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DECAY RESULTS OF WEAK SOLUTIONS TO THE NON-STATIONARY FRACTIONAL NAVIER-STOKES EQUATIONS

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Abstract. The goal of this paper is to study decay properties of weak solutions to Cauchy problem of the non-stationary fractional Navier-Stokes equations. By using the Fourier splitting method, we give the time L^2-decay rate of weak solutions, which reveals that L^2-decay is generally determined by its linear generalized Stokes flow. In second part, we establish various decay results and the uniqueness of the two dimensional fractional Navier-Stokes flows. In the end of this article, as an appendix, the existence of global weak solutions is given by making use of Galerkin' method, weak and strong compact convergence theorems.

1. Statement of the main results

We consider Cauchy problem of the n-dimensional generalized incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t u + \Lambda^{2\theta} u + (u\cdot\nabla)u + \nabla p = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ \nabla\cdot u = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u(x,0) = a & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 2$, $u = (u_1(x,t), u_2(x,t), \ldots, u_n(x,t))$ and $p = p(x,t)$ denote unknown velocity vector and the pressure, respectively, while $a = a(x)$ is a given initial velocity vector field satisfying $\nabla\cdot a = 0$ in the sense of distribution.

$$\partial_t = \frac{\partial}{\partial t}, \quad \nabla = (\partial_1, \partial_2, \ldots, \partial_n), \quad \partial_j = \frac{\partial}{\partial x_j} \quad (j = 1, 2, \ldots, n),$$
$$(u\cdot\nabla)u = \sum_{j=1}^{n} u_j \partial_j u, \quad \nabla\cdot u = \sum_{j=1}^{n} \partial_j u_j,$$

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$\Lambda = (-\Delta)^{\frac{1}{2}}$, and $\Lambda^{2\theta} = (-\Delta)^{\theta}$ $(0 < \theta \leq 1)$, which is defined as follows:

$$\widehat{\Lambda^{\gamma}\varphi}(\xi) = |\xi|^{\gamma}\hat{\varphi}(\xi),$$

where $\gamma > 0$, $\hat{\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot y}\varphi(y)dy$ is the Fourier transform of φ, $i = \sqrt{-1}$.

Definition. Let $a \in L^2_\sigma(\mathbb{R}^n)$, $0 < \theta < 1$. A vector function u is called a weak solution of (1.1) if $u \in L^\infty(0,\infty; L^2_\sigma(\mathbb{R}^n)) \cap L^2_{loc}([0,\infty); H^\theta(\mathbb{R}^n))$ satisfies in the sense of the distribution

$$\frac{d}{dt}(u(t),\varphi) + (\Lambda^\theta u, \Lambda^\theta \varphi) - (u \otimes u, \nabla\varphi) = 0 \text{ for every } \varphi \in H^1_\sigma(\mathbb{R}^2),$$

where $u \otimes u = (u_i u_j)_{n\times n}$ denotes the $n \times n$ matrix. Moreover,

$$\lim_{t\to 0} \|u(t) - a\|_{L^2(\mathbb{R}^n)} = 0,$$

and the energy inequality holds for any $t > 0$

$$\|u(t)\|^2_{L^2(\mathbb{R}^n)} + 2\int_0^t \|\Lambda^\theta u(\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau \leq \|a\|^2_{L^2(\mathbb{R}^n)}. \tag{1.2}$$

The systematic theory on the classical Navier-Stokes equations ($\theta = 1$ in (1.1)) origins from Leray's pioneering work in 1934. In this paper [12], J. Leray proposed one open problem whether the weak solution established by him tends to zero in L^2-norm as $t \to \infty$. About fifty years later, M. Schonbek [14–17] attacked this problem and succeeded for the first time in showing the existence of weak solutions with explicit decay rate. For further relevant topics, please refer to [1–9] and the references therein.

There is abundant literature in studying properties of solutions to the generalized Navier-Stokes equations. In [13], J. L. Lions established the existence of a global classical solution to 3D problem (1.1) with $\theta \geq \frac{5}{4}$. However, for the case $\theta < \frac{5}{4}$, the global well-posedness issue has not been resolved so far. Additionally, the generalized Navier-Stokes equations also own scaling invariance properties and energy estimate as in the classical Navier-Stokes system. Indeed, if (u,p) is a solution to the n-dimensional generalized Navier-Stokes equations, then for any $\lambda > 0$, the scalings

$$u_\lambda(x,t) = \lambda^{2\theta-1}u(\lambda x, \lambda^{2\theta}t), \quad p_\lambda(x,t) = \lambda^{4\theta-2}p(\lambda x, \lambda^{2\theta}t)$$

also solves the generalized Navier-Stokes equations. Namely,

$$\partial_t u_\lambda + \Lambda^{2\theta}u_\lambda + (u_\lambda\cdot\nabla)u_\lambda + \nabla p_\lambda = 0, \quad \nabla\cdot u_\lambda = 0.$$

The corresponding energy is

$$I(u_\lambda) = \sup_{t>0}\int_{\mathbb{R}^n}|u_\lambda(x,t)|^2dx + 2\int_0^\infty\int_{\mathbb{R}^n}|\Lambda^\theta u_\lambda(x,t)|^2dxdt = \lambda^{4\theta-2-n}I(u).$$

Obviously, $I(u_\lambda) \to +\infty$ as $\lambda \to 0$ if $\theta < \frac{n+2}{4}$. In this sense, it says that the n-dimensional generalized Navier-Stokes equation is supercritical if $\theta < \frac{n+2}{4}$, critical for $\theta = \frac{n+2}{4}$, and subcritical with $\theta > \frac{n+2}{4}$. It has been proved that when $\theta \geq \frac{5}{4}$, the three dimensional generalized Navier-Stokes system admits a global and unique regular solution (see [13, 21] for instance). There are also many important achievements on problem (1.1) and relevant viscous fluid models, for examples, see [11], [22–24].

Recently, Q. Jiu and H. Yu [10] obtained L^2-decay of solutions of problem (1.1) with $n = 3$. In this article, we consider the general n-dimensional case, here $n \geq 2$. Especially our results contain the two dimensional case, which is more difficult and challenging to be treated than that of the three dimensional case as in the classical Navier-Stokes system. $L^r(1 \leq r \leq \infty)$-decay properties of weak solutions to problem (1.1) with $n = 2$ are established, see Theorem 1.2. Additionally, the time decay of $\|\nabla u(t)\|_{L^2(\mathbb{R}^2)}$ is also given. It should be pointed out that it's almost impossible to establish these decay properties in 3D case.

More precisely, our mains results are stated as follows:

Theorem 1.1. *Let $a \in L^1(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)$ $(n \geq 2)$, $0 < \theta < 1$. Then there exists a weak solution u of problem* (1.1)*, which satisfies for $t > 0$*

$$\|u(t)\|_{L^2(\mathbb{R}^n)} \leq C(t+1)^{-\frac{n}{4\theta}};$$

and

$$\|u(t) - e^{-t\Lambda^{2\theta}}a\|_{L^2(\mathbb{R}^n)} \leq C(t+1)^{-\frac{n}{4\theta}-\min\{\frac{1}{2\theta},\frac{n+2}{8\theta}-\frac{1}{2}\}},$$

where constant C depends only on n, θ, $\|a\|_{L^2(\mathbb{R}^n)}$, $\|a\|_{L^1(\mathbb{R}^n)}$.

Remark. The L^2-decay of u with initial datum a coincides with that of the corresponding fractional heat flow $e^{-t\Lambda^{2\theta}}a$. Indeed, set $u_0(t) = e^{-t\Lambda^{2\theta}}a$. Then $u_0(t)$ satisfies the generalized heat equations:

$$\begin{cases} \partial_t u_0 + \Lambda^{2\theta} u_0 = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u_0(x,0) = a & \text{in} \quad \mathbb{R}^n. \end{cases}$$

Applying the Fourier transform to this linear equation yields for $t > 0$

$$\partial_t \hat{u}_0 + |\xi|^{2\theta}\hat{u}_0 = 0, \quad \hat{u}_0(\xi,0) = \hat{a}(\xi).$$

A simple calculation shows for $t > 0$

$$\hat{u}_0(\xi,t) = e^{-t|\xi|^{2\theta}}\hat{a}(\xi).$$

Furthermore Plancherel theorem means for $t > 0$

$$\begin{aligned} \|u_0(t)\|_{L^2(\mathbb{R}^n)} &= (2\pi)^{-\frac{n}{2}}\|\hat{u}_0(t)\|_{L^2(\mathbb{R}^n)} \\ &= (2\pi)^{-\frac{n}{2}}\Big(\int_{\mathbb{R}^n} e^{-2t|\xi|^{2\theta}}|\hat{a}(\xi)|^2 d\xi\Big)^{\frac{1}{2}} \\ &= (2\pi)^{-\frac{n}{2}} t^{-\frac{n}{4\theta}}\Big(\int_{\mathbb{R}^n} e^{-2|\eta|^{2\theta}}|\hat{a}(t^{-\frac{1}{2\theta}}\eta)|^2 d\eta\Big)^{\frac{1}{2}} \\ &\leq (2\pi)^{-\frac{n}{2}} t^{-\frac{n}{4\theta}}\Big(\int_{\mathbb{R}^n} e^{-2|\eta|^{2\theta}} d\eta\Big)^{\frac{1}{2}}\|\hat{a}\|_{L^\infty(\mathbb{R}^n)} \\ &\leq C t^{-\frac{n}{4\theta}}\|a\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

As the following theorem shows, various time L^r-decay rates can be obtained for weak solutions to the two-dimensional problem (1.1). The crucial point is that we can establish the decay estimate of $\|\nabla u(t)\|_{L^2(\mathbb{R}^2)}$ by applying the Fourier transform to the corresponding *curl* equation. However, the decay property of the gradient u in L^2-norm is unknown for n-dimensional problem (1.1) with $n \geq 3$, this is why we only have the estimate of $\|u(t)\|_{L^2(\mathbb{R}^n)}$ for $n \geq 3$. Now we state our two-dimensional decay results on problem (1.1) as follows.

Theorem 1.2. *Suppose* $a \in L^1_\sigma(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$, $0 < \theta < 1$. *Then problem* (1.1) *admits a weak solution* (u, p), *which satisfies* $u \in L^\infty(0,\infty; H^1(\mathbb{R}^2)) \cap L^2_{loc}(0,\infty; H^{1+\theta}(\mathbb{R}^2))$, $\partial_t u$, $\nabla p \in L^1_{loc}(0,\infty, L^2(\mathbb{R}^2))$ *and for every* $t > 0$

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^2)} \leq C(t+1)^{-\frac{1}{\theta}};$$

$$\|u(t)\|_{L^r(\mathbb{R}^2)} \leq C(t+1)^{-\frac{1}{\theta}(1-\frac{1}{r})}, \quad 1 \leq r < \infty.$$

Furthermore, if $\theta > \frac{1}{2}$, *then there exists* $t_1 > 0$ *such that for* $t > t_1$

$$\|u(t)\|_{L^\infty(\mathbb{R}^2)} \leq Ct^{-\frac{1}{\theta}},$$

and

$$\|\nabla u(t)\|_{L^r(\mathbb{R}^2)} \leq Ct^{-\frac{1}{2\theta}-\frac{1}{\theta}(1-\frac{1}{r})}, \quad 1 \leq r < \infty,$$

where C *depends only on* θ, $\|a\|_{H^1(\mathbb{R}^2)}$, $\|a\|_{L^1(\mathbb{R}^2)}$.

Remark. In estimating the decay rates of $\|u(t)\|_{L^\infty(\mathbb{R}^2)}$, $\|\nabla u(t)\|_{L^r(\mathbb{R}^2)}$ $(r \neq 2)$, the Fourier splitting method does not work any longer, an alternative effective approach is to use Lemma 3.1, then a kind of integral appears inevitably: $\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2\theta}} f(s)ds$. This is why the assumption of $\theta > \frac{1}{2}$ has to be imposed on to avoid generating the strong singularity. It remains open whether $\|u(t)\|_{L^\infty(\mathbb{R}^2)} \leq Ct^{-\frac{1}{\theta}}$, $\|\nabla u(t)\|_{L^r(\mathbb{R}^2)} \leq Ct^{-\frac{1}{2\theta}-\frac{1}{\theta}(1-\frac{1}{r})}$ $(r \neq 2)$ for $0 < \theta \leq \frac{1}{2}$ and large time t.

Theorem 1.3. *Suppose* $a \in L^1_\sigma(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$, $\frac{1}{2} < \theta < 1$. *Then problem* (1.1) *has at most one weak solution.*

Remark. It follows from Theorems 1.2, 1.3 that the two dimensional problem (1.1) admits the unique weak solution for $\frac{1}{2} < \theta < 1$, however, the uniqueness is still open in the case of $0 < \theta \leq \frac{1}{2}$.

This paper is organized as follows. In Section 2, we consider the general n-dimensional problem (1.1), and give the optimal time L^2-decay of weak solutions u in the sense that which coincides with L^2-decay of the fractional Stokes flow: $e^{-t\Lambda^{2\theta}}a$ with the same initial data a (i.e., Theorem 1.1). Section 3 devotes to dealing with the two-dimensional case of (1.1). We first collect some basic and known results regarding the Stokes semigroup $\{e^{-t\Lambda^{2\theta}}\}_{t\geq 0}$, which will be applied in establishing 2D large time decay properties. A series of decay estimates $\|u(t)\|_{L^r(\mathbb{R}^2)}$ $(1 \leq r \leq \infty)$ are presented, including the crucial decay property $\|\nabla u(t)\|_{L^2(\mathbb{R}^2)}$, which is based on the *curl* operator property and the Fourier transform splitting method (i.e., Theorem 1.2). The uniqueness follows from the application of Biot-Savart formula, Gronwall inequality and Fourier transform. In the last section, for completeness of the reading, as an Appendix, we prove the global existence of weak solutions to problem (1.1) with large initial data by using the Galerkin approximation, weak convergence method and space-time strong compactness theorem.

