2}\rangle$ and $\langle\bar{3}\rangle$ are weakly (m,n)-prime ideals of $\mathbb{Z}_{12}$ for all positive integers n and m, but $\langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$ is not so.

Next, we discuss the behavior of weakly (m,n)-prime ideals under ring homomorphisms and localizations.

Proposition 7. *Let $f : R_1 \to R_2$ be a ring homomorphism and m, n be positive integers.*

(1) *If f is a monomorphism and J is a weakly (m,n)-prime ideal of R_2, then $f^{-1}(J)$ is a weakly (m,n)-prime ideal of R_1.*
(2) *If f is an epimorphism and I is a weakly (m,n)-prime ideal of R_1 containing $Ker(f)$, then $f(I)$ is a weakly (m,n)-prime ideal of R_2.*

Proof. (1) Let $a, b \in R_1$ such that $0 \neq a^m b \in f^{-1}(J)$ and $b \notin f^{-1}(J)$. Since $Ker(f) = 0$, we have $0 \neq f(a^m b) = f(a)^m f(b) \in J$ and $f(b) \notin J$ which imply $f(a)^n = f(a^n) \in J$. Hence $a^n \in f^{-1}(J)$, as required.

(2) Let $a := f(a_1)$, $b := f(b_1) \in R_2$ such that $0 \neq a^m b \in f(I)$ and $b \notin f(I)$. Then $0 \neq f(a_1^m b_1) \in f(I)$ and since $Ker(f) \subseteq I$, we conclude $0 \neq a_1^m b_1 \in I$. Since I is weakly (m,n)-prime, then $a_1^n \in I$ or $b_1 \in I$. Therefore, $a^n = f(a_1^n) \in f(I)$ or $b = f(b_1) \in f(I)$. □

As a consequence of the previous proposition, we have the following corollary.

Corollary 6. *Let I and J be proper ideals of a ring R, m, n be positive integers and X be an indeterminate.*

(1) *If I is a weakly (m,n)-prime ideal of an overring R' of R, then $I \cap R$ is a weakly (m,n)-prime ideal of R.*
(2) *If $I \subseteq J$ and J is a weakly (m,n)-prime ideal of R, then J/I is a weakly (m,n)-prime ideal of R/I.*
(3) *If $I \subseteq J$, J/I is a weakly (m,n)-closed ideal of R/I and I is an (m,n)-prime ideal of R, then J is a weakly (m,n)-prime ideal of R. In particular, in a ring in which the zero ideal is (m,n)-prime, every weakly (m,n)-prime ideal is (m,n)-prime.*
(4) *I is weakly (m,n)-prime in R if and only if $\langle I, X\rangle$ is weakly (m,n)-prime in $R[X]$.*

Proof. (1) and (2) follow clearly by Proposition 7.

(3) Suppose that $0 \neq a^m b \in J$ for some $a, b \in R$. If $a^m b \in I$, then as I is an (m,n)-prime ideal, we have $a^n \in I \subseteq J$ or $b \in I \subseteq J$. Now, assume that $a^m b \notin I$. Then $0 + I \neq (a+I)^m(b+I) \in J/I$ implies $(a+I)^n \in J/I$ or $b + I \in J/I$ as J/I is a weakly (m,n)-prime ideal. Thus, we have either $a^n \in J$ or $b \in J$ as needed.

(4) Since $R[X]/\langle X\rangle \cong R$ and $\langle I, X\rangle/\langle X\rangle \cong I$, the claim follows by (2) of Proposition 7. □

A non-empty subset S of a ring R is said to be a multiplicatively subset if $1 \in S$, and for each $a, b \in S$ we have $ab \in S$. In the following, $Z_I(R)$, where I is an ideal of R, denotes the set $\{x \in R : xy \in I \text{ for some } y \in R \backslash I\}$.

