

ON TRANSLATION LENGTHS OF PSEUDO-ANOSOV MAPS ON THE CURVE GRAPH

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ABSTRACT. We show that a pseudo-Anosov map constructed as a product of the large power of Dehn twists of two filling curves always has a geodesic axis on the curve graph of the surface. We also obtain estimates of the stable translation length of a pseudo-Anosov map, when two filling curves are replaced by multicurves. Three main applications of our theorem are the following: (a) determining which word realizes the minimal translation length on the curve graph within a specific class of words, (b) giving a new class of pseudo-Anosov maps optimizing the ratio of stable translation lengths on the curve graph to that on Teichmüller space, (c) giving a partial answer of how much power is needed for Dehn twists to generate right-angled Artin subgroup of the mapping class group.

1. Introduction

Let $S = S_{g,n}$ be a connected orientable surface with genus g and n punctures. The mapping class group of the surface S , written as $\text{Mod}(S)$, is the group of orientation-preserving homeomorphisms of S up to isotopy. For a surface S , letting each isotopy class of simple closed curve as vertex, and connecting edge between two distinct vertices when they can be represented by pair of disjoint curves, we obtain the curve graph $\mathcal{C}(S)$. By assigning each edge length 1, $\mathcal{C}(S)$ is a δ -hyperbolic space [9] and the natural action of $\text{Mod}(S)$ on curve graph is an isometric action.

The asymptotic translation length (also known as stable translation length) of a mapping class $f \in \text{Mod}(S)$ on the curve graph is defined as

$$l_{\mathcal{C}}(f) = \lim_{n \rightarrow \infty} \frac{d_{\mathcal{C}}(\alpha, f^n(\alpha))}{n},$$

where α is any vertex of $\mathcal{C}(S)$. The triangle inequality guarantees that the definition of $l_{\mathcal{C}}(f)$ does not depend on the choice of α . Masur-Minsky showed that $l_{\mathcal{C}}(f) > 0$ if and only if f is a pseudo-Anosov element [9]. Furthermore, Bowditch showed that $l_{\mathcal{C}}(f)$ is always a rational number by showing that there

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exists n depending only on the type of surface such that f^n preserves a geodesic axis [4]. In other words, the asymptotic translation of a pseudo-Anosov map on the curve graph is always a rational number with a bounded denominator.

While there have been many research works on the asymptotic translation length of pseudo-Anosov maps on the curve graph, there are only a few known results about which pseudo-Anosov maps have integer asymptotic translation lengths. In the case of a torus, authors together with Kwak and Shin proved that for any Anosov map on a torus, there is always a geodesic axis on the curve graph of torus [3]. Thus in this case the asymptotic translation length agrees with the translation length, and they are always positive integers. In the non-sporadic case, Watanabe in [14] obtained certain sufficient conditions for a pseudo-Anosov map to have an integer asymptotic translation length.

Here we construct another class of pseudo-Anosov maps that preserve geodesic axes and explicitly calculate the translation lengths' value from our construction. Our main theorem is the following:

Theorem (Theorem 3.1). *Let α, β be two filling curves with $d_{\mathcal{C}}(\alpha, \beta) = l \geq 3$. Pick a geodesic path p joining α and β in $\mathcal{C}(S)$, and write p as $\alpha, v_1, \dots, v_{l-1}, \beta$. Let $f = T_{\alpha}^{a_1} T_{\beta}^{b_1} \dots T_{\alpha}^{a_n} T_{\beta}^{b_n}$, and for each $1 \leq i \leq 2n$, denote by f_i the partial product of f up to the first i powers of Dehn twists. Then there exists a universal constant $M > 0$ such that the following holds. If $|a_i|, |b_i| > 2M$ for all $1 \leq i \leq 2n$, then the concatenated path \mathcal{P} formed by $p \cdot f_1 p \dots f_{2n-1} p \cdot f p$ is a geodesic. Moreover, f is a pseudo-Anosov map, and it has the orbit of \mathcal{P} as its geodesic axis. In particular, the asymptotic translation length is given by $l_{\mathcal{C}}(f) = 2n(l - 2)$.*

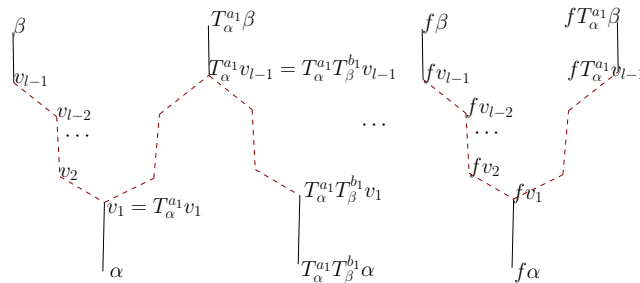


FIGURE 1. Geodesic axis of map f

Gadre, Hironaka, Kent, and Leininger showed that the minimal value of the ratio between the asymptotic translation length on the curve graph and the translation length on the Teichmüller space is asymptotic to $\log |\chi(S)|$ [6]. Aougab-Taylor gave a construction of ratio optimizers which is a class of pseudo-Anosov maps that realize asymptotic minimum [2]. As an application of the first theorem, we give a larger class of ratio optimizers.