Throughout this paper, $C_0^\infty(\mathbb{R}^n)$ denotes the set of all C^∞-real vector-valued functions with compact support in $\mathbb{R}^n$, $\mathscr{S}'(\mathbb{R}^n)$ is the space of all tempered distributions in $\mathbb{R}^n$, and

$$C_{0,\sigma}^\infty(\mathbb{R}^n) = \{\phi = (\phi_1, \dots, \phi_n) \in C_0^\infty(\mathbb{R}^n) : \nabla \cdot \phi = 0 \text{ in } \mathbb{R}^n\},$$

$$L_\sigma^2(\mathbb{R}^n) = \text{the closure of } C_{0,\sigma}^\infty(\mathbb{R}^n) \text{ in } L^2(\mathbb{R}^n),$$

$$H_\sigma^\gamma(\mathbb{R}^n) = \text{the closure of } C_{0,\sigma}^\infty(\mathbb{R}^n) \text{ in } H^\gamma(\mathbb{R}^n),$$

$$H^\gamma(\mathbb{R}^n) = \left\{u \in \mathscr{S}'(\mathbb{R}^n) : \|u\|_{L^2(\mathbb{R}^n)} + \|\Lambda^\gamma u\|_{L^2(\mathbb{R}^n)} < \infty\right\}, \ \gamma > 0.$$

$L^r(\mathbb{R}^n)$ represents the usual Lebesgue space of vector-valued functions. The norm of $L^r(\mathbb{R}^n)$ is denoted by $\|u\|_{L^r(\mathbb{R}^n)} = (\int_{\mathbb{R}^n} |u(x)|^r dx)^{\frac{1}{r}}$ if $1 \le r < \infty$; and $\|u\|_{L^\infty(\mathbb{R}^n)} = ess \sup_{x\in\mathbb{R}^n} |u(x)|$. By symbol C, it means a generic positive constant which may vary from line to line.

2. L^2-decay of Navier-Stokes flows

In this section, we give the detailed proofs of Theorem 1.1. The main method employed by us is the Fourier splitting method, which is founded by M. E. Schonbek in a series of her work, see [14–17] for details. In the following arguments, we concentrate on estimating the approximate solution u_m of problem (1.1) with initial data $a \in L_\sigma^2(\mathbb{R}^n)$. The existence of u_m is established by using Galerkin's method in Proposition A in the Appendix. Recall that the approximate solution $u_m = \sum_{k=1}^m g_k(t)e_k(x)$ satisfies for every $T > 0$: $u_m \in C([0,T], C_{0,\sigma}^\infty(\mathbb{R}^n)) \cap C^1((0,T), C_{0,\sigma}^\infty(\mathbb{R}^n))$. Moreover, it follows from (A.2), (A.3) that there exists a scalar function $p_m(x,t)$ such that

$$\begin{cases} \partial_t u_m + \Lambda^{2\theta} u_m + (u_m \cdot \nabla)u_m + \nabla p_m = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ \nabla \cdot u_m = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u_m(x,0) = a_m & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{2.1}$$

where $a_m \in C_{0,\sigma}^\infty(\mathbb{R}^n)$ satisfies: $\|a_m\|_{L^2(\mathbb{R}^n)} \le \|a\|_{L^2(\mathbb{R}^n)}$, $\lim_{m\to\infty} \|a_m - a\|_{L^2(\mathbb{R}^n)} = 0$.

In order to obtain the desired decay results, we first need to establish a series of *a prior* estimates on the approximate solution u_m. Then by making use of weak and strong compact convergence theorems, we can reach our expected aims. For simplicity of the statement in the proofs of our main results, we always denote (u,p) by dropping the subscript m of (u_m,p_m) in problem (2.1).

Proof of Theorem 1.1. We prove Theorem 1.1 through the following four steps.

Step 1. Let $a \in L^1(\mathbb{R}^n) \cap L^2_\sigma(\mathbb{R}^n)$, $n \geq 2$.

The following estimates hold for $\xi \in \mathbb{R}^n$, $t > 0$

$$|\hat{u}(\xi,t)| \leq \|a\|_{L^1(\mathbb{R}^n)} + 2\|a\|^2_{L^2(\mathbb{R}^n)}|\xi|^{1-2\theta}, \tag{2.2}$$

and

$$|\hat{u}(\xi,t)| \leq \|a\|_{L^1(\mathbb{R}^n)} + 2|\xi| \int_0^t \|u(s)\|^2_{L^2(\mathbb{R}^n)} ds. \tag{2.3}$$

In fact, since $\nabla \cdot u = 0$, we have

$$\begin{aligned}|\widehat{(u\cdot\nabla)u}(\xi,t)| &= \Big|\sum_{j=1}^n \widehat{\partial_j(u_j u)}(\xi,t)\Big| = \Big|\sum_{j=1}^n i\xi_j\widehat{(u_j u)}(\xi,t)\Big| \\ &\leq \sum_{j=1}^n |\xi_j||\widehat{(u_j u)}(\xi,t)| \leq |\xi|\int_{\mathbb{R}^n}|u(x,t)|^2dx = |\xi|\|u(t)\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Whence

$$|\widehat{(u\cdot\nabla)u}(\xi,t)| \leq |\xi|\|u(t)\|^2_{L^2(\mathbb{R}^n)}. \tag{2.4}$$

Note that

$$\widehat{\nabla\cdot\Lambda^{2\theta}u}(\xi,t) = \sum_{j=1}^n i\xi_j|\xi|^{2\theta}\hat{u}_j(\xi,t) = |\xi|^{2\theta}\widehat{\nabla\cdot u}(\xi,t) = 0.$$

By means of the equation in (2.1), we derive the pressure function p satisfies:

$$\Delta p = -\sum_{k=1}^n\sum_{j=1}^n \partial_k\partial_j(u_k u_j),$$

from which,

$$|\xi|^2\widehat{p}(\xi,t) = -\sum_{k=1}^n\sum_{j=1}^n \xi_k\xi_j\widehat{u_k u_j}(\xi,t).$$

Then

$$\begin{aligned}|\xi|^2|\widehat{p}(\xi,t)| &\leq \sum_{k=1}^n\sum_{j=1}^n |\xi_k||\xi_j|\|u_k(t)\|_{L^2(\mathbb{R}^n)}\|u_j(t)\|_{L^2(\mathbb{R}^n)} \\ &= (\sum_{k=1}^n |\xi_k|\|u_k(t)\|_{L^2(\mathbb{R}^n)})^2 \leq |\xi|^2\|u(t)\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

This shows

$$|\xi||\widehat{p}(\xi,t)| \leq |\xi|\|u(t)\|^2_{L^2(\mathbb{R}^n)}. \tag{2.5}$$

Applying the Fourier transform in both sides of the first equation in (2.1), we find for $t > 0$

$$\widehat{u}_t + |\xi|^{2\theta}\widehat{u} + \widehat{(u\cdot\nabla)u} + \widehat{\nabla p} = 0,$$

which is equivalent to

$$\big(\widehat{u}e^{|\xi|^{2\theta}t}\big)_t+\big(\widehat{(u\cdot\nabla)u}+\widehat{\nabla p}\big)e^{|\xi|^{2\theta}t}=0.$$

Integrating by parts on time t, we have for $t>0$

$$\widehat{u}(\xi,t)=e^{-|\xi|^{2\theta}t}\widehat{a}(\xi)-\int_0^t e^{-|\xi|^{2\theta}(t-s)}\big(\widehat{(u\cdot\nabla)u}+\widehat{\nabla p}\big)(\xi,s)ds. \tag{2.6}$$

Observe that

$$|\hat{a}(\xi)|=\Big|\int_{\mathbb{R}^n}e^{-i\xi\cdot y}a(y)dy\Big|\le\int_{\mathbb{R}^n}|a(y)|dy.$$

Recall the energy equality

$$\int_{\mathbb{R}^n}|u(x,t)|^2dx+2\int_0^t\int_{\mathbb{R}^n}|\Lambda^\theta u(x,s)|^2dxds=\int_{\mathbb{R}^n}|a(x)|^2dx,\quad\forall t>0.$$

Combining (2.4), (2.5) and (2.6) yields for $t>0$

$$\begin{aligned}|\widehat{u}(\xi,t)|&\le|\widehat{a}(\xi)|+2|\xi|\int_0^t e^{-|\xi|^{2\theta}(t-s)}\|u(s)\|^2_{L^2(\mathbb{R}^n)}ds\\&\le\|a\|_{L^1(\mathbb{R}^n)}+2|\xi|\|a\|^2_{L^2(\mathbb{R}^n)}\int_0^t e^{-|\xi|^{2\theta}(t-s)}ds\\&\le\|a\|_{L^1(\mathbb{R}^n)}+2\|a\|^2_{L^2(\mathbb{R}^n)}|\xi|^{1-2\theta},\end{aligned}$$

and

$$\begin{aligned}|\widehat{u}(\xi,t)|&\le|\widehat{a}(\xi)|+2|\xi|\int_0^t e^{-|\xi|^{2\theta}(t-s)}\|u(s)\|^2_{L^2(\mathbb{R}^n)}ds\\&\le\|a\|_{L^1(\mathbb{R}^n)}+2|\xi|\int_0^t\|u(s)\|^2_{L^2(\mathbb{R}^n)}ds,\end{aligned}$$

which are (2.2), (2.3), respectively.

Step 2. $n\ge3$, $\|u(t)\|_{L^2(\mathbb{R}^n)}\le C(1+t)^{-\frac{n}{4\theta}}$, $\forall t>0$.

Note that the following energy equality holds for $t>0$

$$\frac{d}{dt}\int_{\mathbb{R}^n}|u(x,t)|^2dx+2\int_{\mathbb{R}^n}|\Lambda^\theta u(x,t)|^2dx=0.$$

Using the Fourier transform, we get for $t>0$

$$\frac{d}{dt}\int_{\mathbb{R}^n}|\widehat{u}(\xi,t)|^2d\xi+2\int_{\mathbb{R}^n}|\xi|^{2\theta}|\widehat{u}(\xi,t)|^2d\xi=0. \tag{2.7}$$

Let $f(t)$ be a continuous differentiable function for $t\ge0$, which satisfies: $f(0)=1$, $f'(t)>0$, $\forall t>0$. Together with (2.7), we have

$$\frac{d}{dt}\Big(f(t)\int_{\mathbb{R}^n}|\widehat{u}(\xi,t)|^2d\xi\Big)+2f(t)\int_{\mathbb{R}^n}|\xi|^{2\theta}|\widehat{u}(\xi,t)|^2d\xi=f'(t)\int_{\mathbb{R}^n}|\widehat{u}(\xi,t)|^2d\xi. \tag{2.8}$$

Set

$$B(t)=\{\xi\in\mathbb{R}^n:2f(t)|\xi|^{2\theta}\le f'(t)\}.$$

Then

$$2f(t)\int_{\mathbb{R}^n}|\xi|^{2\theta}|\widehat{u}(\xi,t)|^2d\xi \tag{2.9}$$
$$= 2f(t)\int_{B(t)}|\xi|^{2\theta}|\widehat{u}(\xi,t)|^2d\xi + 2f(t)\int_{B(t)^c}|\xi|^{2\theta}|\widehat{u}(\xi,t)|^2d\xi$$
$$\geq f'(t)\int_{B(t)^c}|\widehat{u}(\xi,t)|^2d\xi$$
$$= f'(t)\int_{\mathbb{R}^n}|\widehat{u}(\xi,t)|^2d\xi - f'(t)\int_{B(t)}|\widehat{u}(\xi,t)|^2d\xi.$$

Inserting (2.9) into (2.8), together with (2.2), (2.3), we find for $t > 0$

$$\frac{d}{dt}\Big(f(t)\int_{\mathbb{R}^n}|\widehat{u}(\xi,t)|^2d\xi\Big) \tag{2.10}$$
$$\leq f'(t)\int_{B(t)}|\widehat{u}(\xi,t)|^2d\xi$$
$$\leq f'(t)\int_{|\omega|=1}\int_0^A\big(\|a\|_{L^1(\mathbb{R}^n)} + 2\|a\|^2_{L^2(\mathbb{R}^n)}r^{1-2\theta}\big)^2 r^{n-1}drdS_\omega$$
$$\leq Cf'(t)\Big(\big(\frac{f'(t)}{f(t)}\big)^{\frac{n}{2\theta}} + \big(\frac{f'(t)}{f(t)}\big)^{\frac{n+2}{2\theta}-2}\Big),$$

and

$$\frac{d}{dt}\Big(f(t)\int_{\mathbb{R}^n}|\widehat{u}(\xi,t)|^2d\xi\Big) \tag{2.11}$$
$$\leq f'(t)\int_{B(t)}|\widehat{u}(\xi,t)|^2d\xi$$
$$\leq f'(t)\int_{|\omega|=1}\int_0^A\Big(\|a\|_{L^1(\mathbb{R}^n)} + 2r\int_0^t\|u(s)\|^2_{L^2(\mathbb{R}^n)}ds\Big)^2 r^{n-1}drdS_\omega$$
$$\leq 2f'(t)\int_{|\omega|=1}\int_0^A\Big(\|a\|^2_{L^1(\mathbb{R}^n)} + 4r^2t\int_0^t\|u(s)\|^4_{L^2(\mathbb{R}^n)}ds\Big)r^{n-1}drdS_\omega$$
$$\leq Cf'(t)\Big(\int_0^A r^{n-1}dr + \int_0^A r^{n+1}drt\int_0^t\|u(s)\|^4_{L^2(\mathbb{R}^n)}ds\Big)$$
$$\leq Cf'(t)\Big(A^n + A^{n+2}t\int_0^t\|u(s)\|^4_{L^2(\mathbb{R}^n)}ds\Big)$$
$$\leq Cf'(t)\Big(\big(\frac{f'(t)}{f(t)}\big)^{\frac{n}{2\theta}} + t\big(\frac{f'(t)}{f(t)}\big)^{\frac{n+2}{2\theta}}\int_0^t\|u(s)\|^4_{L^2(\mathbb{R}^n)}ds\Big),$$

where $A^{2\theta} = \frac{f'(t)}{2f(t)}$.