Proposition 8. *Let m, n be positive integers, I be a proper ideal of a ring R and S a multiplicatively closed subset of R such that $I \cap S = \emptyset$.*

(1) *If I is a weakly (m,n)-prime ideal of R, then $S^{-1}I$ is a weakly (m,n)-prime ideal of $S^{-1}R$.*
(2) *If $S \subseteq reg(R)$ and $S^{-1}I$ is a weakly (m,n)-prime ideal of $S^{-1}R$ with $S \cap Z_I(R) = \emptyset$, then I is a weakly (m,n)-prime ideal of R.*

Proof. (1) Let $0 \neq \left(\frac{a}{s_1}\right)^m \left(\frac{b}{s_2}\right) \in S^{-1}I$ for $\frac{a}{s_1}, \frac{b}{s_2} \in S^{-1}R$. Then $0 \neq (ua)^m b \in I$ for some $u \in S$ which implies either $(ua)^n \in I$ or $b \in I$. Hence, either $\left(\frac{a}{s_1}\right)^n = \frac{u^n a^n}{u^n s_1^n} \in S^{-1}I$ or $\frac{b}{s_2} \in S^{-1}I$.

(2) Let $a, b \in R$ with $0 \neq a^m b \in I$. Then $\frac{a^m b}{1} = \left(\frac{a}{1}\right)^m \left(\frac{b}{1}\right) \in S^{-1}I$. If $\frac{a^m b}{1} = 0$, then $ua^m b = 0$ for some $u \in S \cap Z(R)$, a contradiction. Thus, $\frac{a^m b}{1}$

is nonzero. This implies either $\left(\frac{a}{1}\right)^n \in S^{-1}I$ or $\left(\frac{b}{1}\right) \in S^{-1}I$. Thus, there are some elements $v, w \in S$ such that $va^n \in I$ or $wb \in I$. Since $S \cap Z_I(R) = \emptyset$, we conclude $a^n \in I$ or $b \in I$. Thus, I is a weakly (m, n)-prime ideal of R. □

Let S be a multiplicatively closed subset of a ring. Now, we give a characterization for a ring which has only one weakly (m, n)-prime ideal disjoint with S.

Proposition 9. *Let R be a ring and S a multiplicatively closed subset of R. Then the following statements are equivalent.*

(1) *The zero ideal is the only weakly (m, n)-prime ideal of R disjoint with S.*
(2) *The zero ideal is the only (m, n)-prime ideal of R disjoint with S.*
(3) *R is a domain and $S^{-1}R$ is a field.*

Proof. (1)⇒(2) It is straightforward.

(2)⇒(3) It is well-known by [19, Proposition 2.12] that there exits a prime ideal I of R such that $I \cap S = \emptyset$. Hence, I is (m, n)-prime and so $I = \{0\}$. Thus, R is a domain. Now, let $\frac{0}{1} \neq \frac{a}{s} \in S^{-1}\dot{R}$. We show that $\frac{a}{s} \in U(S^{-1}\dot{R})$. If $a \in S$, then we are done. Assume that $a \notin S$. If $\langle a \rangle \cap S = \emptyset$, then there exists a prime (so, an (m, n)-prime) ideal J of R including $\langle a \rangle$. But, our assumption yields that $J = \{0\}$, a contradiction. Thus, we have $\langle a \rangle \cap S \neq \emptyset$ and we may choose $r \in \langle a \rangle \cap S$. Choose $r' \in R$ such that $r = ar'$ and put $s' = sr'$. Then $\frac{a}{s}\frac{s'}{r} = \frac{1}{1}$ and $\frac{a}{s} \in U(S^{-1}\dot{R})$. Therefore, $S^{-1}R$ is a field.