Generalizing the first theorem, we also give bounds on the asymptotic translation lengths of pseudo-Anosov maps when the number of curves used in the construction is more than two. We first deal with the case where a map is a product of the powers of Dehn twists, with each neighboring pair of curves in the map filling the surface. We compute how large the exponent should be for each Dehn twist to give a bound of the asymptotic translation lengths.

Two well-known constructions of pseudo-Anosov maps come from two filling multicurves, namely Thurston's [13] and Penner's [10]. In the case of two filling multicurves, we can also give bounds on the asymptotic translation length of a product of Dehn twists, where the exponents in the powers of Dehn twists satisfy a certain condition. This allows us to calculate the asymptotic translation length of pseudo-Anosov maps on a curve graph in Thurston's construction and Penner's construction, as long as we take high enough powers of Dehn twists. Furthermore, we figure out how large the exponent of powers should be in order to give similar bounds when the number of multicurves is more than two. This bound can be used to find words that give minimal translation length in a specific class of words.

It is proven by Koberda that for a finite set of curves C , there exists K such that for $n > K$, $\{T_\gamma^n\}_{\gamma \in C}$ generates right-angled Artin groups [7]. In the same paper, it is asked if one can give an explicit estimate of K . It is also known that K depends on the choice of curves in C . As an application of our bounds on the asymptotic translation lengths on the curve graphs, we give an explicit estimate of K to generate a right-angled Artin group under a specific condition on the configuration of the curves.

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2. Background

2.1. Subsurface projection

Masur and Minsky defined subsurface projection in [9], which is a map from the set of vertices of the curve graph of a surface to the power set of the set of vertices of the curve graph of a subsurface. When the subsurface is annular, subsurface projection sends the core of the annulus and curves not intersecting the core to the empty set.

Theorem 2.1 (Bounded geodesic image theorem, [9]). *Assume for a subsurface Y in S and geodesic g in $\mathcal{C}(S)$, all vertices of g have a nontrivial projection on the curve graph of Y . Then the whole projection of g on curve graph of Y has a diameter smaller than M .*

It is proven by Webb that M can be chosen as a universal constant that does not depend on the type of S or Y [15]. For a simple closed curve α on a surface, let's say annular subsurface around α as Y . We use the notation $\mathcal{C}(\alpha) = \mathcal{C}(Y)$ as the curve graph of subsurface Y and $d_\alpha = d_Y$ as the distance on the curve graph of subsurface Y .

Bounded geodesic image theorem implies that if three curves $\alpha, \beta, \gamma \in \mathcal{C}(S)$ satisfy $d_\alpha(\beta, \gamma) > M$, any geodesic path joining β and γ must intersect one-neighborhood of α in $\mathcal{C}(S)$.

We use the bounded geodesic image theorem to restrict the geodesic path between two points. To enlarge the distance on the curve graph of a subsurface, we need the following lemma.

Lemma 2.2 (Equation 2.6, [9]). *For $\alpha \in \mathcal{C}_0(S)$ and $\beta \in \mathcal{C}_0(\alpha)$, the following holds.*

$$d_\alpha(\beta, T_\alpha^n(\beta)) = |n| + 2.$$

2.2. Pseudo-Anosov construction

We call a set of curves as a multicurve if it is made of non-intersecting essential simple closed curves. For two multicurves $A = \{\alpha_1, \dots, \alpha_m\}$ and $B = \{\beta_1, \dots, \beta_n\}$ on surface S , we say A and B fill S if complement $S \setminus (A \cup B)$ contains only disks and once punctured disks components. Two curves α and β satisfy $d_{\mathcal{C}}(\alpha, \beta) \geq 3$ if and only if when α and β are filling curves. This can be shown by that if $S \setminus (\alpha \cup \beta)$ contains a component that has higher complexity, a curve can be drawn on it avoiding α and β , which makes $d_{\mathcal{C}}(\alpha, \beta) \leq 2$.

