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A DENSITY THEOREM RELATED TO DIHEDRAL GROUPS

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Abstract. For a finite group G, let $\psi(G)$ denote the sum of element orders of G. If $\psi''(G) = \frac{\psi(G)}{|G|^2}$, we show here that the image of ψ'' on the class of all Dihedral groups whose order is twice a composite number greater than 4 is dense in $[0, \frac{1}{4}]$. We also derive some properties of ψ'' on the class of all dihedral groups whose order is twice a prime number.

1. Introduction

Let G be a finite group. If $o(g)$ denotes the order of $g \in G$, Amiri et al. [2] introduced the quantity

$$\psi(G) = \sum_{g \in G} o(g).$$

They proved that for any finite group G with order n, $\psi(G)$ is maximum if and only if $G \simeq \mathbb{Z}_n$, where $\mathbb{Z}_n$ is a cyclic group of order n. Later both Jafarian Amiri and Amiri [6], and Shen et al. [11] independently studied groups G with the second largest value of $\psi(G)$. This function ψ has been considered by various authors for studying different properties it posses. See for instance, [1, 5, 7, 8, 14]. Herzog et al. [4] determined the exact upper bound for $\psi(G)$ for a non-cyclic group G of order n which is sharp.

On the class of all finite groups $\mathcal{G}$, Tărnăuceanu [13] introduced the function $\psi'' : \mathcal{G} \to (0, 1]$ defining $\psi''(G) = \frac{\psi(G)}{|G|^2}$. There he gave some criteria to characterize groups on some of the properties like cyclicity, Abelianess, solvability, nilpotency and supersolvability using ψ''. Lazorec and Tărnăuceanu [9] showed that the image of ψ'' on $\mathcal{G}$ is dense in $[0, 1]$ and in fact, the image of ψ'' on the class of all finite cyclic groups is also dense in $[0, 1]$. Motivated by these results, here we show that the image of ψ'' on the class of all Dihedral groups of order twice a composite number greater than 4 is dense in $[0, \frac{1}{4}]$. Also, we prove that the sequence $(\psi''(D_{p_n}))$ is a strictly decreasing sequence converging

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to $\frac{1}{4}$, where D_n denotes the dihedral group of order $2n$ and p_n denotes the nth prime number.

2. Main results

To start with, we recall the following two results related to the sum of element orders.

Proposition 2.1 ([4, Lemma 2.9])**.**

$$\psi(\mathbb{Z}_{p^\alpha}) = \frac{p^{2\alpha+1}+1}{p+1},$$

where p is a prime and α is a positive integer.

Proposition 2.2 ([1, Lemma 2.1])**.** *Let G and H be two finite groups. Then $\psi(G \times H) = \psi(G)\,\psi(H)$ if and only if $\gcd(|G|, |H|) = 1$.*

Now we have a well-known fact about primes.

Proposition 2.3. *$\sum_{n=1}^{\infty} \frac{1}{p_n}$ diverges to ∞.*

The next statement is a consequence of the proposition mentioned on the page 863 of [10].

Proposition 2.4. *Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n$ is divergent. Then the set containing the sums of all finite subsequences of $(a_n)_{n\geq 1}$ is dense in $[0, \infty)$.*

The following fact about comparing infinite series can be found in standard Analysis textbooks, see for example [3, Theorem 10.9].

Proposition 2.5. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be sequences of positive real numbers such that $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or diverge together.*

We use the following standard result relating closure and image of continuous functions [12, Proposition 6.12].

Proposition 2.6. *Let (X, τ) and (Y, τ') be topological spaces, let $f : X \to Y$ be a continuous function and let $A, B \subseteq X$. If $\overline{A}_\tau = \overline{B}_\tau$, then $\overline{f(A)}_{\tau'} = \overline{f(B)}_{\tau'}$.*

As a consequence of the above propositions, we prove the following. A similar statement with $x_n = p_n$ appears in the proof of [9, Theorem 1.1].