Integrating on t in (2.10), (2.11), respectively, we get for $t > 0$

$$f(t)\int_{\mathbb{R}^n}|\widehat{u}(\xi,t)|^2d\xi \leq \int_{\mathbb{R}^n}|\widehat{a}(\xi)|^2d\xi + C\min\{I_1(t), I_2(t)\}, \tag{2.12}$$

where

$$I_1(t) = \int_0^t f'(s)\Big(\big(\frac{f'(s)}{f(s)}\big)^{\frac{n}{2\theta}} + \big(\frac{f'(s)}{f(s)}\big)^{\frac{n+2}{2\theta}-2}\Big)ds, \tag{2.13}$$

$$I_2(t) = \int_0^t f'(s)\Big(\big(\frac{f'(s)}{f(s)}\big)^{\frac{n}{2\theta}} + s\big(\frac{f'(s)}{f(s)}\big)^{\frac{n+2}{2\theta}} \int_0^s \|u(r)\|^4_{L^2(\mathbb{R}^n)}dr\Big)ds. \tag{2.14}$$

Take $f(t) = (t+1)^\alpha$ in (2.13), (2.14), respectively. Then for $t > 0$

$$f'(t) = \alpha(t+1)^{\alpha-1}, \quad \frac{f'(t)}{f(t)} = \alpha(t+1)^{-1},$$

and

(2.15)

$$\begin{aligned} I_1(t) &\le C\int_0^t (s+1)^{\alpha-1}\big[(s+1)^{-\frac{n}{2\theta}} + (s+1)^{-\frac{n+2}{2\theta}+2}\big]ds \\ &\le C\big((t+1)^{\alpha-\frac{n}{2\theta}} + (t+1)^{\alpha-\frac{n+2}{2\theta}+2}\big), \text{ where } \alpha > \max\big\{\frac{n}{2\theta}, \frac{n+2}{2\theta}-2\big\}, \end{aligned}$$

(2.16)

$$\begin{aligned} I_2(t) &\le C\int_0^t (s+1)^{\alpha-1}\Big((s+1)^{-\frac{n}{2\theta}} + (s+1)^{-\frac{n+2}{2\theta}+1}\int_0^s \|u(r)\|^4_{L^2(\mathbb{R}^n)}dr\Big)ds \\ &\le C\int_0^t (s+1)^{\alpha-1-\frac{n}{2\theta}}\big(1+(s+1)^{-\frac{1}{\theta}+1}\int_0^s \|u(r)\|^4_{L^2(\mathbb{R}^n)}dr\big)ds. \end{aligned}$$

Inserting (2.15) into (2.12), using Plancherel theorem, we find for $t > 0$

$$\begin{aligned} \int_{\mathbb{R}^n} |u(x,t)|^2dx &= (2\pi)^{-n}\int_{\mathbb{R}^n} |\widehat{u}(\xi,t)|^2 d\xi \\ &\le \frac{1}{f(t)}\int_{\mathbb{R}^n} |a(x)|^2dx + \frac{CI_1(t)}{f(t)} \\ &\le C\big((t+1)^{-\frac{n}{2\theta}} + (t+1)^{-\frac{n+2}{2\theta}+2}\big). \end{aligned} \tag{2.17}$$

Applying (2.17) to (2.16), we get for any $t > 0$

$$\begin{aligned} I_2(t) &\le C\int_0^t (s+1)^{\alpha-1-\frac{n}{2\theta}} \\ &\quad \times \Big(1+(s+1)^{-\frac{1}{\theta}+1}\int_0^s \big((r+1)^{-\frac{n}{2\theta}} + (r+1)^{-\frac{n+2}{2\theta}+2}\big)^2dr\Big)ds \\ &\le C\int_0^t (s+1)^{\alpha-1-\frac{n}{2\theta}}\big(1+(s+1)^{-\frac{1}{\theta}+1}\big)ds \\ &\le C(t+1)^{\alpha-\frac{n}{2\theta}}. \end{aligned} \tag{2.18}$$

Here we require $n \ge 3$ to guarantee the integral in (2.18) is finite:

$$\int_0^s (r+1)^{2(-\frac{n+2}{2\theta}+2)}dr \le \int_0^\infty (r+1)^{2(-\frac{n+2}{2\theta}+2)}dr < \infty.$$

Inserting (2.18) into (2.12), together with Plancherel theorem, we conclude for $t > 0$

$$\begin{aligned}(2.19)\qquad \int_{\mathbb{R}^n} |u(x,t)|^2 dx &\le \frac{1}{f(t)} \int_{\mathbb{R}^n} |a(x)|^2 dx + \frac{CI_2(t)}{f(t)} \\ &\le C((t+1)^{-\alpha} + (t+1)^{-\frac{n}{2\theta}}) \\ &\le C(t+1)^{-\frac{n}{2\theta}}.\end{aligned}$$

Step 3. $n = 2$, $\|u(t)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\frac{1}{2\theta}}$, $\forall t > 0$.

Let $a \in L^1(\mathbb{R}^2) \cap L^2_\sigma(\mathbb{R}^2)$. We write (2.12) in the two dimensional case as follows:

$$(2.20)\qquad f(t) \int_{\mathbb{R}^2} |\widehat{u}(\xi,t)|^2 d\xi \le \int_{\mathbb{R}^2} |\widehat{a}(\xi)|^2 d\xi + CJ(t),$$

where

$$J(t) = \int_0^t f'(s)\Big(\big(\frac{f'(s)}{f(s)}\big)^{\frac{1}{\theta}} + s\big(\frac{f'(s)}{f(s)}\big)^{\frac{2}{\theta}} \int_0^s \|u(r)\|^4_{L^2(\mathbb{R}^2)} dr\Big) ds.$$

(a) Take $f(t) = [\log(e+t)]^{\frac{3}{\theta}}$, $t \ge 0$. Then

$$f'(t) = \frac{3}{\theta}[\log(e+t)]^{\frac{3}{\theta}-1}(e+t)^{-1}, \quad \frac{f'(t)}{f(t)} = \frac{3}{\theta}[(e+t)\log(e+t)]^{-1}.$$

Whence

$$\begin{aligned}(2.21)\qquad J(t) &\le \int_0^t f'(s)\Big(\big(\frac{f'(s)}{f(s)}\big)^{\frac{1}{\theta}} + s^2\big|\frac{f'(s)}{f(s)}\big|^{\frac{2}{\theta}} \|a\|^4_{L^2(\mathbb{R}^2)}\Big) ds \\ &\le C\int_0^t \frac{[\log(e+s)]^{\frac{3}{\theta}-1}}{e+s}\Big([(e+s)\log(e+s)]^{-\frac{1}{\theta}} + s^2[(e+s)\log(e+s)]^{-\frac{2}{\theta}}\Big) ds \\ &\le C\int_0^t \Big(\frac{[\log(e+s)]^{\frac{2}{\theta}-1}}{(e+s)^{1+\frac{1}{\theta}}} + \frac{[\log(e+s)]^{\frac{1}{\theta}-1}}{(e+s)^{-1+\frac{2}{\theta}}}\Big) ds \\ &\le C + C[\log(e+t)]^{\frac{1}{\theta}-1} \int_0^t (e+s)^{1-\frac{2}{\theta}} ds \le C[\log(e+t)]^{\frac{1}{\theta}-1},\ t > 0,\end{aligned}$$

where C depends only on θ, $\|a\|_{L^2(\mathbb{R}^2)}$.

Inserting (2.21) into (2.20), we get for $t > 0$

$$\begin{aligned}(2.22)\qquad \int_{\mathbb{R}^2} |u(x,t)|^2 dx &= (2\pi)^{-n} \int_{\mathbb{R}^2} |\widehat{u}(\xi,t)|^2 d\xi \\ &\le C[\log(e+t)]^{-\frac{3}{\theta}} \int_{\mathbb{R}^2} |\widehat{a}(\xi)|^2 d\xi + C[\log(e+t)]^{-1-\frac{2}{\theta}} \\ &\le C[\log(e+t)]^{-1-\frac{2}{\theta}},\ t > 0.\end{aligned}$$

(b) Set $f(t) = (1+t)^{\frac{2}{\theta}},\ t \geq 0$. Then

$$f'(t) = \frac{2}{\theta}(1+t)^{\frac{2}{\theta}-1}, \quad \frac{f'(t)}{f(t)} = \frac{2}{\theta}(1+t)^{-1}.$$

From (2.22), we have for $t > 0$

$$\begin{aligned} J(t) &\leq C\int_0^t (1+s)^{\frac{2}{\theta}-1}\Big((1+s)^{-\frac{1}{\theta}} + s(1+s)^{-\frac{2}{\theta}}\int_0^s \|u(r)\|^4_{L^2(\mathbb{R}^2)}dr\Big)ds \\ &\leq C\int_0^t \Big((1+s)^{\frac{1}{\theta}-1} + \int_0^s \|u(r)\|^4_{L^2(\mathbb{R}^2)}dr\Big)ds \\ &\leq C(1+t)^{\frac{1}{\theta}} + Ct\int_0^t \|u(r)\|^4_{L^2(\mathbb{R}^2)}dr \\ &\leq C(1+t)^{\frac{1}{\theta}} + Ct\int_0^t \|u(r)\|^2_{L^2(\mathbb{R}^2)}[\log(e+r)]^{-1-\frac{2}{\theta}}dr. \end{aligned}$$

Combining with (2.20), we conclude for $t > 0$

$$\begin{aligned} (1+t)^{\frac{1}{\theta}}\|u(t)\|^2_{L^2(\mathbb{R}^2)} &\leq (1+t)^{-\frac{1}{\theta}}\int_{\mathbb{R}^2}|a(x)|^2dx + C(1+t)^{-\frac{1}{\theta}}J(t) \\ &\leq C + C\int_0^t \|u(s)\|^2_{L^2(\mathbb{R}^2)}[\log(e+s)]^{-1-\frac{2}{\theta}}ds. \end{aligned} \tag{2.23}$$

Set

$$g(t) = (1+t)^{\frac{1}{\theta}}\|u(t)\|^2_{L^2(\mathbb{R}^2)}, \quad h(t) = C(1+t)^{-\frac{1}{\theta}}[\log(e+t)]^{-1-\frac{2}{\theta}}.$$

Then it follows from (2.23) that for $t > 0$

$$g(t) \leq C + \int_0^t g(s)h(s)ds.$$

Applying Gronwall inequality yields for $t > 0$

$$g(t) \leq C\exp\Big(\int_0^t h(s)ds\Big).$$

Namely,

$$(1+t)^{\frac{1}{\theta}}\|u(t)\|^2_{L^2(\mathbb{R}^2)} \leq C\exp\Big(C\int_0^\infty (1+t)^{-\frac{1}{\theta}}[\log(e+t)]^{-1-\frac{2}{\theta}}dt\Big) \leq C,\ t > 0.$$

Whence

$$\|u(t)\|_{L^2(\mathbb{R}^2)} \leq C(1+t)^{-\frac{1}{2\theta}},\ t > 0.$$

Step 4. $\|u(t) - u_0(t)\|_{L^2(\mathbb{R}^n)} \leq C(t+1)^{-\frac{n}{4\theta}-\min\{\frac{1}{2\theta},\frac{n+2}{8\theta}-\frac{1}{2}\}}$, $n \geq 2$ for $t > 0$.

In this last step, we give the proof of the second part of Theorem 1.1. That is, we establish the decay rates of $\|(u-u_0)(t)\|_{L^2(\mathbb{R}^n)}$ as $t \to \infty$, where $u_0(t) =$

$e^{-t\Lambda^{2\theta}}a$, $a\in L^2_\sigma(\mathbb{R}^n)$. Set $D(t)=u(t)-u_0(t)$. Then D satisfies in the sense of distribution

$$\tag{2.24}\begin{cases}\partial_t D-\Lambda^{2\theta}D+(u\cdot\nabla)u+\nabla\pi=0 & \text{in} \quad \mathbb{R}^n\times(0,\infty),\\ \nabla\cdot D=0 & \text{in} \quad \mathbb{R}^n\times(0,\infty),\\ D(x,0)=0 & \text{in} \quad \mathbb{R}^n.\end{cases}$$

The energy equality for problem (2.24) is written as follows: For all $t>0$

$$\frac{d}{dt}\|D(t)\|^2_{L^2(\mathbb{R}^n)}+2\|\Lambda^\theta D(t)\|^2_{L^2(\mathbb{R}^n)}-2\int_{\mathbb{R}^n}[(u\cdot\nabla)u\cdot D](x,t)dx=0.$$

Using Plancherel theorem yields for $t>0$

$$\tag{2.25}\frac{d}{dt}\int_{\mathbb{R}^n}|\widehat{D}(\xi,t)|^2d\xi+2\int_{\mathbb{R}^n}|\xi|^{2\theta}|\widehat{D}(\xi,t)|^2d\xi-2(2\pi)^n\int_{\mathbb{R}^n}[(u\cdot\nabla)u\cdot D](x,t)dx=0,$$

and then

$$\tag{2.26}\begin{aligned}&\frac{d}{dt}\Big(g(t)\int_{\mathbb{R}^n}|\widehat{D}(\xi,t)|^2d\xi\Big)+2g(t)\int_{\mathbb{R}^n}|\xi|^{2\theta}|\widehat{D}(\xi,t)|^2d\xi\\ &=g'(t)\int_{\mathbb{R}^n}|\widehat{D}(\xi,t)|^2d\xi-2(2\pi)^ng(t)\int_{\mathbb{R}^n}[(u\cdot\nabla)u\cdot D](x,t)dx,\end{aligned}$$

where the function $g(t)$ is a continuous differentiable function for $t\geq 0$, which satisfies: $g(0)=0$, $g'(t)>0$, $\forall t>0$.

By the Fourier splitting method, we get for $t>0$

$$\tag{2.27}\begin{aligned}&2g(t)\int_{\mathbb{R}^n}|\xi|^{2\theta}|\widehat{D}(\xi,t)|^2d\xi\\ &=2g(t)\int_{B(t)}|\xi|^{2\theta}|\widehat{D}(\xi,t)|^2d\xi+2g(t)\int_{B(t)^c}|\xi|^{2\theta}|\widehat{D}(\xi,t)|^2d\xi\\ &\geq g'(t)\int_{B(t)^c}|\widehat{D}(\xi,t)|^2d\xi\\ &=g'(t)\int_{\mathbb{R}^n}|\widehat{D}(\xi,t)|^2d\xi-g'(t)\int_{B(t)}|\widehat{D}(\xi,t)|^2d\xi,\end{aligned}$$

where $B(t)=\{\xi\in\mathbb{R}^n: 2g(t)|\xi|^{2\theta}\leq g'(t)\}$.