Two well-known pseudo-Anosov constructions are using Dehn twists of two filling multicurves, which are by Thurston and Penner [10, 13]. Further details and explanations of those constructions can be also found in Chapter 14.1 in [5]. For a multicurve $A = \{\alpha_1, \dots, \alpha_m\}$, let T_A denote $T_{\alpha_1} \cdots T_{\alpha_m}$.

Theorem 2.3 ([13]). *Assume $A = \{\alpha_1, \dots, \alpha_m\}$ and $B = \{\beta_1, \dots, \beta_n\}$ be two multicurves that fill a surface S . Let N be the matrix by $N_{i,j} = i(\alpha_i, \beta_j)$ and μ be the largest real eigenvalue of NN^T . Then there exists a representation $\rho : \langle T_A, T_B \rangle \rightarrow \text{PSL}(2, \mathbb{R})$ given by*

$$T_A \mapsto \begin{bmatrix} 1 & \sqrt{\mu} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad T_B \mapsto \begin{bmatrix} 1 & 0 \\ -\sqrt{\mu} & 1 \end{bmatrix},$$

where $f \in \langle T_A, T_B \rangle$ is a pseudo-Anosov map if and only if $\rho(f)$ is hyperbolic. Moreover, if f is a pseudo-Anosov map, the stretch factor of f is equal to $\rho(f)$'s largest real eigenvalue.

Theorem 2.4 ([10]). *Assume $A = \{\alpha_1, \dots, \alpha_m\}$ and $B = \{\beta_1, \dots, \beta_n\}$ be two multicurves that fill a surface S . If f is a word made from the product of positive Dehn twists about α_i and negative Dehn twists about β_j where all α_i, β_j are used at least once, then f is a pseudo-Anosov map.*

Mangahas suggested a new recipe for pseudo-Anosov maps using pure reducible mapping classes [8]. To prove the map is pseudo-Anosov, Mangahas

used the following lemma to show map's stable translation length is positive. We also use a slightly altered version of the lemma which is restricted to the case of annular subsurfaces to construct classes of pseudo-Anosov maps and to calculate stable translation length. This lemma will be used as our main tool in Section 3. Let $N_1(\alpha_i)$ as the 1-neighborhood of α_i in $\mathcal{C}(S)$ and let M be the universal constant from the bounded geodesic image theorem.

Lemma 2.5 ([8]). *For set of curves $\{\alpha_1, \dots, \alpha_n\}$, let $X_i = N_1(\alpha_i)$, Y_i be an annular subsurface around α_i . Assume $d_{\mathcal{C}}(\alpha_i, \alpha_{i+1}) \geq 3$ and $d_{\alpha_i}(\alpha_{i-1}, \alpha_{i+1}) > 2M + 2$ for all i . Then for any $w_i \in X_i$ and $w_{i+k} \in X_{i+k}$, the geodesic $[w_i, w_{i+k}] \cap X_j \neq \emptyset$ for $i \leq j \leq i+k$. Also, the X_j are pairwise disjoint. Furthermore there exists a geodesic $[w_i, w_{i+k}]$, where elements from $X_i, X_{i+1}, \dots, X_{i+k}$ appear in same order in geodesic.*

3. Translation length on curve graph

Using Mangahas' lemma, Aougab-Taylor gave a lower bound for the asymptotic translation length of pseudo-Anosov map that is made from large enough power of two filling curves' Dehn twists [2]. Here we extend that idea to give the exact translation length by showing that it preserves a geodesic.

Let's say Dehn twists around curves α and β as T_α and T_β , and let M as the universal constant from the bounded geodesic image theorem.

Theorem 3.1. *Let α, β be two filling curves with $d_{\mathcal{C}}(\alpha, \beta) = l \geq 3$. Pick a geodesic path p joining α and β in $\mathcal{C}(S)$, and write p as $\alpha, v_1, \dots, v_{l-1}, \beta$. Let $f = T_\alpha^{a_1} T_\beta^{b_1} \dots T_\alpha^{a_n} T_\beta^{b_n}$, and for each $1 \leq i \leq 2n$, denote by f_i the partial product of f up to the first i powers of Dehn twists. Let M be the universal constant from the bounded geodesic image theorem. Then for each integers a_i, b_i whose absolute values are greater than $2M$ for all $1 \leq i \leq 2n$, the concatenated path \mathcal{P} formed by $p \cdot f_1 p \cdot \dots \cdot f_{2n-1} p \cdot f p$ is a geodesic. Moreover, f is a pseudo-Anosov map, and it has the orbit of \mathcal{P} as its geodesic axis. In particular, the asymptotic translation length is given by $l_{\mathcal{C}}(f) = 2n(l - 2)$.*