(3)⇒(1) Assume that I is a nonzero weakly (m, n)-prime ideal of R disjoint with S and let $0 \neq a \in I$. Then $\frac{a}{1} \neq \frac{0}{1}$ as R is a domain. Since $S^{-1}R$ is a field, there exists $0 \neq b \in R$ and $s \in S$ such that $\frac{a}{1}\frac{b}{s} = \frac{1}{1}$. Hence, there is some $u \in S$ with $uab = us$ and so $u(ab - s) = 0$. Since R is a domain, we have $ab = s \in I \cap S$, a contradiction. Thus, the zero ideal is the only weakly (m, n)-prime ideal of R. □

Next, we characterize weakly (m, n)-prime ideals in Cartesian product of rings.

Theorem 6. *Let R_1 and R_2 be rings, $R = R_1 \times R_2$ and m, n be positive integers. A proper ideal I of R is weakly (m, n)-prime if and only if it has one of the following forms*:

(1) *$I = 0$.*
(2) *$I = J \times R_2$, where J is an (m, n)-prime ideal of R_1.*
(3) *$I = R_1 \times K$, where K is an (m, n)-prime ideal of R_2.*

Proof. Let $I = J \times K$ be a nonzero weakly (m, n)-prime ideal of R, where J and K are ideals of R_1 and R_2, respectively. Assume on contrary that both J and K are proper. Without loss of generality, assume that $J \neq \{0\}$ so there exists a nonzero element a in J. Then, $(0, 0) \neq (1, 0)^m(a, 1) \in J \times K$ which implies either $(1, 0)^n \in J \times K$ or $(a, 1) \in J \times \dot{K}$. Thus, $J = R_1$ or $K = R_2$

which is a contradiction. Since I is proper, we may assume that J is proper and $K = R_2$. Let $a, b \in R_1$ and $a^m b \in J$. Then $(0,0) \neq (a,1)^m(b,1) \in J \times R_2$ and it yields either $(a,1)^n \in J \times R_2$ or $(b,1) \in J \times R_2$. Therefore, we have $a^n \in J$ or $b \in J$, and J is an (m,n)-prime ideal of R_1. Similar to the argument used above, if K is proper in R_2 and $J = R_1$, then K is an (m,n)-prime ideal of R_2. Conversely, if $I = 0$, then I is trivially weakly (m,n)-prime. Suppose that $I = J \times R_2$, where J is an (m,n)-prime ideal of R_1 or $I = R_1 \times K$, where K is an (m,n)-prime ideal of R_2. Then the claim follows from [17, Theorem 5]. □

By [17, Corollary 11], we have the following corollary.

Corollary 7. *Let R_1 and R_2 be rings, $R = R_1 \times R_2$ and m, n be positive integers. Then a proper nonzero ideal I of R is weakly (m,n)-prime if and only if it is (m,n)-prime.*

Note that if I and J are weakly (m,n)-prime ideals of R_1 and R_2, respectively, where $I \neq 0$ or $J \neq 0$, then I and J are proper. Thus, $I \times J$ is never weakly (m,n)-prime ideal in $R_1 \times R_2$. In a general manner, we have the following characterization.

Theorem 7. *Let $R_1, R_2, \ldots, R_k$ be rings, $R = R_1 \times R_2 \times \cdots \times R_k$, I be a proper nonzero ideal of R and m and n be positive integers. Then the following statements are equivalent.*

(1) *I is a weakly (m,n)-prime ideal of R.*
(2) *$I = R_1 \times \cdots \times I_j \times \cdots \times R_k$, where I_j is an (m,n)-prime ideal of R_j for some $j \in \{1, 2, \ldots, k\}$.*
(3) *I is an (m,n)-prime ideal of R.*