Theorem 2.7. *If $(x_n)_{n\geq 1}$ is a sequence of positive reals which diverge to ∞ and $\sum_{n=1}^{\infty} \frac{1}{x_n}$ is a divergent series, then $\left\{\prod_{n\in J} \frac{x_n}{x_n+1} : J \subseteq \mathbb{N}, |J| < \infty\right\}$ is dense in $[0, 1]$.*

Proof. Consider the sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ given by $a_n = \ln\left(\frac{x_n+1}{x_n}\right)$ and $b_n = \frac{1}{x_n}$. Now,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} x_n \ln\left(1+\frac{1}{x_n}\right) = \lim_{n\to\infty} \ln\left(1+\frac{1}{x_n}\right)^{x_n} = 1.$$

Since the series $\sum_{n=1}^{\infty} b_n$ diverges, $\sum_{n=1}^{\infty} a_n$ also diverges. Since $a_n > 0$ and $\lim_{n\to\infty} a_n = 0$,

$$\overline{\left\{\sum_{n\in J} a_n : J\subseteq \mathbb{N}, |J|<\infty\right\}} = [0,\infty)$$
$$\iff \overline{\left\{\sum_{n\in J} \ln\left(\frac{x_n+1}{x_n}\right) : J\subseteq \mathbb{N}, |J|<\infty\right\}} = [0,\infty)$$
$$\iff \overline{\left\{\ln\left(\prod_{n\in J} \frac{x_n+1}{x_n}\right) : J\subseteq \mathbb{N}, |J|<\infty\right\}} = \overline{[0,\infty)}.$$

Since the function exp: $\mathbb{R}\to\mathbb{R}$ given by $\exp(x) = e^x$ is continuous,

$$\overline{\left\{\exp\left(\ln\left(\prod_{n\in J} \frac{x_n+1}{x_n}\right)\right) : J\subseteq \mathbb{N}, |J|<\infty\right\}} = \overline{[1,\infty)}$$
$$\iff \overline{\left\{\prod_{n\in J} \frac{x_n+1}{x_n} : J\subseteq \mathbb{N}, |J|<\infty\right\}} = \overline{[1,\infty)}.$$

Define $g : (0,\infty)\to\mathbb{R}$ by $g(x) = \frac{1}{x}$. Since g is continuous, we have

$$\overline{\left\{g\left(\prod_{n\in J} \frac{x_n+1}{x_n}\right) : J\subseteq \mathbb{N}, |J|<\infty\right\}} = \overline{(0,1]} = [0,1]$$
$$\iff \overline{\left\{\prod_{n\in J} \frac{x_n}{x_n+1} : J\subseteq \mathbb{N}, |J|<\infty\right\}} = [0,1].$$

□

Now we derive an identity for $\psi''(D_n)$.

Proposition 2.8. *For $n \geq 2$, $\psi''(D_n) = \frac{1}{2n} + \frac{1}{4n^2}\psi(\mathbb{Z}_n)$.*

Proof. D_n contains n rotations and n reflections. Each reflection has order 2. Also the rotations form a group which is isomorphic to $\mathbb{Z}_n$. Hence

$$\psi''(D_n) = \frac{\psi(D_n)}{4n^2} = \frac{2n+\psi(\mathbb{Z}_n)}{4n^2} = \frac{1}{2n} + \frac{1}{4n^2}\psi(\mathbb{Z}_n).$$

□

We proceed to prove $\psi''(D_n) < \frac{1}{4}$ for $n \geq 5$ and n is composite.

Proposition 2.9. *Let p be a prime. Then $\psi''(D_{p^\alpha}) < \frac{1}{4}$ if and only if*

(a) *p is an odd prime and $\alpha \geq 2$ or*

(b) $p = 2$ *and* $\alpha \geq 3$.