Inserting (2.27) into (2.26), we conclude for $t>0$

$$\tag{2.28}\begin{aligned}&\frac{d}{dt}\Big(g(t)\int_{\mathbb{R}^n}|\widehat{D}(\xi,t)|^2d\xi\Big)\\ &\leq g'(t)\int_{B(t)}|\widehat{D}(\xi,t)|^2d\xi-2(2\pi)^ng(t)\int_{\mathbb{R}^n}[(u\cdot\nabla)u\cdot D](x,t)dx.\end{aligned}$$

Observe that the pressure function π in (2.24) satisfies

$$\Delta\pi=-\sum_{k=1}^n\sum_{j=1}^n\frac{\partial^2}{\partial x_k\partial x_j}(u_ku_j),$$

from which,

$$|\xi||\widehat{\pi}(\xi,t)| \leq |\xi|\|u(t)\|^2_{L^2(\mathbb{R}^n)}. \tag{2.29}$$

Combining (2.4) and (2.29) yields for $t > 0$

$$|\widehat{(u\cdot\nabla)}u(\xi,t)| + |\xi||\widehat{\pi}(\xi,t)| \leq 2|\xi|\|u(t)\|^2_{L^2(\mathbb{R}^n)}. \tag{2.30}$$

Applying the Fourier transform in both sides of the first equation in (2.24), we get for $t > 0$

$$\widehat{D}_t + |\xi|^{2\theta}\widehat{D} + \widehat{(u\cdot\nabla)u} + \widehat{\nabla\pi} = 0.$$

Whence,

$$\big(\widehat{D}e^{|\xi|^{2\theta}t}\big)_t + \big(\widehat{(u\cdot\nabla)u} + \widehat{\nabla\pi}\big)e^{|\xi|^{2\theta}t} = 0,$$

and then for $t > 0$

$$\widehat{D}(\xi,t) = -\int_0^t e^{-|\xi|^{2\theta}(t-s)}\big(\widehat{(u\cdot\nabla)u} + \widehat{\nabla\pi}\big)(\xi,s)ds. \tag{2.31}$$

Inserting (2.30) into (2.31) yields for $t > 0$

$$|\widehat{D}(\xi,t)| \leq 2|\xi|\int_0^t e^{-|\xi|^{2\theta}(t-s)}\|u(s)\|^2_{L^2(\mathbb{R}^n)}ds \leq 2|\xi|\int_0^t \|u(s)\|^2_{L^2(\mathbb{R}^n)}ds. \tag{2.32}$$

Recall we have proved that for $n \geq 2$ and $s > 0$

$$\|u(s)\|_{L^2(\mathbb{R}^n)} \leq C(1+s)^{-\frac{n}{4\theta}}.$$

Note that $0 < \theta < 1 \leq \frac{n}{2}$ for $n \geq 2$. Together with (2.29), we find for $t > 0$

$$\begin{aligned} |\widehat{D}(\xi,t)| &\leq 2|\xi|\int_0^t e^{-|\xi|^{2\theta}(t-s)}\|u(s)\|^2_{L^2(\mathbb{R}^n)} \\ &\leq C|\xi|\int_0^t (1+s)^{-\frac{n}{2\theta}}ds \\ &\leq C|\xi|. \end{aligned} \tag{2.33}$$

Note that $u_0(t) = e^{-t\Lambda^{2\theta}}a$, and for $t > 0$

$$\begin{aligned} \|i\xi_j e^{-t|\xi|^{2\theta}}\|_{L^2(\mathbb{R}^n)} &\leq \Big(\int_{\mathbb{R}^n}|\xi|^2 e^{-2t|\xi|^{2\theta}}d\xi\Big)^{\frac{1}{2}} \\ &= t^{-\frac{n+2}{4\theta}}\Big(\int_{\mathbb{R}^n}|\eta|^2 e^{-2|\eta|^{2\theta}}d\eta\Big)^{\frac{1}{2}} \\ &= Ct^{-\frac{n+2}{4\theta}}. \end{aligned}$$

Whence for $t > 0$

$$\begin{aligned} \|\widehat{\partial_j u_0}(t)\|_{L^1(\mathbb{R}^n)} &= \|i\xi_j \widehat{e^{-t\Lambda^{2\theta}}a}\|_{L^1(\mathbb{R}^n)} \\ &= \|i\xi_j e^{-t|\xi|^{2\theta}}\hat{a}\|_{L^1(\mathbb{R}^n)} \\ &\leq \|i\xi_j e^{-t|\xi|^{2\theta}}\|_{L^2(\mathbb{R}^n)}\|\hat{a}\|_{L^2(\mathbb{R}^n)} \\ &\leq Ct^{-\frac{n+2}{4\theta}}\|a\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Then for $t > 0$

$$
\begin{aligned}
(2.34)\qquad \Big|\int_{\mathbb{R}^n}[(u\cdot\nabla)u\cdot D](x,t)dx\Big| &= \Big|\int_{\mathbb{R}^n}[(u\cdot\nabla)u\cdot u_0](x,t)dx\Big| \\
&= (2\pi)^{-n}\Big|\int_{\mathbb{R}^n}\sum_{j=1}^{n}[\widehat{(u_j u)}\widehat{\partial_j u_0}](\xi,t)d\xi\Big| \\
&\le \sum_{j=1}^{n}\|\widehat{\partial_j u_0}(t)\|_{L^1(\mathbb{R}^n)}\|\widehat{(u_j u)}(t)\|_{L^\infty(\mathbb{R}^n)} \\
&\le Ct^{-\frac{n+2}{4\theta}}\|u(t)\|^2_{L^2(\mathbb{R}^n)} \\
&\le Ct^{-\frac{n+2}{4\theta}}(1+t)^{-\frac{n}{2\theta}} \le Ct^{-\frac{3n+2}{4\theta}}.
\end{aligned}
$$

From (2.28), (2.33) and (2.34), we obtain for $t > 0$

$$
\begin{aligned}
(2.35)\qquad &\frac{d}{dt}\Big(g(t)\int_{\mathbb{R}^n}|\widehat{D}(\xi,t)|^2d\xi\Big) \\
&\le Cg'(t)\int_{|\omega|=1}\int_0^A r^2r^{n-1}drdS_\omega + Cg(t)t^{-\frac{3n+2}{4\theta}} \\
&\le C\Big(g'(t)\big(\frac{g'(t)}{g(t)}\big)^{\frac{n+2}{2\theta}} + g(t)t^{-\frac{3n+2}{4\theta}}\Big),
\end{aligned}
$$

where $A^{2\theta} = \frac{g'(t)}{2g(t)}$.

Integrating on t in (2.35), we get for $t > 0$

$$
(2.36)\qquad \int_{\mathbb{R}^n}|D(x,t)|^2dx = (2\pi)^{-n}\int_{\mathbb{R}^n}|\widehat{D}(\xi,t)|^2d\xi \le CJ(t),
$$

where

$$
(2.37)\qquad J(t) = \frac{1}{g(t)}\int_0^t \Big(g'(s)\big(\frac{g'(s)}{g(s)}\big)^{\frac{n+2}{2\theta}} + g(s)s^{-\frac{3n+2}{4\theta}}\Big)ds.
$$

Take $g(t) = t^\alpha$ in (2.34), where $\alpha > 0$ is sufficiently large. Then

$$
g'(t) = \alpha t^{\alpha-1}, \quad \frac{g'(t)}{g(t)} = \alpha t^{-1}.
$$

Therefore,

$$
\begin{aligned}
(2.38)\qquad J(t) &\le Ct^{-\alpha}\int_0^t \big(s^{\alpha-1-\frac{n+2}{2\theta}} + s^{\alpha-\frac{3n+2}{4\theta}}\big)ds \\
&\le Ct^{-\alpha}\big(t^{\alpha-\frac{n+2}{2\theta}} + t^{\alpha+1-\frac{3n+2}{4\theta}}\big) \\
&\le C\big(t^{-\frac{n+2}{2\theta}} + t^{1-\frac{3n+2}{4\theta}}\big).
\end{aligned}
$$

From (2.36)-(2.38), we derive for $t > 0$

$$
\|D(t)\|^2_{L^2(\mathbb{R}^n)} \le C\big(t^{-\alpha} + t^{-\frac{n+2}{2\theta}} + t^{1-\frac{3n+2}{4\theta}}\big).
$$

Note that

$$\begin{aligned}\|D(t)\|_{L^2(\mathbb{R}^n)} &\le \|u(t)\|_{L^2(\mathbb{R}^n)} + \|u_0(t)\|_{L^2(\mathbb{R}^n)}\\ &\le C(1+t)^{-\frac{n}{4\theta}} + (2\pi)^{-\frac{n}{2}}\|e^{-t|\xi|^{2\theta}}\hat{a}\|_{L^2(\mathbb{R}^n)}\\ &\le C(1+\|a\|_{L^2(\mathbb{R}^n)}), \quad \forall t>0.\end{aligned}$$

Whence

$$\|D(t)\|_{L^2(\mathbb{R}^n)} \le C(t+1)^{-\frac{n}{4\theta}-\min\{\frac{1}{2\theta},\frac{n+2}{8\theta}-\frac{1}{2}\}}, \quad \forall t>0.$$

Applying the standard weak convergence theorem to the estimates on the approximate solution $u=u_m$ in the above Step 1 to Step 3, we finish the proof of Theorem 1.1. □

3. Decay rates of the 2D Navier-Stokes flows

In this section, we first recall some useful known results on the linear fractional heat equation in $\mathbb{R}^2$, which can be found in [20]. Namely,

Lemma 3.1. *Let $1\le r\le q\le\infty$, $0<\theta\le 1$. Then for any $t>0$, the operators $e^{-t\Lambda^{2\theta}}$ and $\nabla e^{-t\Lambda^{2\theta}}$ are bounded operators from $L^r(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$. Furthermore, we have for any $f\in L^r(\mathbb{R}^2)$,*

$$\|e^{-t\Lambda^{2\theta}}f\|_{L^q(\mathbb{R}^2)} \le Ct^{-\frac{1}{\theta}(\frac{1}{r}-\frac{1}{q})}\|f\|_{L^r(\mathbb{R}^2)},$$

and

$$\|\nabla e^{-t\Lambda^{2\theta}}f\|_{L^q(\mathbb{R}^2)} \le Ct^{-\frac{1}{2\theta}-\frac{1}{\theta}(\frac{1}{r}-\frac{1}{q})}\|f\|_{L^r(\mathbb{R}^2)},$$

where C is a constant depending on θ, r and q only.

Proof of Theorem 1.2. It follows from (A.2), (A.3) in Appendix that there exists a classical approximate solution $(u,p)=(u_m,p_m)$ to the following problem with initial data $a=a_m$.

$$\partial_t u + \Lambda^{2\theta}u + (u\cdot\nabla)u + \nabla p = 0, \quad \nabla\cdot u = 0, \quad u(0)=a.$$

Step 1. $\|\nabla u(t)\|_{L^2(\mathbb{R}^2)} \le C(t+1)^{-\frac{1}{\theta}}$, $\forall t>0$.

Let $\omega(x,t)=\partial_2 u_1-\partial_1 u_2$ denote the vorticity of the two-dimensional velocity field $u=(u_1,u_2)$. A direct calculation shows

$$\partial_t\omega + \Lambda^{2\theta}\omega + (u\cdot\nabla)\omega = 0, \tag{3.1}$$

$$\omega(x,0) = \partial_2 a_1 - \partial_1 a_2, \quad a=(a_1,a_2)\in H^1(\mathbb{R}^2). \tag{3.2}$$

The energy equality arising from (3.1) can be written as follows: For $t>0$

$$\frac{d}{dt}\|\omega(t)\|^2_{L^2(\mathbb{R}^2)} + 2\|\Lambda^\theta\omega(t)\|^2_{L^2(\mathbb{R}^2)} = 0,$$

and then

$$\frac{d}{dt}\int_{\mathbb{R}^2}|\widehat{\omega}(\xi,t)|^2d\xi + 2\int_{\mathbb{R}^2}|\xi|^{2\theta}|\widehat{\omega}(\xi,t)|^2d\xi = 0. \tag{3.3}$$

Using (3.3) yields for each $t > 0$

$$\frac{d}{dt}\Big(f(t)\int_{\mathbb{R}^2}|\widehat{\omega}(\xi,t)|^2d\xi\Big)+2f(t)\int_{\mathbb{R}^2}|\xi|^{2\theta}|\widehat{\omega}(\xi,t)|^2d\xi = f'(t)\int_{\mathbb{R}^2}|\widehat{\omega}(\xi,t)|^2d\xi, \tag{3.4}$$

where the function $f(t)$ is a continuous differentiable function for $t \geq 0$, which satisfies: $f(0) = 1$, $f'(t) > 0$, $\forall t > 0$.

Observe that for $t > 0$

$$\begin{aligned} 2f(t)\int_{\mathbb{R}^2}|\xi|^{2\theta}|\widehat{\omega}(\xi,t)|^2d\xi &\geq 2f(t)\int_{B(t)^c}|\xi|^{2\theta}|\widehat{\omega}(\xi,t)|^2d\xi \\ &\geq f'(t)\int_{B(t)^c}|\widehat{\omega}(\xi,t)|^2d\xi \\ &= f'(t)\int_{\mathbb{R}^2}|\widehat{\omega}(\xi,t)|^2d\xi - f'(t)\int_{B(t)}|\widehat{\omega}(\xi,t)|^2d\xi, \end{aligned} \tag{3.5}$$

where $B(t) = \{\xi \in \mathbb{R}^2 : 2f(t)|\xi|^{2\theta} \leq f'(t)\}$.