Proof. (1)⇒(2) Suppose $I = I_1 \times I_2 \times \cdots \times I_k$ is a weakly (m,n)-prime ideal of R. We use the mathematical induction on k. The claim is true for $k = 2$ by Theorem 6. Suppose that the claim is true for $k-1$ and we show that it also holds for k. Put $J = I_1 \times I_2 \times \cdots \times I_{k-1}$. Then $I = J \times I_k$. By Theorem 6, we have either $J = R_1 \times R_2 \times \cdots \times R_{k-1}$ and I_k is an (m,n)-prime ideal of R_k or J is an (m,n)-prime ideal of R_k and $I_k = R_k$. If the former case holds, then $I_j = R_j$ for all $j = 1, \ldots, k-1$ and I_k is an (m,n)-prime ideal of R_k. In the latter case, we conclude from our induction hypothesis that $J = R_1 \times \cdots \times I_j \times \cdots \times R_{k-1}$, where I_j is an (m,n)-prime ideal of R_j and $I_k = R_k$. Thus $I = R_1 \times \cdots \times I_j \times \cdots \times R_{k-1} \times R_k$, where I_j is an (m,n)-prime ideal of R_j, we are done.

(2)⇒(3) [17, Theorem 5].

(3)⇒(1) Clear. □

We end this section by the following corollary.

Corollary 8. *Let R_1, $R_2, \ldots, R_k$ be rings, $R = R_1 \times R_2 \times \cdots \times R_k$ and m, n be positive integers. Then the following statements are equivalent.*

(1) *Every proper ideal of R is a weakly (m,n)-prime ideal.*
(2) *$k=2$ and R_i's are fields.*

Proof. (1)⇒(2) Assume that $k \geq 3$. Let $I = \{0\} \times \{0\} \times R_3 \times \cdots \times R_k$ and $0 \neq a \in R_3$. Then $0 \neq (1,0,1,\ldots,1)^m(0,1,a,\ldots,1,1) \in I$ and since I is weakly (m,n)-prime, then $(1,0,1,\ldots,1)^n \in I$ or $(0,1,a,\ldots,1,1) \in I$, a contradiction. Thus, $k=2$ and $R = R_1 \times R_2$. Now, we show that R_1 and R_2 are fields. If, say, R_1 is not a field, then there is a proper nonzero ideal I_1 of R_1. Then, $I = I_1 \times \{0\}$ is a weakly (m,n)-prime ideal of R which contradicts (2) of Theorem 6. Therefore, R_1 is a field. By a symmetric way, R_2 is a field.

(2)⇒(1) Let $R = R_1 \times R_2$, where R_1 and R_2 are fields. Then, the proper ideals of R are $R_1 \times \{0\}$, $\{0\} \times \{0\}$, $\{0\} \times R_2$ and all of them are weakly (m,n)-prime ideal by Theorem 6. □

3. Weakly (m,n)-prime ideals in extensions of rings

Let R be a ring, M be an R-module and consider the idealization ring $R(+)M$. For positive integers m and n, we start this section by justifying some relations between weakly (m,n)-prime ideals of R and weakly (m,n)-prime ideals of $R(+)M$.

Proposition 10. *Let I be a proper ideal of a ring R, N be a submodule of an R-module M and m,n be positive integers.*

(1) *If $I(+)N$ is a weakly (m,n)-prime ideal of $R(+)M$, then I is a weakly (m,n)-prime ideal of R.*
(2) *If I is a weakly (m,n)-prime ideal of R such that $a \in ann(M)$ for any (m,n)-zero (a,b) of I, then $I(+)M$ is a weakly (m,n)-prime ideal of $R(+)M$.*

Proof. (1) Let $a,b \in R$ with $0 \neq a^m b \in I$. Then $0 \neq (a,0)^m(b,0) \in I(+)M$ and this yields either $(a,0)^n \in I(+)M$ or $(b,0) \in I(+)M$. Thus, $a^n \in I$ or $b \in I$ and I is a weakly (m,n)-prime ideal of R.