Proof.

$$\psi''(D_{p^\alpha}) = \frac{1}{2p^\alpha} + \frac{\psi(\mathbb{Z}_{p^\alpha})}{4p^{2\alpha}} = \frac{1}{2p^\alpha} + \frac{p^{2\alpha+1}+1}{4(p+1)p^{2\alpha}}.$$

Therefore,

$$\begin{aligned}
\psi''(D_{p^\alpha}) < \frac{1}{4} &\iff \frac{1}{2p^\alpha} + \frac{p^{2\alpha+1}+1}{4(p+1)p^{2\alpha}} < \frac{1}{4} \\
&\iff \frac{p^{2\alpha+1}+1}{4(p+1)p^{2\alpha}} < \frac{1}{4} - \frac{1}{2p^\alpha} \\
&\iff \frac{p^{2\alpha+1}+1}{(p+1)p^\alpha} < p^\alpha - 2 \\
&\iff 1 < p^\alpha(p^\alpha - 2(p+1)) \\
&\iff p^\alpha > 2(p+1) \\
&\iff p = 2 \text{ and } \alpha \geq 3 \text{ or } p \text{ is an odd prime and } \alpha \geq 2,
\end{aligned}$$

which is what we claimed. □

Proposition 2.10. *Let $n \geq 3$ be a square free number which is not prime. Then $\psi''(D_n) < \frac{1}{4}$.*

Proof. Assume $n = q_1 q_2 \cdots q_r$ be the prime factorization of n, where $q_1 < q_2 < \cdots < q_r$ and $r \geq 2$. Then

$$\begin{aligned}
\psi''(D_n) &= \frac{1}{2n} + \frac{\psi(\mathbb{Z}_n)}{4n^2} \\
&= \frac{1}{2q_1q_2\cdots q_r} + \frac{1}{4}\prod_{i=1}^{r} \frac{\psi(\mathbb{Z}_{q_i})}{q_i^2}. \qquad (1)
\end{aligned}$$

Now,

$$\frac{1}{2q_1q_2\cdots q_r} + \frac{1}{4}\prod_{i=1}^{r}\frac{\psi(\mathbb{Z}_{q_i})}{(q_i)^2} < \frac{1}{2q_1q_2} + \frac{1}{4}\frac{\psi(\mathbb{Z}_{q_1})}{q_1^2}\frac{\psi(\mathbb{Z}_{q_2})}{q_2^2}. \qquad (2)$$

Again,

$$\begin{aligned}
&\frac{1}{2q_1q_2} + \frac{1}{4}\frac{\psi(\mathbb{Z}_{q_1})}{q_1^2}\frac{\psi(\mathbb{Z}_{q_2})}{q_2^2} < \frac{1}{4} \\
\iff &\frac{1}{4}\frac{\psi(\mathbb{Z}_{q_1})}{q_1^2}\frac{\psi(Z_{q_2})}{q_2^2} < \frac{1}{4} - \frac{1}{2q_1q_2} \\
\iff &\frac{1}{4q_1q_2}\frac{(1+q_1(q_1-1))}{q_1}\frac{(1+q_2(q_2-1))}{q_2} < \frac{q_1q_2-2}{4q_1q_2} \\
\iff &(1+q_1^2-q_1)(1+q_2^2-q_2) < q_1q_2(q_1q_2-2) \\
\iff &1 + q_2^2 - q_2 + q_1^2 - q_2q_1^2 - q_1 - q_1q_2^2 + 3q_1q_2 < 0
\end{aligned}$$

$$(3) \qquad \iff q_1^2(q_2-1)+q_2^2(q_1-1)-3q_1q_2+q_1+q_2-1>0.$$

If $q_1=2$, then

$$q_1^2(q_2-1)+q_2^2(q_1-1)-3q_1q_2+q_1+q_2-1=q_2^2-q_2-3>0.$$

If $2<q_1$, then

$$\begin{aligned}&q_1^2(q_2-1)+q_2^2(q_1-1)-3q_1q_2+q_1+q_2-1\\ &>4q_1^2+2q_2^2-3q_1q_2+q_1+q_2-q_1q_2-1\\ &=2q_1^2+2q_2^2-4q_1q_2+2q_1^2+q_1+q_2-1\\ &=2(q_1-q_2)^2+2q_1^2+q_1+q_2-1>0.\end{aligned}$$

Therefore,

$$q_1^2(q_2-1)+q_2^2(q_1-1)-3q_1q_2+q_1+q_2-1>0$$

and hence from (1) and (2), $\psi''(D_n)<\frac{1}{4}$. □

Theorem 2.11. *Let $n\geq 5$ be a composite number. Then $\psi''(D_n)<\frac{1}{4}$.*

Proof. Assume $n=q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_r^{\alpha_r}$ be the prime factorization of n, where $q_1<q_2<\cdots<q_r$. The result is true when $r=1$ or n is a square free by Proposition 2.9 and Proposition 2.10. So assume that $r\geq 2$ and n is not square free.

