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ON THE GROWTH OF ALGEBROID SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

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Abstract. Using the Nevanlinna value distribution theory of algebroid functions, this paper investigates the growth of two types of complex algebraic differential equation with algebroid solutions and obtains two results, which extend the growth of complex algebraic differential equation with meromorphic solutions obtained by Gao [4].

1. Introduction and results

The algebroid solution of differential equation, originally studied by P. Painleve and P. Bouroux, appears more frequently than the meromorphic solution of a differential equation. For example, the equation $w' = \frac{1+w^4}{2w}$ has 2-valued algebroid solutions. As far as we know, equations with a large range of single valued meromorphic solutions are very special. Thus, complex differential equations with multi-valued algebroid solutions have attracted a great deal of attention. Some scholars studied certain differential equations with single valued meromorphic solutions, at the same time they also discussed equations with multi-valued algebroid solutions on related problems.

Let $w = w(z)$ be the ν-valued algebroid function defined by a irreducible equation

$$A_\nu(z)w^\nu + A_{\nu-1}(z)w^{\nu-1} + \cdots + A_0(z) = 0,$$

where $A_\nu(z), A_{\nu-1}(z), \ldots, A_0(z)$ are entire functions without any common zeros in $|z| < +\infty$.

Let $w(z)$ be a ν-value algebroid function and a be a pole of $w(z)$. Then in a neighbourhood of a, we have the following expansions of w:

$$w(z) = (z-a)^{\frac{-\tau_i}{\beta_i}} S((z-a)^{\frac{1}{\beta_i}}),$$

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where $i = 1, 2, \ldots, \nu(a)(\leq \nu)$, $1 \leq \tau_i$, $1 \leq \beta_i$, $\sum \beta_i = \nu$ and $S(t)$ is a regular power of t such that $S(0) \neq 0$. For more theories and basic results we refer the readers to [8–10, 14].

In 1934, Yosida in [13] considered a type of differential equation $w' = R(z, w)$, and he showed that the equation can be reduced to a new form with $p \leq 2\nu$, $q \leq 2(\nu - 1)$ if it has ν-valued algebriod solutions, which called Malmquist Theorem of equations with algebriod solutions. With the development of this topic, He and Xiao in [7] investigated a type of higher order differential equations with algebriod solutions, and they gave a corresponding Malmquist Theorem. Obviously, it can be viewed as the generalization of K. Yosida's result.

In the year of 1978, Bank ([1]) showed that the growth of meromorphic solutions of linear differential equations, hence of algebraic differential equations, with meromorphic coefficients cannot be estimated uniformly in terms of the growth of the coefficients alone. Two years later, such uniform estimates for the growth of meromorphic solutions were given by Bank ([2]) and Bank-Laine ([3]).

Xiao-He ([12]) and He-Laine ([6]) considered algebraic differential equations of the form

$$\Omega(z, w) = R(z, w),$$

where $\Omega(z, w)$ is a differential polynomial with meromorphic coefficients, $R(z, w)$ is irreducible and rational in w. They gave some estimates for the growth of algebroid solutions of the equation.

Actually, one can see the above equation is only the case that the left hand side of the above equation is a quite general differential polynomial. It is natural to pose the question about the growth of meromorphic solutions on a differential equation with rational left hand side.

In 2002, Gao ([5]) considered the growth of meromorphic solution on two types of differential equation with rational left hand side as follows:

$$\left[\frac{\Omega_1(z, w)}{w^{k_0}(w')^{k_1} \cdots (w^{(n)})^{k_n}}\right]^m = \sum_{i=0}^{k} a_i(z) w^i, \tag{1.1}$$

$$\frac{\Omega_1(z, w)}{(w - \hat{a})^{\lambda_1} \Omega_2(z, w)} = \frac{P(z, w)}{Q(z, w)}, \tag{1.2}$$

where $\hat{a}$ is a nonzero constant.

It is known to us all, for a differential equation a meromorphic solution is a special case of an algebroid solution. Especially, we should considered both poles and branch points of an algebroid solution while we only considered the poles of a meromorphic solution. Therefore, it inspired us to pose the question as follows.

Question 1.1. What can be said for the growth of algebroid solutions on the above two differential equations with rational left hand side?

In this paper, we shall consider the growth of algebroid solutions on the generalized higher-order algebraic differential equations and we give our results as follows:

Theorem 1.1. *Let w be a ν-valued algebroid solution of* (1.1), $2 \le k < m$. *Then*

$$T(r,w) \le K\Big(N(r,w) + N(r,\tfrac{1}{w}) + N_x(r,w)\Big) - K_1 N_b(r,w) + K_2\Big(\sum_{(i)} T(r,a_{(i)}(z)) + \sum_{i=0}^{k} T(r,a_i(z))\Big) + S(r,w),$$

where K, K_1 and K_2 are positive constants.