(2) Let $(a_1,b_1),(a_2,b_2) \in R(+)M$ such that $(0,0) \neq (a_1,b_1)^m(a_2,b_2) = (a_1^m a_2,\ a_1^m b_2 + m a_1^{m-1} a_2 b_1) \in I(+)M$. Then $a_1^m a_2 \in I$. If $a_1^m a_2 \neq 0$, then $a_1^n \in I$ or $a_2 \in I$ and hence, $(a_1,b_1)^n \in I(+)M$ or $(a_2,b_2) \in I(+)M$, we are done. Assume that $a_1^m a_2 = 0$ and neither $a_1^n \in I$ nor $a_2 \in I$. Then (a_1,a_2) is an (m,n)-zero of I and our assumption implies that $a_1 \in ann(M)$. Thus, $a_1^m b_2 + m a_1^{m-1} a_2 b_1 = 0$ and we get $(a_1,b_1)^m(a_2,b_2) = (0,0)$, a contradiction. Therefore, $I(+)M$ is a weakly (m,n)-prime ideal of $R(+)M$. □

Remark 2. The condition "$a \in ann(M)$ for any (m,n)-zero element (a,b) of I" in (2) of Proposition 10 can not be discarded. For example, consider the ideal $\langle \bar{4} \rangle (+)\mathbb{Z}_8$ of the idealization ring $\mathbb{Z}_8(+)\mathbb{Z}_8$. Now, $\langle \bar{4} \rangle$ is a weakly $(3,1)$-prime ideal of $\mathbb{Z}_8$ (Example 2). However, $\langle \bar{4} \rangle (+)\mathbb{Z}_8$ is not a weakly $(3,1)$-prime ideal of $\mathbb{Z}_8(+)\mathbb{Z}_8$ as $(\bar{0},\bar{0}) \neq (\bar{2},\bar{1})^3 = (\bar{0},\bar{4}) \in \langle \bar{4} \rangle (+)\mathbb{Z}_8$ but $(\bar{2},\bar{1}) \notin \langle \bar{4} \rangle (+)\mathbb{Z}_8$. Note that $(\bar{2},\bar{2})$ is clearly a $(3,1)$-zero of $\langle \bar{4} \rangle$ but $\bar{2} \notin ann(\mathbb{Z}_8)$.

For rings R and R', let $f : R \to R'$ be a ring homomorphism and J be an ideal of R'. The amalgamation of R and R' along J with respect to f is the subring $R \ltimes^f J = \{(a, f(a) + j) : a \in R,\, j \in J\}$ of $R \times R'$. The amalgamated duplication of a ring R along an ideal J is $R \ltimes J = R \ltimes^{Id_R} J = \{(r, r + j) : r \in R,\, j \in J\}$ corresponds to the identity homomorphism $Id_R : R \to R$. For further details and many properties of this ring, we refer the reader to [14] and [15]. For an ideal I of R and an ideal K of $f(R) + J$, two corresponding ideals of $R \ltimes^f J$ can be defined, [14]: $I \ltimes^f J = \{(i, f(i) + j) : i \in I,\, j \in J\}$ and $\overline{K}^f = \{(a, f(a)+j) : a \in R,\, j \in J,\, f(a) + j \in K\}$.

Next, we determine when the ideal $I \ltimes^f J$ is a weakly (m,n)-prime ideal in $R \ltimes^f J$ for positive integers m and n.

Theorem 8. *Consider the amalgamation of rings R and R' along the ideal J of R' with respect to a homomorphism f. For positive integers m and n and any ideal I of R, the following are equivalent.*

1. *$I \ltimes^f J$ is a weakly (m,n)-prime ideal of $R \ltimes^f J$.*
2. *I is a weakly (m,n)-prime ideal of R and for any (m,n)-zero (a,b) of I, we have $(f(a) + j_1)^m(f(b) + j_2) = 0$ for all $j_1, j_2 \in J$.*