Case (i) $q_1=2$ and $\alpha_1=1$.

In this case $\alpha_k\geq 2$ for some $1<k\leq r$.

$$\begin{aligned}\psi''(D_n)&=\frac{1}{2n}+\frac{1}{4}\frac{\psi(\mathbb{Z}_n)}{n^2}\\ &=\frac{1}{2q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_r^{\alpha_r}}+\frac{1}{4}\prod_{i=1}^{r}\frac{\psi(\mathbb{Z}_{q_i^{\alpha_i}})}{q_i^{2\alpha_i}}\\ &\leq\frac{1}{2q_k^{\alpha_k}}+\frac{1}{4}\frac{\psi(\mathbb{Z}_{q_k^{\alpha_k}})}{q_k^{2\alpha_k}}\\ &=\psi''(D_{q_k^{\alpha_k}})<\frac{1}{4}.\end{aligned}$$

Case (ii) $q_1=2$ and $\alpha_1=2$.

Here

$$\begin{aligned}\psi''(D_n)&=\frac{1}{2n}+\frac{1}{4}\frac{\psi(\mathbb{Z}_n)}{n^2}\\ &=\frac{1}{2\times 2^2q_2^{\alpha_2}\cdots q_r^{\alpha_r}}+\frac{1}{4}\prod_{i=1}^{r}\frac{\psi(\mathbb{Z}_{q_i^{\alpha_i}})}{q_i^{2\alpha_i}}\\ &\leq\frac{1}{2\times 2^2q_2}+\frac{1}{4}\frac{\psi(\mathbb{Z}_{2^2})}{2^4}=\frac{1}{8q_2}+\frac{11}{64}\\ &<\frac{1}{4}.\end{aligned}$$

Case (iii) $q_1 = 2$ and $\alpha_1 > 2$.

$$\begin{aligned}\psi''(D_n) &= \frac{1}{2n} + \frac{1}{4}\frac{\psi(\mathbb{Z}_n)}{n^2} \\ &= \frac{1}{2q_1^{\alpha_1} q_2^{\alpha_2}\cdots q_r^{\alpha_r}} + \frac{1}{4}\prod_{i=1}^{r} \frac{\psi(\mathbb{Z}_{q_i^{\alpha_i}})}{q_i^{2\alpha_i}} \\ &\leq \frac{1}{2\times 2^{\alpha_1}} + \frac{1}{4}\frac{\psi(\mathbb{Z}_{2^{\alpha_1}})}{2^{2\alpha_1}} \\ &= \psi''(D_{2^{\alpha_1}}) < \frac{1}{4}.\end{aligned}$$

Case (iv) $q_1 > 2$. Since n is not square free, some $\alpha_k \geq 2$ for some $1 \leq k \leq r$.

$$\begin{aligned}\psi''(D_n) &= \frac{1}{2n} + \frac{1}{4}\frac{\psi(\mathbb{Z}_n)}{n^2} \\ &= \frac{1}{2q_1^{\alpha_1} q_2^{\alpha_2}\cdots q_r^{\alpha_r}} + \frac{1}{4}\prod_{i=1}^{r} \frac{\psi(\mathbb{Z}_{q_i^{\alpha_i}})}{q_i^{2\alpha_i}} \\ &\leq \frac{1}{2\times q_k^{\alpha_k}} + \frac{1}{4}\frac{\psi(\mathbb{Z}_{q_k^{\alpha_k}})}{q_k^{2\alpha_k}} \\ &= \psi''(D_{q_k^{\alpha_k}}) < \frac{1}{4}.\end{aligned}$$

□

Theorem 2.12. *For each $k \geq 3$, the image of ψ'' on $\mathfrak{D}^{(k)}$ is dense in $[0, \frac{1}{4}]$, where $\mathfrak{D}^{(k)} = \{D_{n^k} : n \geq 2\}$.*