Theorem 1.2. *Let $w(z)$ be an algebroid solution of* (1.2) *with ν branches and $p > q + \lambda_1 + \lambda_2$. Then for any $\sigma > 1$, there exist positive constants K_0 and r_0 such that for all $r \ge r_0$,*

$$T(r,w) \le K_0 F(\sigma r),$$

where

$$F(r) = \overline{N}(r,w) + N_x(r,w) + N_b(r,w) + \sum_{(i)} T(r,a_{(i)}) + \sum_{(j)} T(r,b_{(j)}) + \sum_{i=0}^{p} T(r,a_i) + \sum_{j=0}^{q} T(r,b_j) + 1.$$

To arrive at our results, we introduce some definitions and notations.

Let

$$\Omega_1(z,w) = \sum_{(i)\in I} a_{(i)}(z) w^{i_0}(w')^{i_1}\cdots(w^{(n)})^{i_n}$$

and

$$\Omega_2(z,w) = \sum_{(j)\in J} b_{(j)}(z) w^{j_0}(w')^{j_1}\cdots(w^{(n)})^{j_n} \ (n \ge 1)$$

be differential polynomials. We denote $P(z,w) = \sum_{i=0}^{p} a_i(z)w^i$, $Q(z,w) = \sum_{j=0}^{q} b_j(z)w^j$, where $\{a_{(i)}(z)\}$, $\{b_{(j)}(z)\}$, $\{a_i(z)\}$ and $\{b_j(z)\}$ are meromorphic functions, $I = (i_0, i_1, \ldots, i_n)$, $J = (j_0, j_1, \ldots, j_n)$ are multi-indices of nonnegative integer for $a_{(i)} \ne 0$, $b_{(j)} \ne 0$, respectively, and $a_p b_q \ne 0$.

The term $\Omega_{(i)} = c_{(i)} w^{i_0}(w')^{i_1}\cdots(w^{(n)})^{i_n}$ is a differential monomial in w, the degree $\lambda_{(i)}$ and the weight $\Delta_{(i)}$ of $\Omega_{(i)}$ are defined by $\lambda_{(i)} = \sum_{t=0}^{n} i_t$, $\Delta_{(i)} = \sum_{t=0}^{n}(t+1)i_t$ in $\Omega_1(z,w)$ or $\Omega_2(z,w)$. We denote $\sigma_{(i)} = \sum_{t=1}^{n}(2t-1)i_t$, $l_{(i)} = \sum_{t=2}^{n}(t-1)i_t$.

The degrees λ_1, λ_2 and the weights Δ_1, Δ_2 of Ω_1, Ω_2 are defined by

$$\lambda_1 = \max\{\lambda_{(i)}\},\ \Delta_1 = \max\{\Delta_{(i)}\},\ \lambda_2 = \max\{\lambda_{(j)}\},\ \Delta_2 = \max\{\Delta_{(j)}\}.$$

Let

$$\sigma_1 = \max\{\sigma_{(i)}\},\ l_1 = \max\{l_{(i)}\},\ \sigma_2 = \max\{\sigma_{(j)}\},\ l_2 = \max\{l_{(j)}\}.$$

In addition, for an algebroid function, we put

$$n_b(r,w)=\sum_{|a|\le r}\sum_{i=1}^{\nu(a)}(\beta_i-1),$$

$$\nu N_b(r,w)=\int_0^r\frac{n_b(t,w)-n_b(r,w)}{t}dt+n_b(0,w)\log r.$$

2. Some lemmas

In this section, we are devoted to proving several technical lemmas. The proof of the first lemma for algebroid functions is similar to that for meromorphic functions in [11], and we omit the proof here.

Lemma 2.1. *Let $g_0(z)$ and $g_1(z)$ be ν-valued algebroid functions and linearly independent over $\mathbb{C}$, and put*

$$g_0(z)+g_1(z)=\Phi.$$

Then we have

$$T(r,g_0)\le m(r,\Phi)+\overline{N}(r,g_0)+\overline{N}(r,g_1)+\overline{N}(r,\tfrac{1}{g_0})+\overline{N}(r,\tfrac{1}{g_1})+S(r),$$

where

$$S(r)=\begin{cases}O(1), \text{ when } g_0 \text{ and } g_1 \text{ are rational};\\ O(\log^+T(r,g_0)+\log^+T(r,g_1))+O(\log r)\ (r\to\infty, r\notin E), \text{ otherwise.}\end{cases}$$

Lemma 2.2. *Let w be a ν-valued algebroid function in $\mathbb{C}$ and k a positive integer. Then*

$$N(r,\tfrac{1}{w^{(k)}})\le N(r,\tfrac{1}{w})+k\overline{N}(r,w)+(2k-1)N_x(r,w)-(k-1)N_b(r,w)+S(r,w).$$

Proof. Let $w(z_0)=a\in\mathbb{C}$.

Case (i): If $a\ne\infty$, in a neighbourhood of z_0, we have

$$w^{(\alpha)}(z)=(z-z_0)^{(\tau-\alpha\beta)/\beta}w_1(z),\ w_1(z_0)\ne 0,\infty\ (\tau\ge 1,\alpha\ge 1,\beta>1).$$

When $\tau-\alpha\beta<0$, z_0 is a pole of $w^{(\alpha)}(z)$ with multiplicity $\alpha\beta-\tau$.