Inserting (3.5) into (3.4), we obtain for $t > 0$

$$\frac{d}{dt}\Big(f(t)\int_{\mathbb{R}^2}|\widehat{\omega}(\xi,t)|^2d\xi\Big) \leq f'(t)\int_{B(t)}|\widehat{\omega}(\xi,t)|^2d\xi. \tag{3.6}$$

The fact of $\nabla \cdot u = 0$ yields for $t > 0$

$$\widehat{(u\cdot\nabla)\omega}(\xi,t) = \sum_{j=1}^{2}\widehat{\partial_j(u_j\omega)}(\xi,t) = \sum_{j=1}^{2} i\xi_j\widehat{u_j\omega}(\xi,t),$$

and then

$$\begin{aligned} |\widehat{(u\cdot\nabla)\omega}(\xi,t)| &\leq |\xi|\int_{\mathbb{R}^2}|u(x,t)||\omega(x,t)|dx \\ &\leq |\xi|\|u(t)\|_{L^2(\mathbb{R}^2)}\|\omega(t)\|_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{3.7}$$

Applying the Fourier transform to the equation (3.1), we find for $t > 0$

$$\widehat{\omega}_t + |\xi|^{2\theta}\widehat{\omega} + \widehat{(u\cdot\nabla)\omega} = 0,$$

and

$$\widehat{\omega}(\xi,t) = e^{-|\xi|^{2\theta}t}\widehat{\omega}(\xi,0) - \int_0^t e^{-|\xi|^{2\theta}(t-s)}\widehat{(u\cdot\nabla)\omega}(\xi,s)ds. \tag{3.8}$$

Recall the proved result for any $\tau > 0$

$$\|u(\tau)\|_{L^2(\mathbb{R}^2)} \leq C(1+\tau)^{-\frac{1}{2\theta}}.$$

Since

$$|\widehat{\omega}(\xi,0)| = |\widehat{\partial_2 a_1}(\xi) - \widehat{\partial_1 a_2}(\xi)| = |i\xi_2\widehat{a_1}(\xi) - i\xi_1\widehat{a_2}(\xi)| \leq |\xi||\hat{a}(\xi)| \leq 2|\xi|\|a\|_{L^1(\mathbb{R}^2)},$$

inserting (3.7) into (3.8) yields for $t > 0$

$$|\widehat{\omega}(\xi,t)| \le e^{-|\xi|^{2\theta}t}|\widehat{\omega}(\xi,0)| + |\xi|\int_0^t \|u(\tau)\|_{L^2(\mathbb{R}^2)}\|\omega(\tau)\|_{L^2(\mathbb{R}^2)}d\tau \tag{3.9}$$
$$\le 2|\xi|\|a\|_{L^1(\mathbb{R}^2)} + C|\xi|\int_0^t (1+\tau)^{-\frac{1}{2\theta}}\|\omega(\tau)\|_{L^2(\mathbb{R}^2)}d\tau.$$

Combining (3.6) and (3.9), we deduce for $t > 0$

$$\frac{d}{dt}\Big(f(t)\int_{\mathbb{R}^2}|\widehat{\omega}(\xi,t)|^2 d\xi\Big) \tag{3.10}$$
$$\le Cf'(t)\Big(\int_{B(t)}|\xi|^2 d\xi + \int_{B(t)}|\xi|^2 d\xi\Big(\int_0^t (1+\tau)^{-\frac{1}{2\theta}}\|\omega(\tau)\|_{L^2(\mathbb{R}^2)}d\tau\Big)^2\Big)$$
$$\le Cf'(t)\Big(\Big(\frac{f'(t)}{f(t)}\Big)^{\frac{2}{\theta}} + \Big(\frac{f'(t)}{f(t)}\Big)^{\frac{2}{\theta}}\Big(\int_0^t (1+\tau)^{-\frac{1}{2\theta}}\|\omega(\tau)\|_{L^2(\mathbb{R}^2)}d\tau\Big)^2\Big).$$

Integrating on time t on both sides in (3.10) yields for $t > 0$

$$\int_{\mathbb{R}^2}|\widehat{\omega}(\xi,t)|^2 d\xi \le \frac{1}{f(t)}\int_{\mathbb{R}^2}|\widehat{\omega}(\xi,0)|^2 d\xi + C\frac{L(t)}{f(t)}, \tag{3.11}$$

where

$$L(t) = \int_0^t f'(s)\Big(\Big(\frac{f'(s)}{f(s)}\Big)^{\frac{2}{\theta}} + \Big(\frac{f'(s)}{f(s)}\Big)^{\frac{2}{\theta}}\Big(\int_0^s (1+\tau)^{-\frac{1}{2\theta}}\|\omega(\tau)\|_{L^2(\mathbb{R}^2)}d\tau\Big)^2\Big)ds. \tag{3.12}$$

Using the energy equality for $t > 0$

$$\|\omega(t)\|^2_{L^2(\mathbb{R}^2)} + 2\int_0^t \|\Lambda^\theta\omega(s)\|^2_{L^2(\mathbb{R}^2)}ds = \|\omega(0)\|^2_{L^2(\mathbb{R}^2)},$$

and taking $f(s) = (1+s)^\alpha$ in (3.12) for large number $\alpha > 0$, we have for $t > 0$

$$L(t) \le C\int_0^t (1+s)^{\alpha-1}\Big((1+s)^{-\frac{2}{\theta}} + (1+s)^{-\frac{2}{\theta}}\Big(\int_0^s (1+\tau)^{-\frac{1}{2\theta}}d\tau\Big)^2\Big)ds \tag{3.13}$$
$$\le C(1+t)^{\alpha-\frac{2}{\theta}} + \begin{cases} C(1+t)^{\alpha+2-\frac{3}{\theta}} & \text{if} \quad \theta > \frac{1}{2}, \\ C(1+t)^{\alpha-4}[\log(1+t)]^2 & \text{if} \quad \theta = \frac{1}{2}, \\ C(1+t)^{\alpha-\frac{2}{\theta}} & \text{if} \quad \theta < \frac{1}{2} \end{cases}$$
$$\le \begin{cases} C(1+t)^{\alpha+2-\frac{3}{\theta}} & \text{if} \quad \theta > \frac{1}{2}, \\ C(1+t)^{\alpha-4}[\log(1+t)]^2 & \text{if} \quad \theta = \frac{1}{2}, \\ C(1+t)^{\alpha-\frac{2}{\theta}} & \text{if} \quad \theta < \frac{1}{2}. \end{cases}$$

From (3.11) and (3.13), we get for $t > 0$

$$\|\omega(t)\|^2_{L^2(\mathbb{R}^2)} \le \begin{cases} C(1+t)^{2-\frac{3}{\theta}} & \text{if} \quad \theta > \frac{1}{2}, \\ C(1+t)^{-4}[\log(1+t)]^2 & \text{if} \quad \theta = \frac{1}{2}, \\ C(1+t)^{-\frac{2}{\theta}} & \text{if} \quad \theta < \frac{1}{2}. \end{cases}$$

Then a direct calculation shows

$$\int_0^\infty (1+\tau)^{-\frac{1}{2\theta}} \|\omega(\tau)\|_{L^2(\mathbb{R}^2)} d\tau < \infty. \tag{3.14}$$

By the choice of $f(s) = (1+s)^\alpha$ for large number $\alpha > 0$, it follows from (3.12), (3.14) that for $t > 0$

$$\begin{aligned} &L(t) \\ &\le C\int_0^t (1+s)^{\alpha-1}\Big((1+s)^{-\frac{2}{\theta}} + (1+s)^{-\frac{2}{\theta}}\Big(\int_0^\infty (1+\tau)^{-\frac{1}{2\theta}}\|\omega(\tau)\|_{L^2(\mathbb{R}^2)}d\tau\Big)^2\Big)ds \\ &\le C\int_0^t (1+s)^{\alpha-1-\frac{2}{\theta}}ds \\ &\le C(1+t)^{\alpha-\frac{2}{\theta}}. \end{aligned} \tag{3.15}$$

Combining (3.11) and (3.15), we obtain for $t > 0$

$$\|\omega(t)\|^2_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\alpha}(1+L(t)) \le C(1+t)^{-\frac{2}{\theta}}. \tag{3.16}$$

On the other hand, since $\partial_1 u_1 + \partial_2 u_2 = 0$, we get for $t > 0$

$$\begin{aligned} \|\omega(t)\|^2_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}^2} |\partial_2 u_1 - \partial_1 u_2|^2 dx \\ &= \int_{\mathbb{R}^2} (|\partial_2 u_1|^2 + |\partial_1 u_2|^2)dx - 2\int_{\mathbb{R}^2} \partial_2 u_1 \partial_1 u_2 dx \\ &= \int_{\mathbb{R}^2} (|\partial_2 u_1|^2 + |\partial_1 u_2|^2)dx - 2\int_{\mathbb{R}^2} \partial_1 u_1 \partial_2 u_2 dx \\ &= \int_{\mathbb{R}^2} (|\partial_2 u_1|^2 + |\partial_1 u_2|^2)dx + \int_{\mathbb{R}^2} |\partial_1 u_1|^2 dx + \int_{\mathbb{R}^2} |\partial_2 u_2|^2 dx \\ &= \|\nabla u(t)\|^2_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{3.17}$$

Combining (3.16) and (3.17) yields for $t > 0$

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{-\frac{1}{\theta}}.$$

Step 2. $\|u(t)\|_{L^r(\mathbb{R}^2)} \le C(1+t)^{-\frac{1}{\theta}(1-\frac{1}{r})}$, $\forall t > 0$, $1 \le r < \infty$.

By the interpolation inequality, together with the result of Step 1, we have for each $2 < r < \infty$, $t > 0$

$$\begin{aligned} \|u(t)\|_{L^r(\mathbb{R}^2)} &\le C\|u(t)\|^{\frac{2}{r}}_{L^2(\mathbb{R}^2)}\|\nabla u(t)\|^{1-\frac{2}{r}}_{L^2(\mathbb{R}^2)} \\ &\le C(1+t)^{-\frac{1}{2\theta}\times\frac{2}{r}-\frac{1}{\theta}(1-\frac{2}{r})} = C(1+t)^{-\frac{1}{\theta}(1-\frac{1}{r})}. \end{aligned} \tag{3.18}$$

Now we treat the case of $r = 1$. Let $1 < r < \infty$, it is known that $P = P_r$ is the bounded projection from L^r to the closed subspace L^r_σ of L^r consisting of all the solenoidal vector fields, not bounded for $r = 1$ any more. In fact, using the Fourier transform, the projection P can be represented as follows:

$$\widehat{Pf}(\xi) = (I - \frac{\xi \otimes \xi}{|\xi|^2})\hat{f}(\xi),$$

where I denotes the unit $n \times n$ matrix, $\xi \otimes \xi = (\xi_j \xi_k)_{n\times n}$.

By the inverse Fourier transform, we find $(Pf)_j = \sum_{k=1}^{n}(\delta_{jk} + R_j R_k f_k)$ $(j = 1, 2, \dots, n)$. This shows that the projection P contains the Riesz operators R_j $(j = 1, 2, \dots, n)$, which is defined by

$$\widehat{R_j f}(\xi) = \frac{i\xi_j}{|\xi|}\hat{f}(\xi), \quad i = \sqrt{-1}.$$

The situation becomes complicated and challenging because using Lemma 3.1 will inevitably produces the crucial term $\|P(u\cdot\nabla)u\|_{L^1}$. In order to overcome this difficulty, we use the Hardy space $\mathcal{H}^1$ to replace L^1. We first recall the definition of the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$.

A function $f \in L^1(\mathbb{R}^n)$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ if $\sup\limits_{s>0}|G_s * f| \in L^1(\mathbb{R}^n)$, where the symbol $*$ denotes the convolution with respect to the space variable, G_s is the Gaussian kernel for $s > 0$: $G_s(x) = (4\pi s)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4s}} \in \mathscr{S}(\mathbb{R}^n)$, which denotes Schwartz space of rapidly decreasing smooth functions in $\mathbb{R}^n$. $\mathcal{H}^1(\mathbb{R}^n)$ is a Banach space, the norm of $f \in \mathcal{H}^1(\mathbb{R}^n)$ is defined by

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} := \|\sup_{s>0}|G_s * f|\|_{L^1(\mathbb{R}^n)},$$

and the injection $\mathcal{H}^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$ is continuous. Importantly, P is bounded on $\mathcal{H}^1(\mathbb{R}^n)$.

To proceed, we additionally need an important inequality due to [8], which can be used to treat effectively the convection term $(u\cdot\nabla)u$ on Hardy space $\mathcal{H}^1(\mathbb{R}^n)$.

Let $n \geq 2$, $u \in L^2_\sigma(\mathbb{R}^n)$ and $\nabla v \in L^2(\mathbb{R}^n)$. Then $(u\cdot\nabla)v \in \mathcal{H}^1(\mathbb{R}^n)$, and we have the estimate

$$\|(u\cdot\nabla)v\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C\|u\|_{L^2(\mathbb{R}^n)}\|\nabla v\|_{L^2(\mathbb{R}^n)}, \tag{3.19}$$

with C independent of u and v.

Using (3.19) yields for every $u \in L^2_\sigma(\mathbb{R}^2)$ and $\nabla v \in L^2(\mathbb{R}^2)$

$$\begin{aligned} \|P(u\cdot\nabla)v\|_{L^1(\mathbb{R}^2)} &\leq \|P(u\cdot\nabla)v\|_{\mathcal{H}^1(\mathbb{R}^2)} \\ &\leq C\|(u\cdot\nabla)v\|_{\mathcal{H}^1(\mathbb{R}^2)} \\ &\leq C\|u\|_{L^2(\mathbb{R}^2)}\|\nabla v\|_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{3.20}$$

Note that

$$u(t) = e^{-t\Lambda^{2\theta}}a - \int_0^t e^{-(t-s)\Lambda^{2\theta}}[P(u\cdot\nabla)u](s)ds, \ t > 0. \tag{3.21}$$

Using Lemma 3.1, Theorem 1.1, the result of Step 1, (3.20) and (3.21), we have for $t>0$

(3.22)

$$\begin{aligned}\|u(t)\|_{L^1(\mathbb{R}^2)} &\le \|e^{-t\Lambda^{2\theta}}a\|_{L^1(\mathbb{R}^2)} + \int_0^t \|e^{-(t-s)\Lambda^{2\theta}}[P(u\cdot\nabla)u](s)\|_{L^1(\mathbb{R}^2)}ds\\ &\le C\|a\|_{L^1(\mathbb{R}^2)} + C\int_0^t \|P[(u\cdot\nabla)u](s)\|_{L^1(\mathbb{R}^2)}ds\\ &\le C\|a\|_{L^1(\mathbb{R}^2)} + C\int_0^t \|u(s)\|_{L^2(\mathbb{R}^2)}\|\nabla u(s)\|_{L^2(\mathbb{R}^2)}ds\\ &\le C\|a\|_{L^1(\mathbb{R}^2)} + C\int_0^t (1+s)^{-\frac{3}{2\theta}}ds \le C.\end{aligned}$$

It follows from Theorem 1.1 and (3.22) that for $1<r<2$ and $t>0$

$$\|u(t)\|_{L^r(\mathbb{R}^2)} \le \|u(t)\|_{L^1(\mathbb{R}^2)}^{\frac{2}{r}-1}\|u(t)\|_{L^2(\mathbb{R}^2)}^{2-\frac{2}{r}} \le C(1+t)^{-\frac{1}{\theta}(1-\frac{1}{r})}. \tag{3.23}$$

By (3.18), (3.22) and (3.23), we complete the proof of Step 2.