Proof. Let p_n denote the nth prime number and J be a finite subset of $\mathbb{N}$. Fix $k \geq 3$. Consider the sequence $(\delta_i)_{i\geq 1}$, where $\delta_i = \prod_{n\in J} p_n^{ik}$. Then $\psi''(D_{\delta_i}) \in \psi''(\mathfrak{D}^{(k)})$ for all $i \in \mathbb{N}$. Now,

$$\begin{aligned}\psi''(D_{\delta_i}) &= \frac{1}{2\delta_i} + \frac{1}{4}\psi''(\mathbb{Z}_{\delta_i}) \\ &= \frac{1}{2\delta_i} + \frac{1}{4}\psi''(\prod_{n\in J}\mathbb{Z}_{p_n^{ik}}) \\ &= \frac{1}{2\delta_i} + \frac{1}{4}\prod_{n\in J}\psi''(\mathbb{Z}_{p_n^{ik}}) \\ &= \frac{1}{2\delta_i} + \frac{1}{4}\prod_{n\in J}\left(\frac{p_n^{2ik+1}+1}{(p_n+1)p_n^{2ik}}\right) \\ &= \frac{1}{2}\prod_{n\in J}\left(\frac{1}{p_n^{ik}}\right) + \frac{1}{4}\prod_{n\in J}\left(\frac{p_n + \frac{1}{p_n^{2ik}}}{(p_n+1)}\right)\end{aligned}$$

so that

$$\lim_{i\to\infty}\psi''(D_{\delta_i}) = \frac{1}{4}\prod_{n\in J}\frac{p_n}{p_n+1}.$$

Hence

$$\left\{\frac{1}{4}\prod_{n\in J}\frac{p_n}{p_n+1} : J\subseteq\mathbb{N}, |J|<\infty\right\} \subseteq \overline{\psi''(\mathfrak{D}^{(k)})} \subseteq \overline{[0,\frac{1}{4})}.$$

Since $(p_n)_{n\geq 1}$ is a strictly increasing sequence diverging to ∞ and $\sum_{n=1}^{\infty}\frac{1}{p_n}$ is a divergent series, we have

$$\begin{aligned}
&\overline{\left\{\frac{1}{4}\prod_{n\in J}\frac{p_n}{p_n+1} : J\subseteq\mathbb{N}, |J|<\infty\right\}} \subseteq \overline{\psi''(\mathfrak{D}^{(k)})} \subseteq [0,\frac{1}{4}],\\
&[0,\frac{1}{4}] \subseteq \overline{\psi''(\mathfrak{D}^{(k)})} \subseteq [0,\frac{1}{4}],\\
&\overline{\psi''(\mathfrak{D}^{(k)})} = [0,\frac{1}{4}].
\end{aligned}$$

□

Corollary 2.13. *The image of ψ'' on $\mathfrak{D}$ is dense in $[0,\frac{1}{4}]$, where $\mathfrak{D} = \{D_n : n\geq 5, n$ is a composite number$\}$.*

Proof. For each $k\geq 3$,

$$\begin{aligned}
&\psi''(\mathfrak{D}^{(k)}) \subseteq \psi''(\mathfrak{D}) \subseteq [0,\frac{1}{4}],\\
&\overline{\psi''(\mathfrak{D}^{(k)})} \subseteq \overline{\psi''(\mathfrak{D})} \subseteq \overline{[0,\frac{1}{4}]},\\
&[0,\frac{1}{4}] \subseteq \overline{\psi''(\mathfrak{D})} \subseteq [0,\frac{1}{4}].
\end{aligned}$$

□

Finally, we derive some properties of ψ'' on the class of all dihedral groups whose order is twice a prime number.