Case (ii): If $a=\infty$, then

$$w^{(\alpha)}(z)=(z-z_0)^{-(\tau+\alpha\beta)/\beta}w_1(z),\ w_1(z_0)\ne 0,\infty,$$

which implies that z_0 is a pole of $w^{(\alpha)}(z)$ with multiplicity $\alpha\beta+\tau$. Combining Case (i) with Case (ii), we get

$$n(r,w^{(\alpha)})=\sum_{w=\infty}(\tau+\alpha\beta)+\sum_{w\ne\infty}(-\tau+\alpha\beta)^+,$$

where $(-\tau+\alpha\beta)^+=\max\{0,-\tau+\alpha\beta\}$. Since $\beta>1$, $\alpha\ge 1$, $\tau\ge 1$, note that

$$-\tau+\alpha\beta\le\alpha\beta-1\le(2\alpha-1)(\beta-1).$$

Thus

$$\begin{aligned}
n(r,w^{(\alpha)}) \le & \sum_{w=\infty}(\tau+\alpha\beta)+(2\alpha-1)\sum_{w\neq\infty}(\beta-1)\\
=& \sum_{w=\infty}(\tau+\alpha+\alpha(\beta-1))+(2\alpha-1)\sum_{w\neq\infty}(\beta-1)\\
=& \sum_{w=\infty}(\tau+\alpha)+(2\alpha-1)\sum_{w=\infty}(\beta-1)\\
&+(2\alpha-1)\sum_{w\neq\infty}(\beta-1)-(\alpha-1)\sum_{w=\infty}(\beta-1)\\
=& \sum_{w=\infty}(\tau+\alpha)+(2\alpha-1)\sum_{a\in\mathbb{C}}(\beta-1)-(\alpha-1)\sum_{w=\infty}(\beta-1)\\
=& \, n(r,w)+\alpha\overline{n}(r,w)+(2\alpha-1)n_x(r,w)-(\alpha-1)n_b(r,w),
\end{aligned}$$

i.e.,

$$N(r,w^{(\alpha)})\le N(r,w)+\alpha\overline{N}(r,w)+(2\alpha-1)N_x(r,w)-(\alpha-1)N_b(r,w).$$

Further, the following inequality gives that

$$T(r,w)-N(r,\tfrac{1}{w})\le T(r,\tfrac{1}{w^{(k)}})-N(r,\tfrac{1}{w^{(k)}})+S(r,w).$$

We obtain

$$\begin{aligned}
&N(r,\tfrac{1}{w^{(k)}})\\
\le\; & T(r,w^{(k)})+N(r,\tfrac{1}{w})-T(r,w)+S(r,w)\\
\le\; & m(r,\tfrac{w^{(k)}}{w})+m(r,w)+N(r,w^{(k)})+N(r,\tfrac{1}{w})-T(r,w)+S(r,w)\\
\le\; & m(r,w)+N(r,w)+k\overline{N}(r,w)+(2k-1)N_x(r,w)-(k-1)N_b(r,w)\\
&+N(r,\tfrac{1}{w})-T(r,w)+S(r,w)\\
\le\; & N(r,\tfrac{1}{w})+k\overline{N}(r,w)+(2k-1)N_x(r,w)-(k-1)N_b(r,w)+S(r,w). \qquad \square
\end{aligned}$$

Lemma 2.3 ([7]). *Let* $R(z,w)=\frac{\sum_{i=0}^{p}a_i(z)w^i}{\sum_{j=0}^{q}b_j(z)w^j}$ *be an irreducible rational function in* $w(z)$ *with the meromorphic coefficients* $\{a_i(z)\}$ *and* $\{b_j(z)\}$*. If* $w(z)$ *is a* ν*-valued algebroid function, then*

$$T(r,R(z,w))=\max\{p,q\}T(r,w)+O\big\{\sum T(r,a_i)+\sum T(r,b_j)\big\}.$$

Lemma 2.4. *Let* w *be a* ν*-valued algebroid function. Then*

$$\begin{aligned}
&N(r,[\tfrac{\Omega_1(z,w)}{w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}}]^m)\\
\le\; & C_5\left(N(r,w)+N(r,\tfrac{1}{w})+N_x(r,w)\right)-C_4N_b(r,w)+\sum_{(i)}N(r,a_{(i)})+S(r,w)
\end{aligned}$$

for some positive constants C_4*,* C_5*.*

Proof. Let

$$\begin{aligned} &N(r,[\tfrac{\Omega_1(z,w)}{w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}}]^m,[w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}]^m) \\ =\ &N(r,[\tfrac{\Omega_1(z,w)}{w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}}]^m)+N(r,[w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}]^m) \\ &-N(r,\tfrac{1}{[w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}]^m}). \end{aligned}$$