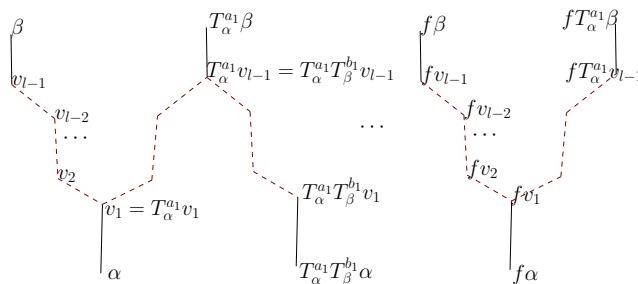


FIGURE 2. Geodesic axis of map f

Proof. Let $C_A = N_1(\alpha)$, $C_B = N_1(\beta)$, and let $X_0 = C_B$, $X_1 = C_A$, $X_{2i} = f_{2i}C_B$, $X_{2i+1} = f_{2i+1}C_A$. Now the shortest path between X_i and X_{i+1} is length $l - 2$, since it is an isomorphic image of C_A and C_B by f_i . This path can be realized as $f_i p = (f_i v_1, \dots, f_i v_{l-1})$. By canceling f_{2i-1} and using $f_{2i}\beta = f_{2i-1}\beta$ and $f_{2i+1}\alpha = f_{2i}\alpha$, we get following by Lemma 2.2:

$$d_{f_{2i}\beta}(f_{2i-1}\alpha, f_{2i+1}\alpha) = d_\beta(\alpha, T_\beta^{b_i}\alpha) = |b_i| + 2 > 2M + 2.$$

Similarly $d_{f_{2i+1}\alpha}(f_{2i}\beta, f_{2i+2}\beta) > 2M + 2$, thus conditions of Lemma 2.5 are satisfied. Let's pick a geodesic path between v_1 and $f v_1$. Then by Lemma 2.5 this path must meet all X_i between them. It must contain $2n$ paths of X_i and X_{i+1} and they should be all disjoint, so has a length of at least $2n(l - 2)$. Concatenated path of $f_1 p$ to $f_{2n} p$ realizes this path, so we have $d(v_1, f v_1) = 2n(l - 2)$. By our setting, same logic holds for f^m for any positive m , thus it makes $d(v_1, f^m v_1) = 2mn(l - 2)$, which proves rest. \square

This can be also applied to give bound to pseudo-Anosov maps made with more than two Dehn twists.

Theorem 3.2. *Consider curves $\alpha_1, \alpha_2, \dots, \alpha_k$ which satisfy $d_C(\alpha_i, \alpha_{i+1}) \geq 3$, where we set $\alpha_k = \alpha_0$ and $\alpha_{k+1} = \alpha_1$. For $f = T_{\alpha_1}^{n_1} T_{\alpha_2}^{n_2} \dots T_{\alpha_k}^{n_k}$ with $|n_i| > 2M + 2 + d_{\alpha_i}(\alpha_{i-1}, \alpha_{i+1})$, its translation length is bounded as follows:*

$$\sum_{i=1}^k d_C(\alpha_i, \alpha_{i+1}) - 2k \leq l_C(f) \leq \sum_{i=1}^k d_C(\alpha_i, \alpha_{i+1}).$$

In particular, f is a pseudo-Anosov map.

Proof. Let $f_i = T_{\alpha_1}^{n_1} T_{\alpha_2}^{n_2} \dots T_{\alpha_i}^{n_i}$, $X_i = f_i N_1(\alpha_i)$ with Y_i be an annular subsurface that has core $\beta_i = f_i \alpha_i$, where index is given modulo k . Now $d_{\beta_i}(\beta_{i-1}, \beta_{i+1}) = d_{f_i \alpha_i}(f_{i-1} \alpha_{i-1}, f_{i+1} \alpha_{i+1})$. Since $\text{Mod}(S)$ preserves relation between curves on $\mathcal{C}(S)$, deleting f_{i-1} gives $d_{f_i \alpha_i}(f_{i-1} \alpha_{i-1}, f_{i+1} \alpha_{i+1}) = d_{\alpha_i}(\alpha_{i-1}, T_{\alpha_i}^{n_i} \alpha_{i+1})$. Similarly $d_C(\beta_i, \beta_{i+1}) = d_C(f_i \alpha_i, f_{i+1} \alpha_{i+1}) = d_C(\alpha_i, \alpha_{i+1})$. From triangle inequality and Lemma 2.2 we get

$$d_{\alpha_i}(\alpha_{i-1}, T_{\alpha_i}^{n_i} \alpha_{i+1}) \geq d_{\alpha_i}(\alpha_{i+1}, T_{\alpha_i}^{n_i} \alpha_{i+1}) - d_{\alpha_i}(\alpha_{i-1}, \alpha_{i+1}) > 2M + 2.$$