Proposition 2.14. *$\psi''(D_n) > \frac{1}{4}$ if n is a prime or $n=4$.*

Proof. Let n be a prime number. Then

$$\begin{aligned}
\psi''(D_n) &= \frac{1}{2n}+\frac{1}{4}\frac{\psi(\mathbb{Z}_n)}{n^2}\\
&= \frac{1}{2n}+\frac{1+n(n-1)}{4n^2}\\
&= \frac{1+n+n^2}{4n^2} > \frac{1}{4}.
\end{aligned}$$

Also,

$$\psi''(D_4) = \frac{1}{2\times 4}+\frac{1}{4}\frac{\psi(\mathbb{Z}_4)}{16} = \frac{19}{64} > \frac{1}{4}.$$

□

Theorem 2.15. *$(\psi''(D_{p_n}))_{n=1}^{\infty}$ is a strictly decreasing sequence converging to $\frac{1}{4}$.*

Proof. Let $p < q$ be primes. Then

$$\begin{aligned}\psi''(D_p) &= \frac{1+p+p^2}{4p^2} \longrightarrow \frac{1}{4} \text{ as } p \to \infty \\ &= \frac{1}{4}\left(1+\frac{1}{p}+\frac{1}{p^2}\right) \\ &> \frac{1}{4}\left(1+\frac{1}{q}+\frac{1}{q^2}\right) \\ &= \psi''(D_q).\end{aligned}$$

Hence $(\psi''(D_{p_n}))_{n=1}^{\infty}$ is a strictly decreasing sequence converging to $\frac{1}{4}$. □

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References

[1] H. Amiri and S. M. Jafarian Amiri, *Sum of element orders on finite groups of the same order*, J. Algebra Appl. **10** (2011), no. 2, 187–190. https://doi.org/10.1142/S0219498811004057

[2] H. Amiri, S. M. Jafarian Amiri, and I. M. Isaacs, *Sums of element orders in finite groups*, Comm. Algebra **37** (2009), no. 9, 2978–2980. https://doi.org/10.1080/00927870802502530

[3] T. M. Apostol, *Calculus. Vol. I*, second edition, Blaisdell Publishing Co., Waltham, MA, 1967.

[4] M. Herzog, P. Longobardi, and M. Maj, *An exact upper bound for sums of element orders in non-cyclic finite groups*, J. Pure Appl. Algebra **222** (2018), no. 7, 1628–1642. https://doi.org/10.1016/j.jpaa.2017.07.015

[5] M. Herzog, P. Longobardi, and M. Maj, *Sums of element orders in groups of order* $2m$ *with* m *odd*, Comm. Algebra **47** (2019), no. 5, 2035–2048. https://doi.org/10.1080/00927872.2018.1527924

[6] S. M. Jafarian Amiri and M. Amiri, *Second maximum sum of element orders on finite groups*, J. Pure Appl. Algebra **218** (2014), no. 3, 531–539. https://doi.org/10.1016/j.jpaa.2013.07.003

[7] S. M. Jafarian Amiri and M. Amiri, *Characterization of* p*-groups by sum of the element orders*, Publ. Math. Debrecen **86** (2015), no. 1-2, 31–37. https://doi.org/10.5486/PMD.2015.5961

[8] M. Jahani, Y. Marefat, H. Refaghat, and B. V. Fasaghandisi, *The minimum sum of element orders of finite groups*, Int. J. Group Theory **10** (2021), no. 2, 55–60.

[9] M.-S. Lazorec and M. Tărnăuceanu, *A density result on the sum of element orders of a finite group*, Arch. Math. (Basel) **114** (2020), no. 6, 601–607. https://doi.org/10.1007/s00013-020-01437-4

[10] Z. Nitecki, *Cantorvals and subsum sets of null sequences*, Amer. Math. Monthly **122** (2015), no. 9, 862–870. https://doi.org/10.4169/amer.math.monthly.122.9.862

[11] R. Shen, G. Chen, and C. Wu, *On groups with the second largest value of the sum of element orders*, Comm. Algebra **43** (2015), no. 6, 2618–2631. https://doi.org/10.1080/00927872.2014.900686

[12] W. A. Sutherland, *Introduction to Metric and Topological Spaces*, Oxford Univ. Press, Oxford, 2009.

[13] M. Tărnăuceanu, *Detecting structural properties of finite groups by the sum of element orders*, Israel J. Math. **238** (2020), no. 2, 629–637. https://doi.org/10.1007/s11856-020-2033-9

[14] M. Tărnăuceanu and D. G. Fodor, *On the sum of element orders of finite abelian groups*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **60** (2014), no. 1, 1–7. https://doi.org/10.2478/aicu-2013-0013

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