Using the similar method as the proof of Theorem 1 in [4], we can obtain

$$\begin{aligned} &N(r,[\tfrac{\Omega_1(z,w)}{w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}}]^m,[w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}]^m) \\ \le\ &\max\{N(r,\Omega_1(z,w)^m),N(r,[w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}]^m)\}. \end{aligned}$$

Further

$$\begin{aligned} &N(r,[\tfrac{\Omega_1(z,w)}{w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}}]^m) \\ \le\ &\max\{N(r,\Omega_1(z,w)^m),N(r,[w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}]^m)\} \\ &+N(r,\tfrac{1}{[w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}]^m})-N(r,[w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}]^m). \end{aligned} \tag{2.1}$$

Next, we will give the estimation of

$$N(r,\Omega_1(z,w)^m),N(r,[w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}]^m),N(r,\tfrac{1}{[w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}]^m}).$$

First,

$$\begin{aligned} N(r,\Omega_1(z,w)^m) &\le m[\lambda_1 N(r,w)+(\Delta_1-\lambda_1)\overline{N}(r,w)+\sigma_1 N_x(r,w) \\ &\qquad -l_1N_b(r,w)]+m\sum_{(i)}N(r,a_{(i)}(z)) \\ &\le m\Big(\Delta_1 N(r,w)+\sigma_1N_x(r,w)-l_1N_b(r,w)+\sum_{(i)}N(r,a_{(i)})\Big), \end{aligned}$$

Next,

$$\begin{aligned} N(r,[w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}]^m) \le\ &m(\sum_{i=0}^{n}(i+1)k_i)N(r,w)-m(\sum_{i=1}^{n}ik_i)N_b(r,w) \\ &+m(\sum_{i=1}^{n}(2i-1)k_i)N_x(r,w). \end{aligned}$$

By using Lemma 2.2, it yields

$$\begin{aligned} &N(r,\tfrac{1}{[w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}]^m}) \\ &\le m\{k_0N(r,\tfrac{1}{w})+k_1N(r,\tfrac{1}{w'})+\cdots+k_nN(r,\tfrac{1}{w^{(n)}})\} \\ &\le m\{k_0N(r,\tfrac{1}{w})+\cdots+k_n[N(r,\tfrac{1}{w})+n\overline{N}(r,w) \\ &\quad +(2n-1)N_x(r,w)-(n-1)N_b(r,w)]\} \\ &\le C_1N(r,w)+C_2N(r,\tfrac{1}{w})+C_3N_x(r,w)-C_4N_b(r,w) \end{aligned} \tag{2.2}$$

for some positive constants C_i $(i=1,2,3,4)$.

It follows from (2.1) and (2.2) that

$$\begin{aligned}
&N(r,[\tfrac{\Omega_1(z,w)}{w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}}]^m)\\
\le\ &\max\{N(r,\Omega_1(z,w)^m),N(r,[(w')^{k_1}\cdots(w^{(n)})^{k_n}]^m)\}\\
&+C_1N(r,w)+C_2N(r,\tfrac{1}{w})+C_3N_x(r,w)-C_4N_b(r,w)\\
\le&\Delta N(r,w)+C_1N(r,w)+C_2N(r,\tfrac{1}{w})+C_3N_x(r,w)\\
&-C_4N_b(r,w)+\sum_{(i)}N(r,a_{(i)})+S(r,w)\\
\le\ &C_5(N(r,w)+N(r,\tfrac{1}{w})+N_x(r,w))-C_4N_b(r,w)\\
&+\sum_{(i)}N(r,a_{(i)})+S(r,w),
\end{aligned}$$

where $\Delta=\max\{m\Delta_1-m(k_0+2k_1+\cdots+(n+1)k_n),0\}$, $C_5=\max\{\Delta+C_1,C_2,C_3\}$.

This completes the proof of Lemma 2.4. □

Lemma 2.5. *Let w be a ν-valued algebroid function and $\hat{a}$ be a nonzero constant. Then*

$$\begin{aligned}
N(r,\tfrac{\Omega_1}{(w-\hat{a})^{\lambda_1}\Omega_2})\le\ &\lambda_1N(r,\tfrac{1}{w-\hat{a}})+N(r,\tfrac{1}{\Omega_2})\\
&+(\Delta_1-\lambda_1)(\overline{N}(r,w)+N_b(r,w))+\sum_{(i)}N(r,a_{(i)}).
\end{aligned}$$

Proof. We denote the order of pole of w at $z=z_0$ as $n(r,w)$.