Proof. (1)⇒(2) Suppose $I \ltimes^f J$ is a weakly (m,n)-prime ideal of $R \bowtie^f J$. Let $a, b \in R$ such that $0 \neq a^m b \in I$ and $b \notin I$. Then $(0,0) \neq (a, f(a))^m(b, f(b)) \in I \ltimes^f J$ with $(b, f(b)) \notin I \ltimes^f J$ and so by assumption, $(a, f(a))^n \in I \ltimes^f J$. Thus, $a^n \in I$ and I is a weakly (m,n)-prime ideal of R. Now, let (a,b) be an (m,n)-zero of I. Then for every $j_1, j_2 \in J$, we have $(a, f(a) + j_1)^m(b, f(b) + j_2) \in I \ltimes^f J$ but $(a, f(a) + i)^n \notin I \ltimes^f J$ and $(b, f(b)) \notin I \ltimes^f J$. Therefore, we get $(f(a) + j_1)^m(f(b) + j_2) = 0$ since $I \ltimes^f J$ is weakly (m,n)-prime in $R \bowtie^f J$.

(2)⇒(1) Let $(a, f(a)+j_1), (b, f(b)+j_2) \in R \ltimes^f J$ such that $(0,0) \neq (a, f(a)+j_1)^m(b, f(b) + j_2) = (a^m b, (f(a) + j_1)^m(f(b) + j_2)) \in I \ltimes^f J$. If $a^m b \neq 0$, then $a^n \in I$ or $b \in I$ as I is weakly (m,n)-prime in R. Hence, $(a, f(a)+j_1)^n \in I \ltimes^f J$ or $(b, f(b) + j_2) \in I \ltimes^f J$ as required. Now, suppose $a^m b = 0$. Then $(f(a) + j_1)^m(f(b) + j_2) \neq 0$ and so (a,b) is not an (m,n)-zero of I. Therefore, either $a^n \in I$ or $b \in I$. Hence, again $(a, f(a)+j_1)^n \in I \ltimes^f J$ or $(b, f(b)+j_2) \in I \ltimes^f J$ and $I \ltimes^f J$ is a weakly (m,n)-prime ideal of $R \bowtie^f J$. □

In particular, we have:

Corollary 9. *Let I and J be ideals of a ring R and m, n be positive integers. Then $I \ltimes J$ is a weakly (m,n)-prime ideal of $R \ltimes J$ if and only if I is a weakly (m,n)-prime ideal of R and for any (m,n)-zero (a,b) of I, we have $(a + j_1)^m(b + j_2) = 0$ for all $j_1, j_2 \in J$.*

Corollary 10. *Let m, n, R, R', J and f be as in Theorem 8. Then any weakly (m,n)-prime ideal of $R \ltimes^f J$ containing $\{0\} \times J$ is of the form $I \ltimes^f J$, where I is a weakly (m,n)-prime ideal of R.*

Proof. Let K be a weakly (m,n)-prime ideal of $R \ltimes^f J$ containing $\{0\} \times J$. Consider the surjective homomorphism $\varphi : R \ltimes^f J \to R$ defined by $\varphi(a, f(a) +$

$j) = a$. Then $Ker(\varphi) = \{0\} \times J \subseteq K$ and so $I := \varphi(K)$ is a weakly (m, n)-prime ideal of R by Proposition 7. Since $\{0\} \times J \subseteq K$, we conclude that $K = I \ltimes^f J$. Moreover, I is a weakly (m, n)-prime ideal of R by Theorem 8. □

Theorem 9. *Consider the amalgamation of rings R and R' along the ideal J of R' with respect to an epimorphism f. Let K be an ideal of R' and m, n be positive integers. Then the following are equivalent.*

(1) *$\bar{K}^f$ is a weakly (m, n)-prime ideal of $R \ltimes^f J$.*

(2) *K is a weakly (m, n)-prime ideal of R' and for every $j_1, j_2 \in J$, if $(f(a) + j_1, f(b) + j_2)$ is an (m, n)-zero of K, we have $a^m b = 0$.*