Step 3. Let $\theta>\frac{1}{2}$. Then there exists a $t_0>0$ such that for $t>t_0$,

$$\|u(t)\|_{L^\infty(\mathbb{R}^2)} \le Ct^{-\frac{1}{\theta}},\quad \|\nabla u(t)\|_{L^r(\mathbb{R}^2)} \le Ct^{-\frac{1}{2\theta}-\frac{1}{\theta}(1-\frac{1}{r})},\quad 1\le r<\infty.$$

From Lemma 3.1, Theorem 1.1, the result of Step 1 and (3.20), we conclude for $t>0$

(3.24)

$$\begin{aligned}\|u(t)\|_{L^\infty(\mathbb{R}^2)} &\le \|e^{-t\Lambda^{2\theta}}a\|_{L^\infty(\mathbb{R}^2)} + \int_0^t \|e^{-(t-s)\Lambda^{2\theta}}[P(u\cdot\nabla)u](s)\|_{L^\infty(\mathbb{R}^2)}ds\\ &\le Ct^{-\frac{1}{\theta}}\|a\|_{L^1(\mathbb{R}^2)} + C\int_0^{\frac{t}{2}}(t-s)^{-\frac{1}{\theta}}\|P[(u\cdot\nabla)u](s)\|_{L^1(\mathbb{R}^2)}ds\\ &\quad + C\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2\theta}}\|P[(u\cdot\nabla)u](s)\|_{L^2(\mathbb{R}^2)}ds\\ &\le Ct^{-\frac{1}{\theta}} + Ct^{-\frac{1}{\theta}}\int_0^{\frac{t}{2}}\|u(s)\|_{L^2(\mathbb{R}^2)}\|\nabla u(s)\|_{L^2(\mathbb{R}^2)}ds\\ &\quad + C\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2\theta}}\|u(s)\|_{L^\infty(\mathbb{R}^2)}\|\nabla u(s)\|_{L^2(\mathbb{R}^2)}ds\\ &\le Ct^{-\frac{1}{\theta}} + Ct^{-\frac{1}{\theta}}\int_0^\infty (1+s)^{-\frac{3}{2\theta}}ds\\ &\quad + C\sup_{s>\frac{t}{2}}[s^{\frac{1}{\theta}}\|u(s)\|_{L^\infty(\mathbb{R}^2)}]\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2\theta}}s^{-\frac{1}{\theta}}(1+s)^{-\frac{1}{\theta}}ds\\ &\le Ct^{-\frac{1}{\theta}} + C_0\sup_{s>\frac{t}{2}}[s^{\frac{1}{\theta}}\|u(s)\|_{L^\infty(\mathbb{R}^2)}]t^{1-\frac{3}{2\theta}}(1+t)^{-\frac{1}{\theta}},\end{aligned}$$

where $1-\frac{1}{2\theta}>0$ for $\theta>\frac{1}{2}$, and C_0 is independent of t,u.

The estimate (3.24) implies for any $t>0$

$$t^{\frac{1}{\theta}}\|u(t)\|_{L^\infty(\mathbb{R}^2)}\le C+C_0t^{1-\frac{3}{2\theta}}\sup_{s>\frac{t}{2}}[s^{\frac{1}{\theta}}\|u(s)\|_{L^\infty(\mathbb{R}^2)}]. \tag{3.25}$$

It follows from (3.25) that for any $t>t_0$ (determined later)

$$t^{\frac{1}{\theta}}\|u(t)\|_{L^\infty(\mathbb{R}^2)}\le C+C_0t_0^{1-\frac{3}{2\theta}}\sup_{s>\frac{t_0}{2}}[s^{\frac{1}{\theta}}\|u(s)\|_{L^\infty(\mathbb{R}^2)}].$$

Furthermore,

$$\sup_{t>t_0}[t^{\frac{1}{\theta}}\|u(t)\|_{L^\infty(\mathbb{R}^2)}]\le C+C_0t_0^{1-\frac{3}{2\theta}}\sup_{s>\frac{t_0}{2}}[s^{\frac{1}{\theta}}\|u(s)\|_{L^\infty(\mathbb{R}^2)}]. \tag{3.26}$$

On the other hand, from (3.25), we have for $\frac{t_0}{2}<t\le t_0$

$$t^{\frac{1}{\theta}}\|u(t)\|_{L^\infty(\mathbb{R}^2)}\le C+C_02^{\frac{3}{2\theta}-1}t_0^{1-\frac{3}{2\theta}}\sup_{s>\frac{t_0}{2}}[s^{\frac{1}{\theta}}\|u(s)\|_{L^\infty(\mathbb{R}^2)}],$$

from which,

$$\sup_{t_0>t>\frac{t_0}{2}}[t^{\frac{1}{\theta}}\|u(t)\|_{L^\infty(\mathbb{R}^2)}]\le C+C_02^{\frac{3}{2\theta}-1}t_0^{1-\frac{3}{2\theta}}\sup_{s>\frac{t_0}{2}}[s^{\frac{1}{\theta}}\|u(s)\|_{L^\infty(\mathbb{R}^2)}]. \tag{3.27}$$

Combining (3.26) and (3.27), we conclude

$$\sup_{t>\frac{t_0}{2}}[t^{\frac{1}{\theta}}\|u(t)\|_{L^\infty(\mathbb{R}^2)}]\le 2C+C_0(1+2^{\frac{3}{2\theta}-1})t_0^{1-\frac{3}{2\theta}}\sup_{s>\frac{t_0}{2}}[s^{\frac{1}{\theta}}\|u(s)\|_{L^\infty(\mathbb{R}^2)}]. \tag{3.28}$$

Take $t_0>1$ in (3.28), such that $C_0(1+2^{\frac{3}{2\theta}-1})t_0^{1-\frac{3}{2\theta}}\le\frac{1}{2}$. Then it follows from (3.28) that

$$\sup_{t>\frac{t_0}{2}}[t^{\frac{1}{\theta}}\|u(t)\|_{L^\infty(\mathbb{R}^2)}]\le 4C,$$

which implies that for any $t>t_0$

$$\|u(t)\|_{L^\infty(\mathbb{R}^2)}\le Ct^{-\frac{1}{\theta}}.$$

Similar to the proof of (3.24), we have for $t>0$

$$\begin{aligned}\|\nabla u(t)\|_{L^1(\mathbb{R}^2)}&\le\|\nabla e^{-t\Lambda^{2\theta}}a\|_{L^1(\mathbb{R}^2)}+\int_0^t\|\nabla e^{-(t-s)\Lambda^{2\theta}}[P(u\cdot\nabla)u](s)\|_{L^1(\mathbb{R}^2)}ds\\&\le Ct^{-\frac{1}{2\theta}}\|a\|_{L^1(\mathbb{R}^2)}+C\int_0^t(t-s)^{-\frac{1}{2\theta}}\|P[(u\cdot\nabla)u](s)\|_{L^1(\mathbb{R}^2)}ds\\&\le Ct^{-\frac{1}{2\theta}}+Ct^{-\frac{1}{2\theta}}\int_0^{\frac{t}{2}}\|u(s)\|_{L^2(\mathbb{R}^2)}\|\nabla u(s)\|_{L^2(\mathbb{R}^2)}ds\\&\quad+C\sup_{\frac{t}{2}<s<t}[\|u(s)\|_{L^2(\mathbb{R}^2)}\|\nabla u(s)\|_{L^2(\mathbb{R}^2)}]\int_{\frac{t}{2}}^t(t-s)^{-\frac{1}{2\theta}}ds\end{aligned}$$

$$\leq Ct^{-\frac{1}{2\theta}} + Ct^{-\frac{1}{2\theta}}\int_0^\infty (1+s)^{-\frac{3}{2\theta}}ds + C(t+1)^{-\frac{3}{2\theta}}t^{1-\frac{1}{2\theta}}$$
$$\leq \overline{C}t^{-\frac{1}{2\theta}}.$$

Let $1 < r < \infty$. Then for $t > 2t_0$

$$\|\nabla u(t)\|_{L^r(\mathbb{R}^2)}$$
$$\leq \|\nabla e^{-t\Lambda^{2\theta}}a\|_{L^r(\mathbb{R}^2)} + \int_0^t \|\nabla e^{-(t-s)\Lambda^{2\theta}}[P(u\cdot\nabla)u](s)\|_{L^r(\mathbb{R}^2)}ds$$
$$\leq Ct^{-\frac{1}{2\theta}-\frac{1}{\theta}(1-\frac{1}{r})}\|a\|_{L^1(\mathbb{R}^2)} + C\int_0^{\frac{t}{2}}(t-s)^{-\frac{1}{2\theta}-\frac{1}{\theta}(1-\frac{1}{r})}\|P[(u\cdot\nabla)u](s)\|_{L^1(\mathbb{R}^2)}ds$$
$$+ C\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2\theta}}\|P[(u\cdot\nabla)u](s)\|_{L^r(\mathbb{R}^2)}ds$$
$$\leq Ct^{-\frac{1}{2\theta}-\frac{1}{\theta}(1-\frac{1}{r})} + Ct^{-\frac{1}{2\theta}-\frac{1}{\theta}(1-\frac{1}{r})}\int_0^{\frac{t}{2}}\|u(s)\|_{L^2(\mathbb{R}^2)}\|\nabla u(s)\|_{L^2(\mathbb{R}^2)}ds$$
$$+ C\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2\theta}}\|u(s)\|_{L^\infty(\mathbb{R}^2)}\|\nabla u(s)\|_{L^r(\mathbb{R}^2)}ds$$
$$\leq Ct^{-\frac{1}{2\theta}-\frac{1}{\theta}(1-\frac{1}{r})} + Ct^{-\frac{1}{2\theta}-\frac{1}{\theta}(1-\frac{1}{r})}\int_0^\infty (1+s)^{-\frac{3}{2\theta}}ds$$
$$+ C\sup_{s>\frac{t}{2}}[s^{\frac{1}{2\theta}+\frac{1}{\theta}(1-\frac{1}{r})}\|\nabla u(s)\|_{L^r(\mathbb{R}^2)}]t^{-\frac{3}{2\theta}-\frac{1}{\theta}(1-\frac{1}{r})}\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2\theta}}ds$$
$$\leq Ct^{-\frac{1}{2\theta}-\frac{1}{\theta}(1-\frac{1}{r})} + C\sup_{s>\frac{t}{2}}[s^{\frac{1}{2\theta}+\frac{1}{\theta}(1-\frac{1}{r})}\|\nabla u(s)\|_{L^r(\mathbb{R}^2)}]t^{1-\frac{3}{2\theta}-\frac{1}{2\theta}-\frac{1}{\theta}(1-\frac{1}{r})}.$$

Similar to the proof of (3.28), we get for $t > t_1 > 2t_0$

$$\sup_{t>\frac{t_1}{2}}[t^{\frac{1}{2\theta}+\frac{1}{\theta}(1-\frac{1}{r})}\|\nabla u(t)\|_{L^r(\mathbb{R}^2)}] \leq C + C_1 t_1^{1-\frac{3}{2\theta}}\sup_{s>\frac{t_1}{2}}[s^{\frac{1}{2\theta}+\frac{1}{\theta}(1-\frac{1}{r})}\|\nabla u(s)\|_{L^r(\mathbb{R}^2)}].$$

Whence there exists $t_1 > 2t_0$ such that $C_1 t_1^{1-\frac{3}{2\theta}} \leq \frac{1}{2}$, and then

$$\|\nabla u(t)\|_{L^r(\mathbb{R}^2)} \leq Ct^{-\frac{1}{2\theta}-\frac{1}{\theta}(1-\frac{1}{r})}, \quad \forall t > t_1.$$

Step 4. $u \in L^2_{loc}(0,\infty,H^{1+\theta}(\mathbb{R}^2))$, $\partial_t u$, $\nabla p \in L^1_{loc}(0,\infty,L^2(\mathbb{R}^2))$.

Note that the solution $u = (u_1, u_2)$ satisfies the following Biot-Savart formula

$$u = \nabla^\perp(-\Delta)^{-1}\omega = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} * \omega,$$

where $\nabla^\perp = (\partial_2, -\partial_1)$, $x^\perp = (x_2, -x_1)$, $\omega = \partial_2 u_1 - \partial_1 u_2$, $*$ denotes the convolution operator.

Using Fourier transform property, we have

$$\widehat{u}(\xi) = \frac{1}{2\pi}\widehat{\frac{x^\perp}{|x|^2}}(\xi)\hat{\omega}(\xi).$$

Whence

$$\widehat{\Lambda^\theta u}(\xi) = |\xi|^\theta \hat{u}(\xi) = \frac{1}{2\pi}\widehat{\frac{x^\perp}{|x|^2}}(\xi)|\xi|^\theta \hat{\omega}(\xi) = \frac{1}{2\pi}\widehat{\frac{x^\perp}{|x|^2}}(\xi)\widehat{\Lambda^\theta \omega}(\xi).$$

From which,

$$\Lambda^\theta u = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} * (\Lambda^\theta \omega).$$

By means of Calderon-Zygmund inequality (see [18]), we get

$$\int_0^T \|\nabla \Lambda^\theta u(s)\|^2_{L^r(\mathbb{R}^2)} ds \le C \int_0^T \|\Lambda^\theta \omega(s)\|^2_{L^r(\mathbb{R}^2)} ds, \ 1 < r < \infty.$$

In particular,

$$\begin{aligned} \int_0^T \|\nabla \Lambda^\theta u(s)\|^2_{L^2(\mathbb{R}^2)} ds &\le C \int_0^T \|\Lambda^\theta \omega(s)\|^2_{L^2(\mathbb{R}^2)} ds \\ &\le C\|\omega(0)\|^2_{L^2(\mathbb{R}^2)} \le C\|\nabla a\|^2_{L^2(\mathbb{R}^2)}. \end{aligned} \tag{3.29}$$

Using the Sobolev imbedding theorem: $H^{1+\theta}(\mathbb{R}^2) \hookrightarrow C^\theta(\mathbb{R}^2)$ for every $0 < \theta < 1$, together with (3.29), we find for each $0 < T < \infty$

$$\begin{aligned} \int_0^T \|u(s)\|^2_{L^\infty(\mathbb{R}^2)} ds &\le C \int_0^T \|u(s)\|^2_{H^{1+\theta}(\mathbb{R}^2)} ds \\ &= C \int_0^T (\|u(s)\|^2_{L^2(\mathbb{R}^2)} ds + \|\nabla \Lambda^\theta u(s)\|^2_{L^2(\mathbb{R}^2)}) ds \\ &\le C(T\|a\|^2_{L^2(\mathbb{R}^2)} + \|\nabla a\|^2_{L^2(\mathbb{R}^2)}) \le C(1+T)\|a\|^2_{H^1(\mathbb{R}^2)}. \end{aligned} \tag{3.30}$$

Furthermore,

$$\begin{aligned} \int_0^T \|(u\cdot\nabla)u\|_{L^2(\mathbb{R}^2)} ds &\le \Big(\int_0^T \|u(s)\|^2_{L^\infty(\mathbb{R}^2)} ds\Big)^{\frac12} \Big(\int_0^T \|\nabla u(s)\|^2_{L^2(\mathbb{R}^2)} ds\Big)^{\frac12} \\ &\le C(1+\sqrt{T})\|a\|^2_{H^1(\mathbb{R}^2)}. \end{aligned} \tag{3.31}$$

Note that the pressure p of (1.1) satisfies

$$-\Delta p = div((u\cdot\nabla)u).$$

Combining (3.31), we get

$$\begin{aligned} \int_0^T \|\nabla p(s)\|_{L^2(\mathbb{R}^2)} ds &\le C \int_0^T \|(u\cdot\nabla)u\|_{L^2(\mathbb{R}^2)} ds \\ &\le C(1+\sqrt{T})\|a\|^2_{H^1(\mathbb{R}^2)}. \end{aligned} \tag{3.32}$$

Whence, from (3.30)-(3.32), we conclude

$$\int_0^T \|\partial_t u(t)\|_{L^2(\mathbb{R}^2)} dt$$

$$\begin{aligned}
&\leq \int_0^T (\|\Lambda^{2\theta}u(t)\|_{L^2(\mathbb{R}^2)} + \|(u\cdot\nabla)u\|_{L^2(\mathbb{R}^2)} + \|\nabla p(t)\|_{L^2(\mathbb{R}^2)})dt \\
&\leq C\int_0^T \||\xi|^{2\theta}\hat{u}(t)\|_{L^2(\mathbb{R}^2)}dt + C(1+\sqrt{T})\|a\|^2_{H^1(\mathbb{R}^2)} \\
&\leq C\int_0^T \|(1+|\xi|^{1+\theta})\hat{u}(t)\|_{L^2(\mathbb{R}^2)}dt + C(1+\sqrt{T})\|a\|^2_{H^1(\mathbb{R}^2)} \\
&\leq C\int_0^T \|u(t)\|_{H^{1+\theta}(\mathbb{R}^2)}dt + C(1+\sqrt{T})\|a\|^2_{H^1(\mathbb{R}^2)}.
\end{aligned}$$

From the above arguments in Step 1 to Step 4, through a standard weak convergence and compact argument on the approximate solution $u = u_m$, we complete the proof of Theorem 1.2. □

Now we give the proof of the uniqueness of solutions of (1.1) by using Sobolev inequality, interpolation inequality and Gronwall inequality.