Case (i): when z_0 is not a pole of w,

$$n(r,\tfrac{\Omega_1}{(w-\hat{a})^{\lambda_1}\Omega_2})\le n(r,\tfrac{1}{(w-\hat{a})^{\lambda_1}})+n(r,\tfrac{1}{\Omega_2})+\sum_{(i)}n(r,a_{(i)}).\tag{2.3}$$

Case (ii): when z_0 is a pole of w, in a neighbourhood of $\hat{a}$,

$$n(r,(\tfrac{w^{(l)}}{w-\hat{a}})^{i_l})=n(r,(\tfrac{(w-\hat{a})^{(l)}}{w-\hat{a}})^{i_l})=\beta l i_l.$$

Therefore

$$\begin{aligned}
&n(r,\tfrac{a_{(i)}w^{i_0}\cdots(w^{(n)})^{i_n}}{(w-\hat{a})^{\lambda_1}\Omega_2})\\
\le\ &\beta\sum_{l=1}^{n}li_l+n(r,\tfrac{1}{\Omega_2})+n(r,a_{(i)})\\
=\ &\beta\Big(\sum_{l=1}^{n}(l+1)i_l-\sum_{l=1}^{n}i_l\Big)+n(r,\tfrac{1}{\Omega_2})+n(r,a_{(i)}).
\end{aligned}\tag{2.4}$$

In the following, we will proof that claim:

$$n(r,\tfrac{\Omega_1}{(w-\hat{a})^{\lambda_1}\Omega_2})\le(\Delta_1-\lambda_1)\beta+\sum n(r,a_{(i)})+n(r,\tfrac{1}{\Omega_2}).\tag{2.5}$$

In order to proof our claim, we make use of the methods of mathematical induction. In fact, for $i = 1$, one can immediately see that $\Omega_1(z,w)$ is a differential monomial. Thus, it follows from (2.4), we can obtain that inequality (2.5) holds for $i = 1$.

Now, we suppose inequality (2.5) holds for $i = n$. For convenience, let $\Omega = \sum_{i=1}^{n} a_{(i)} w^{i_0} \cdots (w^{(n)})^{i_n}$ and Δ, λ be the weight and the degree of $\sum_{i=1}^{n} a_{(i)} w^{i_0} \cdots (w^{(n)})^{i_n}$, respectively.

For $i = n+1$, we have $\Omega_1 = \Omega + a_{(n+1)} w^{i_0} \cdots (w^{(n)})^{i_n}$ and

$$\frac{\Omega_1}{(w-\hat{a})^{\lambda_1}\Omega_2} = \frac{\Omega}{(w-\hat{a})^{\lambda_1}\Omega_2} + \frac{a_{(n+1)} w^{i_0} \cdots (w^{(n)})^{i_n}}{(w-\hat{a})^{\lambda_1}\Omega_2}.$$

Thus

$$\begin{aligned}
& n\big(r, \tfrac{\Omega_1}{(w-\hat{a})^{\lambda_1}\Omega_2}\big) \\
\le\ & \max\Big\{ n(r, \tfrac{\Omega}{(w-\hat{a})^{\lambda_1}\Omega_2}), n(r, \tfrac{a_{(n+1)} w^{i_0} \cdots (w^{(n)})^{i_n}}{(w-\hat{a})^{\lambda_1}\Omega_2}) \Big\} \\
\le\ & \max\Big\{ (\Delta - \lambda)\beta + \sum_{i=1}^{n} n(r, a_{(i)}) + n(r, \tfrac{1}{\Omega_2}), \\
& (\Delta_{(n+1)} - \lambda_{(n+1)})\beta + n(r, a_{(n+1)}) + n(r, \tfrac{1}{\Omega_2}) \Big\},
\end{aligned} \tag{2.6}$$

where $\Delta_{(n+1)}, \lambda_{(n+1)}$ are the weight and the degree of $a_{(n+1)} w^{i_0} \cdots (w^{(n)})^{i_n}$, respectively.

Now, we discuss inequality (2.6) by the following two cases.

Case (a): If $\lambda \ge \lambda_{(n+1)}$, we can get

$$\begin{aligned}
& (\Delta - \lambda)\beta + \sum n(r, a_{(i)}) + n(r, \tfrac{1}{\Omega_2}) \\
=\ & (\Delta - \lambda_{(n+1)})\beta + (\lambda_{(n+1)} - \lambda)\beta + \sum n(r, a_{(i)}) + n(r, \tfrac{1}{\Omega_2}) \\
\le\ & (\Delta - \lambda_{(n+1)})\beta + \sum n(r, a_{(i)}) + n(r, \tfrac{1}{\Omega_2}).
\end{aligned}$$

Let $\Delta_0 = \max\{\Delta_{(n+1)}, \Delta\}$. Then

$$n(r, \Omega) \le (\Delta_0 - \lambda_{(n+1)})\beta + \sum n(r, a_{(i)}) + n(r, \tfrac{1}{\Omega_2}).$$

Case (b): If $\lambda_{(n+1)} \ge \lambda$, we have

$$\begin{aligned}
& (\Delta_{(n+1)} - \lambda_{(n+1)})\beta + n(r, a_{(n+1)}) + n(r, \tfrac{1}{\Omega_2}) \\
=\ & (\Delta_{(n+1)} - \lambda)\beta + (\lambda - \lambda_{(n+1)})\beta + n(r, a_{(n+1)}) + n(r, \tfrac{1}{\Omega_2}) \\
\le\ & (\Delta_{(n+1)} - \lambda)\beta + n(r, a_{(n+1)}) + n(r, \tfrac{1}{\Omega_2}).
\end{aligned}$$