Proof. (1)⇒(2) Suppose $\bar{K}^f$ is a weakly (m, n)-prime ideal of $R \ltimes^f J$. Let $a' = f(a)$ and $b' = f(b)$ be any two elements in R' such that $0' \neq f(a)^m f(b) \in K$, where $a, b \in R$. Then $(a, f(a)), (b, f(b)) \in R \ltimes^f J$ with $(0, 0) \neq (a, f(a))^m (b, f(b)) = (a^m b, f(a^m b)) \in \bar{K}^f$. By assumption, we have either $(a, f(a))^n \in \bar{K}^f$ or $(b, f(b)) \in \bar{K}^f$. Thus, $f(a)^n \in K$ or $f(b) \in K$ and K is a weakly (m, n)-prime ideal of R'. Now, let $j_1, j_2 \in J$ and $f(a), f(b) \in R'$ such that $(f(a)+j_1, f(b)+j_2)$ is an (m, n)-zero of K. Then $(f(a)+j_1)^m (f(b)+j_2) = 0'$ with $(f(a)+j_1)^n \notin K$ and $(f(b) + j_2) \notin K$. Hence, $(a, f(a) + j_1)^m (b, f(b) + j_2) \in \bar{K}^f$ with $(a, f(a) + j_1)^n \notin \bar{K}^f$ and $(b, f(b) + j_2) \notin \bar{K}^f$. Since $\bar{K}^f$ is weakly (m, n)-prime, then $(a, f(a) + j_1)^m (b, f(b) + j_2) = (0, 0)$ and so $a^m b = 0$ as needed.

(2)⇒(1) Let $(0, 0) \neq (a, f(a)+j_1)^m (b, f(b)+j_2) = (a^m b, (f(a)+j_1)^m (f(b)+j_2)) \in \bar{K}^f$ for $(a, f(a)+j_1), (b, f(b)+j_2) \in R \ltimes^f J$. Then $(f(a)+j_1)^m (f(b)+j_2) \in K$. If $(f(a)+j_1)^m (f(b)+j_2) \neq 0'$, then $(f(a)+j_1)^n \in K$ or $f(b)+j_2 \in K$. Thus, $(a, f(a) + j_1)^n \in \bar{K}^f$ or $(b, f(b) + j_2) \in \bar{K}^f$ and the result follows. Suppose $(f(a) + j_1)^m (f(b) + j_2) = 0'$. Then $a^m b \neq 0$ and so by our assumption, we conclude that $(f(a) + j_1, f(b) + j_2)$ is not an (m, n)-zero of K. Thus, again either $(f(a) + j_1)^n \in K$ or $f(b) + j_2 \in K$ and so $(a, f(a) + j_1)^n \in \bar{K}^f$ or $(b, f(b)+j_2) \in \bar{K}^f$. Therefore, $\bar{K}^f$ is a weakly (m, n)-prime ideal of $R \ltimes^f J$. □

In general, if I (resp. K) is a weakly (m, n)-prime ideal of a ring R, then $I \ltimes J$ (resp. $\bar{K}$) need not be weakly (m, n)-prime in $R \ltimes J$.

Example 4. Consider the ideals $I = K = \langle \bar{4} \rangle$ of the ring $R = \mathbb{Z}_8$ which are weakly $(3, 1)$-prime (Example 2). Then for $J = R$, $I \ltimes J$ and $\bar{K}$ are not weakly $(3, 1)$-prime ideals of $R \ltimes^f J$. Indeed, $(\bar{2}, \bar{3}) \in R \ltimes^f J$ with $(\bar{0}, \bar{0}) \neq (\bar{2}, \bar{3})^3 = (\bar{0}, \bar{3}) \in I \ltimes J$ but, $(\bar{2}, \bar{3}) \notin I \ltimes J$. Also, $(\bar{3}, \bar{2}) \in R \ltimes^f J$ with $(\bar{0}, \bar{0}) \neq (\bar{3}, \bar{2})^3 = (\bar{3}, \bar{0}) \in \bar{K}$ but, $(\bar{3}, \bar{2}) \notin \bar{K}$. We note that $(\bar{2}, \bar{1})$ is clearly an $(3, 1)$-zero of I but $(\bar{2} + \bar{1})^3 (\bar{1} + \bar{1}) \neq 0$.

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