Proof of Theorem 1.3. Let $u \in L^\infty(0,\infty;H^1(\mathbb{R}^2)) \cap L^2_{loc}(0,\infty;H^{1+\theta}(\mathbb{R}^2))$ be the solution of (1.1), which is given in Theorem 1.2. Assume

$$v \in L^\infty(0,\infty;L^2(\mathbb{R}^2)) \cap L^2_{loc}(0,\infty;H^\theta(\mathbb{R}^2))$$

is another solution of (1.1) with the same initial data a. Set $w = v - u$. Then w satisfies in the sense of distributions

$$\partial_t w + \Lambda^{2\theta}w + (w\cdot\nabla)w + (u\cdot\nabla)w + (w\cdot\nabla)u + \nabla\pi = 0,\ w(0) = 0.$$

In the above equations on the weak solution w, taking u as the cut-off function. Then for every $t > 0$

$$\|w(t)\|^2_{L^2(\mathbb{R}^2)} + 2\int_0^t \|\Lambda^\theta w(s)\|^2_{L^2(\mathbb{R}^2)}ds + \int_0^t\int_{\mathbb{R}^2}(w\cdot\nabla)u\cdot w dxds = 0. \tag{3.33}$$

Using Sobolev inequality and interpolation inequality, we have for $\frac{1}{2} < \theta < 1$

$$\begin{aligned}
\|f\|_{L^4(\mathbb{R}^2)} &\leq \|f\|^{2-\frac{1}{\theta}}_{L^2(\mathbb{R}^2)}\|f\|^{\frac{1}{\theta}-1}_{L^{\frac{2}{1-\theta}}(\mathbb{R}^2)} \\
&\leq C\|f\|^{2-\frac{1}{\theta}}_{L^2(\mathbb{R}^2)}\|\Lambda^\theta f\|^{\frac{1}{\theta}-1}_{L^2(\mathbb{R}^2)}, \quad \forall f \in H^\theta(\mathbb{R}^2).
\end{aligned} \tag{3.34}$$

It has been verified that the solution u obtained in Theorem 1.2 satisfies

$$\sup_{t>0}\|\nabla u(t)\|_{L^2(\mathbb{R}^2)} \leq \|\nabla a\|_{L^2(\mathbb{R}^2)}.$$

Using (3.34) yields for $t > 0$

(3.35)

$$\begin{aligned}
\Big|\int_0^t\int_{\mathbb{R}^2}(w\cdot\nabla)u\cdot w dxds\Big| &\leq \sup_{s>0}\|\nabla u(s)\|_{L^2(\mathbb{R}^2)}\int_0^t \|w(s)\|^2_{L^4(\mathbb{R}^2)}ds \\
&\leq \|\nabla a\|_{L^2(\mathbb{R}^2)}\int_0^t \|w(s)\|^{2(2-\frac{1}{\theta})}_{L^2(\mathbb{R}^2)}\|\Lambda^\theta w(s)\|^{2(\frac{1}{\theta}-1)}_{L^2(\mathbb{R}^2)}ds
\end{aligned}$$

$$\leq \int_0^t \|\Lambda^\theta w(s)\|^2 ds + C\int_0^t \|w(s)\|^2_{L^2(\mathbb{R}^2)} ds.$$

Inserting (3.35) into (3.33) yields for $t > 0$

$$\|w(t)\|^2_{L^2(\mathbb{R}^2)} \leq C\int_0^t \|w(s)\|^2_{L^2(\mathbb{R}^2)} ds.$$

Note that $w(0) = 0$. Using the Gronwall inequality yields $w(t) = 0$. That is, $u = v$. □

Appendix

In this section, by using Galerkin method, weak and compact convergence theorems, we establish the global existence of weak solutions to problem (1.1).

Proposition A. *Let* $a \in L^2_\sigma(\mathbb{R}^n)$ $(n \geq 2)$ *and* $0 < \theta < 1$. *Then problem* (1.1) *admits a weak solution.*

Proof. We prove Proposition A in four steps.

Step 1. Existence of approximate solutions.

Since $H^\theta_\sigma(\mathbb{R}^n)$ is separable and $C^\infty_{0,\sigma}(\mathbb{R}^n)$ is dense in $H^\theta_\sigma(\mathbb{R}^n)$, a sequence of elements $e_1, e_2, \ldots, e_m, \ldots$ can be selected in $C^\infty_{0,\sigma}(\mathbb{R}^n)$, which is free, linearly independent and complete in $H^\theta_\sigma(\mathbb{R}^n)$.

For each integer m, and every $0 < T_1 < \infty$, define the approximate solution u_m of (1.1) as follows:

$$u_m = \sum_{i=1}^m g_{im}(t) e_i, \quad t \in (0, T_1), \tag{A.1}$$

where u_m satisfies for $j = 1, 2, \ldots,$

$$(u'_m(t), e_j) + (\Lambda^\theta u_m(t), \Lambda^\theta e_j) + b(u_m(t), u_m(t), e_j) = 0, \tag{A.2}$$

where $(\cdot,\cdot)$ denotes the inner product of $L^2(\mathbb{R}^n)$; the trilinear form of b is defined as follows:

$$b(u, v, e) = -\sum_{k,\ell=1}^n \int_{\mathbb{R}^n} u_k v_\ell \partial_k e_\ell dx.$$

The initial condition is given as follows:

$$u_m(0) = a_m, \tag{A.3}$$

where a_m is the orthogonal projection in $L^2_\sigma(\mathbb{R}^n)$ of a onto the space spanned by $e_1, e_2, \ldots, e_m$. That is,

$$a = a_m + a_m^\perp,$$

$$a_m \in \overline{span\ \{e_1, e_2, \ldots, e_m\}_{L^2}}, \quad a_m^\perp \in \overline{span\ \{e_1, e_2, \ldots, e_m\}_{L^2}}^\perp.$$

This implies $\|a_m\|_{L^2(\mathbb{R}^n)} \leq \|a\|_{L^2(\mathbb{R}^n)}$ and $\lim_{m\to\infty} \|a_m - a\|_{L^2(\mathbb{R}^n)} = 0$.

The equations (A.2) form a nonlinear differential system for the functions $g_{1m},\dots,g_{mm}$:

(A.4)
$$\sum_{i=1}^{m}(e_i,e_j)g'_{im}(t)+\sum_{i=1}^{m}(\Lambda^\theta e_i,\Lambda^\theta e_j)g_{im}(t)$$
$$+\sum_{i,\ell=1}^{m}b(e_i,e_\ell,e_j)g_{im}(t)g_{\ell m}(t)=0,\ j=1,2,\dots,m.$$

Set

$A=\big((e_i,e_j)\big)_{m\times m},\ D=\big((\Lambda^\theta e_i,\Lambda^\theta e_j)\big)_{m\times m},\ B_j=\big(b(e_i,e_\ell,e_j)\big)_{m\times m},\ 1\le j\le m;$

$$B=\begin{pmatrix}B_1\\B_2\\\vdots\\B_m\end{pmatrix},\ G(t)=\begin{pmatrix}g_{1m}(t)\\g_{2m}(t)\\\vdots\\g_{mm}(t)\end{pmatrix}_{m\times 1},\ G^T(t)BG(t)=\begin{pmatrix}G^T(t)B_1G(t)\\G^T(t)B_2G(t)\\\vdots\\G^T(t)B_mG(t)\end{pmatrix}_{m\times 1},$$

where $G^T(t)$ denotes the transpose of $G(t)$:
$$G^T(t)=(g_{1m}(t),g_{2m}(t),\dots,g_{mm}(t)).$$
Then (A.4) can be written as follows:
$$AG'(t)+DG(t)+G^T(t)BG(t)=0.$$
Now we show the matrix A is positive, that is, $A>0$. In fact, for any $X=(x_1,x_2,\dots,x_m)^T\neq 0$, $Y=(y_1,y_2,\dots,y_m)^T\neq 0$. Set $\alpha=\sum_{k=1}^m x_ke_k$, $\beta=\sum_{k=1}^m y_ke_k$. Then $\alpha=\beta$ if and only if $X=Y$ due to the linear independence of $e_1,e_2,\dots,e_m$. Moreover, $\alpha\neq 0$ for $X\neq 0$. Additionally,
$$(\alpha,\beta)=(\sum_{k=1}^m x_ke_k,\sum_{j=1}^m y_je_j)=\sum_{k,j=1}^m x_ky_j(e_k,e_j)=X^TAY.$$
In particular,
$$X^TAX=(\alpha,\alpha)>0\quad\text{for any}\quad X\neq 0,$$
which implies $A>0$.

Inverting the nonsingular matrix A, we write the above differential system in the usual form

(A.5)
$$G'(t)+A^{-1}DG(t)+A^{-1}G^T(t)BG(t)=0,\quad t\in[0,T_1].$$

On the other hand, it follows from (A.1) and (A.3) that $a_m=\sum_{i=1}^m g_{im}(0)e_i$, which implies for $j=1,2,\dots,m$
$$(a,e_j)=(a_m,e_j)+(a_m^\perp,e_j)=(a_m,e_j)=(\sum_{i=1}^m g_{im}(0)e_i,e_j)=\sum_{i=1}^m g_{im}(0)(e_i,e_j).$$
Whence the initial condition for the nonlinear differential system (A.5) is given as follows:

(A.6)
$$G(0)=\vec{c},$$

where $\vec{c} = (c_{1m}, c_{2m}, \dots, c_{mm})^T$, and $(c_{1m}, c_{2m}, \dots, c_{mm})$ is the unique root of the following linear algebraic system:

$$\sum_{i=1}^{m} \lambda_i (e_i, e_j) = (a, e_j), \quad j = 1, 2, \dots, m.$$

ODE theory tells us that problem (A.5), (A.6) admits a unique solution $G \in C[0, t_m] \cap C^1(0, t_m)$, where t_m is the maximum survival time for the existence of solution $G(t)$, that is, $t_m = \sup\{t; \sup_{0<s<t} |G(s)| < \infty\}$. Since $A > 0$, there exist $d_k > 0$, $k = 1, 2, \dots, m$, and an $m \times m$ orthogonal matrix Q: $Q^T Q = QQ^T = I_{m\times m}$, such that

$$A = Q^T \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_m \end{pmatrix} Q.$$

Note that

$$\begin{aligned} \|u_m(t)\|^2_{L^2(\mathbb{R}^n)} &= \sum_{i,j=1}^{m} g_{im}(t) g_{jm}(t)(e_i, e_j) \\ &= G(t)^T A G(t) = [QG(t)]^T \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_m \end{pmatrix} QG(t) \\ &= \sum_{k=1}^{m} d_k [QG(t)]^2_k \ge d_0 \sum_{k=1}^{m} [QG(t)]^2_k \\ &= d_0 [QG(t)]^T [QG(t)] = d_0 \sum_{k=1}^{m} [g_k(t)]^2, \end{aligned}$$

where $d_0 = \min\{d_1, d_2, \dots, d_m\} > 0$.

Combining with the *a priori* estimate obtained in Step 2: $\|u_m(t)\|_{L^2(\mathbb{R}^n)} \le \|a\|_{L^2(\mathbb{R}^n)}$, we conclude that

$$\lim_{t\to t_m} \sum_{k=1}^{m} [g_k(t)]^2 \le d_0^{-1} \|a\|^2_{L^2(\mathbb{R}^n)},$$

which implies $t_m = T_1$. For the simplicity of writing, in the next arguments, we replace T_1 by T.

Step 2. The approximate energy equality.

Multiplying (A.2) by $g_{jm}(t)$ and add these equations for $j = 1, \dots, m$, we get

$$\frac{d}{dt} \|u_m(t)\|^2_{L^2(\mathbb{R}^n)} + 2\|\Lambda^\theta u_m(t)\|^2_{L^2(\mathbb{R}^n)} = 0. \tag{A.7}$$

Integrating (A.7) from 0 to t yields

$$\|u_m(t)\|^2_{L^2(\mathbb{R}^n)} + 2\int_0^t \|\Lambda^\theta u_m(s)\|^2_{L^2(\mathbb{R}^n)} ds \le \|a_m\|^2_{L^2(\mathbb{R}^n)} \le \|a\|^2_{L^2(\mathbb{R}^n)}.$$

From the above estimate, we obtain

$$\sup_{0<t<T} \|u_m(t)\|^2_{L^2(\mathbb{R}^n)} + 2\int_0^T \|\Lambda^\theta u_m(s)\|^2_{L^2(\mathbb{R}^n)} ds \le \|a\|^2_{L^2(\mathbb{R}^n)}. \tag{A.8}$$

Step 3. Estimate of fractional derivatives.