Again, $\Delta_0 = \max\{\Delta_{(n+1)}, \Delta\}$. Then

$$n(r, \Omega) \le (\Delta_0 - \lambda)\beta + \sum_{i=1}^{n+1} n(r, a_{(i)}) + n(r, \tfrac{1}{\Omega_2}).$$

Combining Case (a) with Case (b), we can obtain that inequality (2.6) holds for $i = n+1$. Thus, inequality (2.5) is proved. From (2.3) and (2.5), we completes the proof of Lemma 2.5. □

Lemma 2.6 ([6]). *Let $U(r), H(r)$ ($r \in [0,\infty)$) be two nonnegative and non-decreasing functions, $H(r) \to \infty$ as $r \to \infty$, $\tilde{a}$ and $\tilde{b}$ two positive numbers, $H(r_0) \geq \max\{(\tilde{a}+\tilde{b})\log 2, 2^{2+\frac{\tilde{b}}{\tilde{a}}}\tilde{a}(\tilde{a}+\tilde{b})\}$, for all r and t, $0 < r_0 \leq r < t$, if the following inequality satisfies*

$$U(r) < \tilde{a}\log^+ U(t) + \tilde{b}\log\frac{t}{t-r} + H(r),$$

then we have for $0 < r_0 \leq r < t$,

$$U(r) < (\tilde{a}+\tilde{b})\log\frac{t}{t-r} + 2H(t).$$

3. Proof of Theorem 1.1

Proof. We rewrite equation (1.1) as

$$\left[\tfrac{\Omega_1(z,w)}{w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}}\right]^m = a_k(w+d(z))^k + \sum_{s=0}^{t} d_s(z)w^s,\ 0 \leq t \leq k-2,$$

where $d(z) = \frac{a_{k-1}}{ka_k}$ and d_s is a rational function of a_i ($0 \leq i \leq k$). Let

$$A = -a_k(z)(w+d(z))^k,\ B = \left[\tfrac{\Omega_1(z,w)}{w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}}\right]^m,\ \Phi = \sum_{s=0}^{t} d_s(z)w^s.$$

Then

$$A + B = \Phi.$$

Next, we claim that A and B are linearly independent.

We will prove this claim by contradiction. If A and B are linearly dependent, by the knowledge of linear algebra, there exist a and b such that

$$aA(z) + bB(z) = 0, \quad |a| + |b| \neq 0. \tag{3.1}$$

From (1.1) and (3.1), one can deduce that

$$aa_kw^k + aa_{k-1}w^{k-1} + \cdots + aa_kd^k = ba_kw^k + ba_{k-1}w^{k-1} + \cdots + ba_0.$$

Let D be a field of meromorphic functions a_i satisfying $T(r, a_i) = S(r, w)$. Then $1, w, w^2, \ldots, w^k$ are linear independent over D. It shows that $a = b$.

But

$$aA(z) + bB(z) = a(A(z) + B(z)) = a\sum_{s=0}^{t} d_s(z)w^s.$$

Since $\sum_{s=0}^{t} d_s(z)w^s \not\equiv 0$, this is a contradiction. Thus we have $a = b = 0$.

By Lemma 2.1, we obtain

$$T(r,A) \leq m(r,\Phi) + \overline{N}(r,A) + \overline{N}(r,B) + \overline{N}(r,\tfrac{1}{A}) + \overline{N}(r,\tfrac{1}{B}) + S(r). \tag{3.2}$$

Now, we give the estimation of each term of (3.2).

From Lemma 2.3, we obtain

$$T(r,A)=kT(r,w)+T(r,a_k)+T(r,a_{k-1}),$$

$$T(r,\Phi)=tT(r,w)+\sum_{s=0}^{t}T(r,d_s),$$

$$N(r,A)\le kN(r,w)+N(r,a_k)+N(r,\tfrac{1}{a_k})+N(r,a_{k-1}),$$

$$T(r,\tfrac{\Omega_1(z,w)}{w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}})=\frac{T(r,\Phi-A)}{m}=\frac{k}{m}T(r,w)+\frac{1}{m}\sum_{i=0}^{k}T(r,a_i).$$

By Lemma 2.4, we have

$$N(r,B)\le C_5(N(r,w)+N(r,\tfrac{1}{w})+N_x(r,w))-C_4N_b(r,w)+\sum_{(i)}T(r,a_{(i)})+S(r,w).$$

Note that

$$\begin{aligned}\overline{N}(r,\tfrac{1}{A})&\le\overline{N}(r,\tfrac{1}{w+d(z)})+\overline{N}(r,\tfrac{1}{a_k})\\&\le T(r,w)+T(r,d(z))+T(r,a_k)+O(1),\end{aligned}$$

$$\begin{aligned}\overline{N}(r,\tfrac{1}{B})&=\overline{N}(r,1/\tfrac{\Omega_1(z,w)}{w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}})\\&\le T(r,\tfrac{\Omega_1(z,w)}{w^{k_0}(w')^{k_1}\cdots(w^{(n)})^{k_n}})+O(1)\\&\le\frac{k}{m}T(r,w)+\frac{1}{m}\sum_{j=0}^{k}T(r,a_j)+O(1).\end{aligned}$$

By the above estimation of each term and combining with (3.2), it gives

$$\begin{aligned}kT(r,w)\le\,&T(r,w)+\frac{k}{m}T(r,w)+tT(r,w)+(k+1)N(r,w)\\&-C_4N_b(r,w)+C_5(N(r,w)+N(r,\tfrac{1}{w})+N_x(r,w))\\&+K_1\Big[\sum_{(i)}T(r,a_{(i)})+\sum_{j=0}^{k}T(r,a_j)\Big]+S(r,w).\end{aligned}$$

Further,

$$\begin{aligned}&\left(k-1-\frac{k}{m}-t\right)T(r,w)\\&\le(k+1)N(r,w)+C_5(N(r,w)+N(r,\tfrac{1}{w})+N_x(r,w))\\&\quad-C_4N_b(r,w)+K_1\Big[\sum_{(i)}T(r,a_{(i)})+\sum_{j=0}^{k}T(r,a_j)\Big]+S(r,w).\end{aligned}$$

By the assumption that $m>k\ge 2$ and $0\le t\le k-2$, we can give that $(k-1-\frac{k}{m}-t)>0$. Immediately, we have for constants K, K_1 and K_2

$$T(r,w)\le K(N(r,w)+N(r,\tfrac{1}{w})+N_x(r,w))-K_1N_b(r,w)$$

$$+ K_2\Big[\sum_{(i)} T(r, a_{(i)}) + \sum_{j=0}^{k} T(r, a_j)\Big] + S(r, w).$$

This completes the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

Proof. First, suppose that $w(z)$ is a ν-valued algebriod function satisfying $\sum_{i=0}^{p} a_i(z)w^i \equiv 0$. Then we have

$$a_p(z)w^p = -a_{p-1}(z)w^{p-1} - \cdots - a_0(z).$$

By applying Lemma 2.3 on the above equation, one can see that there exists a positive constant K_0 such that

$$pT(r, w) + T(r, a_p) \leq (p-1)T(r, w) + \sum_{i=0}^{p-1} T(r, a_i(z)),$$

$$T(r, w) \leq K \sum_{i=0}^{p} T(r, a_i(z)) \leq K_0 F(r).$$

If $\sum_{i=0}^{p} a_i(z)w^i \not\equiv 0$, we rewrite equation (1.2) as follows

$$Q(z, w) \cdot \frac{\Omega_1(z, w)}{(w - \hat{a})^{\lambda_1}\Omega_2} = P(z, w). \tag{4.1}$$

Since

$$\frac{\Omega_1}{(w - \hat{a})^{\lambda_1}\Omega_2} = \sum a_{(i)} \Big(\frac{w}{w - \hat{a}}\Big)^{i_0} \Big(\frac{(w - \hat{a})'}{w - \hat{a}}\Big)^{i_1} \cdots \Big(\frac{(w - \hat{a})^{(n)}}{w - \hat{a}}\Big)^{i_n} \frac{1}{(w - \hat{a})^{\lambda_1 - \sum_{l=0}^{n} i_l}} \frac{1}{\Omega_2},$$

we have

$$\begin{aligned} & m(r, \tfrac{\Omega_1}{(w-\hat{a})^{\lambda_1}\Omega_2}) \\ & \leq \lambda_1 m(r, \tfrac{1}{w-\hat{a}}) + m(r, \tfrac{1}{\Omega_2}) + m(r, \sum a_{(i)}) + \sum_{\alpha=1}^{n} m(r, \tfrac{(w-\hat{a})^{(\alpha)}}{w-\hat{a}}). \end{aligned} \tag{4.2}$$

By Lemma 2.5, we get

$$\begin{aligned} N(r, \tfrac{\Omega_1}{(w-\hat{a})^{\lambda_1}\Omega_2}) \leq & \lambda_1 N(r, \tfrac{1}{w-\hat{a}}) + (\Delta_1 - \lambda_1)(\overline{N}(r, w) + N_b(r, w)) \\ & + N(r, \tfrac{1}{\Omega_2}) + n(r, \sum a_{(i)}). \end{aligned} \tag{4.3}$$

From (4.2) and (4.3), we obtain

$$\begin{aligned} T(r, \tfrac{\Omega_1}{(w-\hat{a})^{\lambda_1}\Omega_2}) \leq & \lambda_1 T(r, \tfrac{1}{w-\hat{a}}) + (\Delta_1 - \lambda_1)(\overline{N}(r, w) + N_b(r, w)) \\ & + T(r, \Omega_2) + T(r, \sum a_{(i)}) + \sum_{\alpha=1}^{n} m(r, \tfrac{(w-\hat{a})^{(\alpha)}}{w-\hat{a}}). \end{aligned}$$