Suppose $\tilde{u}_m$ is a function from $\mathbb{R}$ to $H^\theta(\mathbb{R}^n)$, which is equal to u_m on $[0,T]$, and 0 outside of $[0,T]$. The Fourier transform on t of $\tilde{u}_m$ is denoted by $\hat{u}_m$. Namely,

$$\hat{\tilde{u}}_m(x,t) = \int_{-\infty}^{+\infty} e^{-ist}\tilde{u}_m(x,s)ds.$$

Now we prove that there exists a number $\gamma > 0$ such that

$$\int_{-\infty}^{+\infty} |\tau|^{2\gamma} \|\hat{\tilde{u}}_m(\tau)\|^2_{L^2(\mathbb{R}^n)} dt \le C. \tag{A.9}$$

In order to prove (A.9), we note that (A.2) can be written as

$$\frac{d}{dt}(\tilde{u}_m, e_j) = \langle \tilde{f}_m, e_j\rangle + (a_m, e_j)\delta_0 - (u_m(T), e_j)\delta_T, \quad j = 1, 2, \ldots, \tag{A.10}$$

where δ_0, δ_T denote Dirac functions at 0 and T, respectively, and

$$\begin{aligned} f_m &= -\Lambda^{2\theta} u_m - (u_m \cdot \nabla)u_m, \\ \tilde{f}_m &= \begin{cases} f_m & \text{on} \quad [0,T], \\ 0 & \text{outside of} \quad [0,T]. \end{cases} \end{aligned} \tag{A.11}$$

Applying the Fourier transform to both sides of (A.10) yields for $\tau > 0$

$$i\tau(\hat{\tilde{u}}_m, e_j) = \langle \hat{\tilde{f}}_m, e_j\rangle + (a_m, e_j) - (u_m(T), e_j)\exp(-iT\tau), \tag{A.12}$$

where $\hat{\tilde{f}}_m$ denotes the Fourier transforms of $\tilde{f}_m$.

Set

$$\tilde{g}_{jm} = \begin{cases} g_{jm} & \text{on} \quad [0,T], \\ 0 & \text{outside of} \quad [0,T]. \end{cases}$$

Multiply $\hat{\tilde{g}}_{jm}$(the Fourier transform of $\tilde{g}_{jm}$), and sum on j from 1 to m, in both sides of (A.12). We have for $\tau > 0$

$$i\tau\|\hat{\tilde{u}}_m(\tau)\|^2_{L^2} = \langle \hat{\tilde{f}}_m(\tau), \hat{\tilde{u}}_m(\tau)\rangle + (a_m, \hat{\tilde{u}}_m(\tau)) - (u_m(T), \hat{\tilde{u}}_m(\tau))\exp(-iT\tau). \tag{A.13}$$

It follows from (A.8) that for $\tau > 0$

$$\|u_m(0)\|_{L^2(\mathbb{R}^n)} + \|u_m(T)\|_{L^2(\mathbb{R}^n)} \le 2\|a\|_{L^2(\mathbb{R}^n)},$$

and

$$\begin{aligned} &\left|(a_m, \hat{\tilde{u}}_m(\tau)) - (u_m(T), \hat{\tilde{u}}_m(\tau))\exp(-iT\tau)\right| \\ &\le (\|u_m(0)\|_{L^2(\mathbb{R}^n)} + \|u_m(T)\|_{L^2(\mathbb{R}^n)})\|\hat{\tilde{u}}_m(\tau)\|_{L^2(\mathbb{R}^n)} \end{aligned} \tag{A.14}$$

$$\leq 2\|a\|_{L^2(\mathbb{R}^n)}\|\hat{\tilde{u}}_m(\tau)\|_{L^2(\mathbb{R}^n)}.$$

Since $\nabla \cdot u_m = 0$, and then for $\tau > 0$

$$\int_{\mathbb{R}^n} [(u_m \cdot \nabla)u_m \cdot u_m](x,\tau)dx = 0.$$

Whence,

$$\langle \hat{\tilde{f}}_m(\tau), \hat{\tilde{u}}_m(\tau)\rangle = \|\Lambda^\theta \hat{\tilde{u}}_m(\tau)\|^2_{L^2(\mathbb{R}^n)},\ \tau > 0. \tag{A.15}$$

From (A.8), (A.13)-(A.15), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^1} \|\partial_\tau^{\frac{1}{2}} \tilde{u}_m(\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau \\
&= (2\pi)^{-1} \int_{\mathbb{R}^1} \|\widehat{\partial_\tau^{\frac{1}{2}} \tilde{u}_m}(\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau \\
&= (2\pi)^{-1} \int_{\mathbb{R}^1} |\tau| \|\hat{\tilde{u}}_m(\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau \\
&\leq (2\pi)^{-1} \int_{\mathbb{R}^1} \|\Lambda^\theta \hat{\tilde{u}}_m(\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau + 2(2\pi)^{-1}\|a\|_{L^2(\mathbb{R}^n)} \int_{\mathbb{R}^1} \|\hat{\tilde{u}}_m(\tau)\|_{L^2(\mathbb{R}^n)} d\tau \\
&= \int_{\mathbb{R}^1} \|\Lambda^\theta \tilde{u}_m(\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau + \sqrt{\frac{2}{\pi}}\|a\|_{L^2(\mathbb{R}^n)} \int_{\mathbb{R}^1} \|\tilde{u}_m(\tau)\|_{L^2(\mathbb{R}^n)} d\tau \\
&= \int_0^T \|\Lambda^\theta u_m(\tau)\|^2_{L^2(\mathbb{R}^n)} d\tau + \sqrt{\frac{2}{\pi}}\|a\|_{L^2(\mathbb{R}^n)} \int_0^T \|u_m(\tau)\|_{L^2(\mathbb{R}^n)} d\tau \\
&\leq \left(1 + T\sqrt{\frac{2}{\pi}}\right)\|a\|^2_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Namely, $\partial_\tau^{\frac{1}{2}} \tilde{u}_m \in L^2(\mathbb{R}^1, L^2(\mathbb{R}^n))$. Moreover,

$$\|\partial_\tau^{\frac{1}{2}} \tilde{u}_m\|_{L^2(\mathbb{R}^1, L^2(\mathbb{R}^n))} \leq \sqrt{1 + T\sqrt{\frac{2}{\pi}}}\|a\|_{L^2(\mathbb{R}^n)}. \tag{A.16}$$

Step 4. Existence of weak solutions.

Using (A.8) and (A.16), together with the weak convergence theorem and the compact embedding theory (see [19]), we infer that there exists a vector function $u \in L^\infty(0,T; L^2_\sigma(\mathbb{R}^n)) \cap L^2(0,T; H^\theta(\mathbb{R}^n))$, such that (selecting a subsequence if necessary)

$$\begin{cases} u_m \rightharpoonup u \ \text{ weakly in } \ L^2(0,T; H^\theta(\mathbb{R}^n)), \\ u_m \rightharpoonup u \ \text{ weakly in } \ L^\infty(0,T; L^2(\mathbb{R}^n)), \\ u_m \to u \ \text{ strongly in } \ L^2(0,T; L^2_{loc}(\mathbb{R}^n)). \end{cases} \tag{A.17}$$

Let $\psi \in C_0^\infty[0,T)$, multiply both sides of (A.2) by $\psi(t)$, and integrate by parts. This leads to the equation

$$\text{(A.18)} \qquad -\int_0^T (u_m(t), e_j\psi'(t))dt + \int_0^T (\Lambda^\theta u_m(t), \psi(t)\Lambda^\theta e_j)dt + \int_0^T b(u_m(t), u_m(t), e_j\psi(t))dt = (a_m, e_j)\psi(0).$$

Recall the definition of b:

$$b(u_m, u_m, w) = -\sum_{i,k=1}^n \int_0^T \int_{\mathbb{R}^n} (u_m)_i (u_m)_k \partial_i w_k dxdt = -\sum_{i,k=1}^n \int_0^T \int_{supp\ w_k} (u_m)_i (u_m)_k \partial_i w_k dxdt.$$

For simplicity, write $e_j = w$ for $e_j \in C_{0,\sigma}^\infty(\mathbb{R}^n)$ $(j = 1, 2, \ldots)$. Then by (A.8), we find

$$\begin{aligned}
&\Big| \int_0^T [b(u_m, u_m, w\psi(t))dt - \int_0^T b(u, u, w\psi(t))dt \Big| \\
&\le \Big| \sum_{i,k=1}^n \int_0^T \int_\Omega (u_m - u)_i (u_m)_k \partial_i w_k \psi(t) dxdt \Big| \\
&\quad + \Big| \sum_{i,k=1}^n \int_0^T \int_\Omega u_i (u_m - u)_k \partial_i w_k \psi(t) dxdt \Big| \\
&\le \|u_m - u\|_{L^2(0,T;L^2(\Omega))} \|u_m\|_{L^2(0,T;L^2(\mathbb{R}^n))} \|\psi\|_{C([0,T])} \sum_{i,k=1}^n \|\partial_i w_k\|_{C(\mathbb{R}^n)} \\
&\quad + \|u_m - u\|_{L^2(0,T;L^2(\Omega))} \|u\|_{L^2(0,T;L^2(\mathbb{R}^n))} \|\psi\|_{C([0,T])} \sum_{i,k=1}^n \|\partial_i w_k\|_{C(\mathbb{R}^n)} \\
&\le 2T^{\frac{1}{2}} \|a\|_{L^2(\mathbb{R}^n)} \|u_m - u\|_{L^2(0,T;L^2(\Omega))} \|\psi\|_{C([0,T])} \|\nabla w\|_{C(\mathbb{R}^n)},
\end{aligned}$$

where $\Omega = \text{supp}\ w$ is a compact set in $\mathbb{R}^n$.

Whence, using (A.17), we conclude for each $e_j \in C_{0,\sigma}^\infty(\mathbb{R}^n)$, $\psi \in C_0^\infty[0,T)$

$$\text{(A.19)} \qquad \lim_{m\to\infty} \int_0^T b(u_m(t), u_m(t), e_j\psi(t))dt = \int_0^T b(u(t), u(t), e_j\psi(t))dt.$$

Applying (A.17), (A.19) to (A.18) yields as $m \to \infty$

$$\text{(A.20)} \qquad -\int_0^T (u(t), v\psi'(t))dt + \int_0^T (\Lambda^\theta u(t), \psi(t)\Lambda^\theta v)dt + \int_0^T b(u(t), u(t), v\psi(t))dt = (a, v)\psi(0),$$

where $v = e_1, e_2, \ldots$; by linearity this equation (A.20) holds for $v =$ any finite linear combination of e_j, and by a continuity argument, (A.20) is still true for any $v \in C_{0,\sigma}^\infty(\mathbb{R}^n)$.

In particular, taking $\psi \in C_0^\infty(0,T)$ in (A.20), we find u satisfies in the sense of distribution:

$$\frac{d}{dt}(u(t), v) + (\Lambda^\theta u(t), \Lambda^\theta v) + b(u(t), u(t), v) = 0, \quad \forall\, v \in C_{0,\sigma}^\infty(\mathbb{R}^n). \tag{A.21}$$

Note that $u \in L^\infty(0,T; L_\sigma^2(\mathbb{R}^n))$, $\Lambda^\theta u \in L^2(\mathbb{R}^n \times (0,T))$. This yields for each $v \in C_{0,\sigma}^\infty(\mathbb{R}^n)$

$$|(\Lambda^\theta u(t), \Lambda^\theta v)| \le \|\Lambda^\theta u(t)\|_{L^2(\mathbb{R}^n)} \|\Lambda^\theta v\|_{L^2(\mathbb{R}^n)} \in L^2(0,T),$$

and

$$\begin{aligned} |b(u(t), u(t), v)| &= \Big| - \sum_{i,k=1}^n \int_{\mathbb{R}^n} u_i u_k \partial_i v_k dx \Big| \\ &\le \sum_{i,k=1}^n \|\partial_i v_k\|_{C(\mathbb{R}^n)} \|u(t)\|_{L^2(\mathbb{R}^n)}^2 \in L^\infty(0,T). \end{aligned}$$

Together with (A.21), we conclude

$$\frac{d}{dt}(u(t), v) \in L^2(0,T) \text{ for every } v \in C_{0,\sigma}^\infty(\mathbb{R}^n).$$

Since $|(u(t), v)| \le \|u(t)\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)} \in L^2(0,T)$, we conclude

$$(u(t), v) \in W^{1,2}(0,T) \hookrightarrow C^{\frac{1}{2}}([0,T]), \quad \forall v \in C_{0,\sigma}^\infty(\mathbb{R}^n). \tag{A.22}$$

Multiplying the both sides of (A.21) by $\psi \in C_0^\infty[0,T)$, and integrating by parts, we get

$$\begin{aligned} &-\int_0^T (u(t), v\psi'(t)) dt + \int_0^T (\Lambda^\theta u(t), \psi(t) \Lambda^\theta v) dt \\ &+ \int_0^T b(u(t), u(t), v\psi(t)) dt = (u(0), v)\psi(0). \end{aligned} \tag{A.23}$$

Combining (A.20) and (A.23), we conclude

$$(u(0) - a, v)\psi(0) = 0, \quad \forall \psi \in C_0^\infty([0,T)).$$

Take ψ in $C_0^\infty[0,T)$, such that $\psi(0) \ne 0$. Then

$$(u(0) - a, v) = 0, \quad \forall v \in C_{0,\sigma}^\infty(\mathbb{R}^n).$$

Since $u(0)$, $a \in L_\sigma^2(\mathbb{R}^n)$, we infer $u(0) = a$ a.e. in $\mathbb{R}^n$.

It follows from (A.8) and (A.17) that for each $t > 0$

$$\|u(t)\|_{L^2(\mathbb{R}^n)}^2 \le \underline{\lim}_{m\to\infty} \|u_m(t)\|_{L^2(\mathbb{R}^n)}^2 \le \overline{\lim}_{m\to\infty} \|u_m(t)\|_{L^2(\mathbb{R}^n)}^2 \le \|a\|_{L^2(\mathbb{R}^n)}^2.$$

Whence,

$$\|a\|_{L^2(\mathbb{R}^n)}^2 = \|u(0)\|_{L^2(\mathbb{R}^n)}^2 \le \underline{\lim}_{t\to 0} \|u(t)\|_{L^2(\mathbb{R}^n)}^2 \le \overline{\lim}_{t\to 0} \|u(t)\|_{L^2(\mathbb{R}^n)}^2 \le \|a\|_{L^2(\mathbb{R}^n)}^2,$$

and then

$$\lim_{t\to 0} \|u(t)\|_{L^2(\mathbb{R}^n)} = \|a\|_{L^2(\mathbb{R}^n)}. \tag{A.24}$$

On the other hand, it follows from (A.22) that u is weakly continuous from $[0,\infty)$ into $L^2(\mathbb{R}^n)$. Together with (A.24), we conclude that $u(t) \to a$ in L^2 as $t \to 0$. □

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