Further, we have

$$\begin{aligned}&T(r,\tfrac{\Omega_1}{(w-\hat{a})^{\lambda_1}\Omega_2})\\ &\leq \lambda_1 T(r,\tfrac{1}{w-\hat{a}})+\lambda_2 T(r,w)+(\Delta_2-\lambda_2+\Delta_1-\lambda_1)\overline{N}(r,w)\\ &\quad+(\Delta_1-\lambda_1-l_2)N_b(r,w)+\sigma_2 N_x(r,w)+T(r,\sum a_{(i)})\\ &\quad+T(r,\sum b_{(j)})+\sum_{\alpha=1}^{n} m(r,\tfrac{(w-\hat{a})^{(\alpha)}}{w-\hat{a}}).\end{aligned}$$

Using Lemma 2.3 and together with the above inequality, we get

$$\begin{aligned}&T(r,Q(z,w)\tfrac{\Omega_1(z,w)}{(w-\hat{a})^{\lambda_1}\Omega_2(z,w)})\\ &\leq T(r,Q(z,w))+T(r,\tfrac{\Omega_1(z,w)}{(w-\hat{a})^{\lambda_1}\Omega_2(z,w)})\\ &\leq qT(r,w)+(\lambda_1+\lambda_2)T(r,w)+(\Delta_2-\lambda_2+\Delta_1-\lambda_1)\overline{N}(r,w)\\ &\quad+(\Delta_1-\lambda_1-l_2)N_b(r,w)+\sigma_2 N_x(r,w)\\ &\quad+\sum_{(i)} T(r,a_{(i)})+\sum_{(j)} T(r,b_{(j)})+\sum_{j=0}^{q} T(r,b_j)\\ &\quad+\sum_{\alpha=1}^{n} m(r,\tfrac{w^{(\alpha)}}{w})+\sum_{\alpha=1}^{n} m(r,\tfrac{(w-\hat{a})^{(\alpha)}}{w-\hat{a}}).\end{aligned}\tag{4.4}$$

By means of Lemma 2.3, we get

$$T(r,P(z,w))=pT(r,w)+O\big\{\sum_{i=0}^{p} T(r,a_i)\big\}.\tag{4.5}$$

From (4.4) and (4.5), it yields

$$\begin{aligned}pT(r,w)<{}&(q+\lambda_1+\lambda_2)T(r,w)+(\Delta_2-\lambda_2+\Delta_1-\lambda_1)\overline{N}(r,w)\\ &+(\Delta_1-\lambda_1-l_2)N_b(r,w)+\sigma_2 N_x(r,w)\\ &+\sum_{(i)} T(r,a_{(i)})+\sum_{(j)} T(r,b_{(j)})+\sum_{j=0}^{q} T(r,b_j)\\ &+\sum_{\alpha=1}^{n} m(r,\tfrac{w^{(\alpha)}}{w})+\sum_{\alpha=1}^{n} m(r,\tfrac{(w-\hat{a})^{(\alpha)}}{w-\hat{a}}).\end{aligned}$$

Noting that $p>q+\lambda_1+\lambda_2$. One can observe

$$\begin{aligned}T(r,w)<{}&\frac{\Delta_2-\lambda_2+\Delta_1-\lambda_1}{p-(q+\lambda_1+\lambda_2)}\overline{N}(r,w)+\frac{\sigma_2}{p-(q+\lambda_1+\lambda_2)}N_x(r,w)\\ &+\frac{(\Delta_1-\lambda_1-l_2)}{p-(q+\lambda_1+\lambda_2)}N_b(r,w)+Q_1(r)+D(r),\end{aligned}\tag{4.6}$$

where

$$Q_1(r) = \frac{1}{p-(q+\lambda_1+\lambda_2)}\Big\{\sum_{(i)} T(r, a_{(i)}) + \sum_{(j)} T(r, b_{(j)}) + \sum_{j=0}^{q} T(r, b_j)\Big\},$$

$$D(r) = \sum_{\alpha=1}^{n} m(r, \tfrac{w^{(\alpha)}}{w}) + \sum_{\alpha=1}^{n} m(r, \tfrac{(w-\hat{a})^{(\alpha)}}{w-\hat{a}}).$$

By applying the generalized lemma of logarithmic derivative to $D(r)$, it is easy to get from the inequality (4.6) that

$$T(r, w) < \tilde{a}\log T(t, w) + \tilde{b}\log\frac{t}{t-r} + H(r), \tag{4.7}$$

where $\tilde{a}$ and $\tilde{b}$ are positive constants, and

$$H(r) = \frac{\Delta_2-\lambda_2+\Delta_1-\lambda_1}{p-(q+\lambda_1+\lambda_2)}\overline{N}(r, w) + \frac{\sigma_2}{p-(q+\lambda_1+\lambda_2)}N_x(r, w) + Q_1(r).$$

Applying Lemma 2.4 to (4.7) and we get

$$T(r, w) < (\tilde{a}+\tilde{b})\log\frac{t}{t-r} + 2H(t).$$

Set $t = \sigma r$, $\sigma > 1$. Then $T(r, w) \le K_0 F(\sigma r)$. The proof is completed. □

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