Thus it satisfies the condition of Lemma 2.5. Now pick any $\omega_0 \in X_0 = N_1(\alpha_0)$. Then we can see geodesic path from ω_0 to $f^t \omega_0$ have an intersection with all X_i for all $0 < i < tk$, where index is given in modulo k . Thus we get

$$t \left(\sum_{i=1}^k d_C(\alpha_i, \alpha_{i+1}) - 2k \right) \leq d_C(\omega_0, f^t \omega_0).$$

Dividing into t and sending it to infinity gives us the desired lower bound. From $\beta_1 = T_{\alpha_1}^{n_1} \alpha_1 = \alpha_1$ and $\beta_{k+1} = f \alpha_1$ upper bound comes from inequality

$$d_C(\alpha_1, f \alpha_1) \leq \sum_{i=1}^k d_C(\beta_i, \beta_{i+1}) = \sum_{i=1}^k d_C(\alpha_i, \alpha_{i+1}). \quad \square$$

Here we can see that a pair of filling curves is a special case of the theorem, where the lower bound coincides with the exact translation length.

Though it may not preserve a geodesic like in two curve case, we can still estimate stable translation lengths of two well-known constructions of pseudo-Anosov maps, Thurston's and Penner's construction, under some conditions. To track the length on the curve graph of the subsurface in case of multicurve's Dehn twist, we need the following lemma from [14].

Lemma 3.3 ([14]). *For multicurve $A = \{\alpha_1, \dots, \alpha_n\}$ let $f = T_{\alpha_1}^{a_1} \cdots T_{\alpha_n}^{a_n}$. Then for any curve γ intersecting α_i transversely*

$$d_{\alpha_i}(\gamma, f(\gamma)) \geq |a_i|.$$

On the curve graph of an annular subsurface, while n times of Dehn twist about the core of an annulus sends an image to distance $|n|+2$ position from the original, Dehn twists about curves that do not intersect the core have a minor effect. For $A, B \subset \mathcal{C}(S)$, let's define $\text{dist}(A, B) = \inf\{d_{\mathcal{C}}(a, b) \mid a \in A, b \in B\}$.

Theorem 3.4. *Let filling multicurves $A = \{\alpha_1, \dots, \alpha_m\}$, $B = \{\beta_1, \dots, \beta_n\}$ satisfy $\text{dist}(A, B) = l \geq 3$. Let $f = f_1 \cdots f_{2k}$, where f_{2i-1}, f_{2i} are products of Dehn twists by curves in A, B , respectively, and each f_i contains at least one Dehn twist T_{γ}^t with $|t| > 2M + 3$ for some $\gamma \in A \cup B$. Then*

$$2kl - 4k \leq l_{\mathcal{C}}(f) \leq 2kl.$$

In particular f is a pseudo-Anosov map.

Proof. Let $F_i = f_1 \cdots f_i$, and $X_i = F_i N_1(\gamma_i)$, where γ_i is a curve from f_i which Dehn twist's exponent's absolute value is largest. Here γ_i is either element of A or B , thus $A \subset N_1(\gamma_i)$ or $B \subset N_1(\gamma_i)$ since A and B are multicurves. From Lemma 3.3 we get $d_{\gamma_i}(\gamma_{i+1}, f_i(\gamma_{i+1})) > 2M + 3$ and $d_{\gamma_i}(\gamma_{i-1}, \gamma_{i+1}) \leq 1$, thus $d_{\gamma_i}(\gamma_{i-1}, f_i(\gamma_{i+1})) > 2M + 2$. Letting Y_i as an annular subsurface around core $F_i \gamma_i$, X_i, Y_i satisfy conditions of Lemma 2.5. Any geodesic connecting $N_1(\gamma_1)$ to $f N_1(\gamma_1)$ must have nonempty intersection with X_i . On our geodesic, distance between intersection of X_i and X_{i+1} is bounded between $l - 2$ and l . Thus we get following

$$2kl - 4k \leq \text{dist}(X_1, f X_1) \leq 2kl.$$

All our logic still holds for f^t , thus we can expand the situation to

$$t(2kl - 4k) \leq \text{dist}(X_1, f^t X_1) \leq 2tkl.$$

Dividing all sides by t gives us the desired result. □

Theorem 3.5. *Let multicurves $C_1 \cdots C_n$ satisfy $\text{dist}(C_i, C_{i+1}) \geq 3$, $C_n = C_0$ and $C_{n+1} = C_1$. Let $f = f_1 \cdots f_n$, where f_i is a product of Dehn twists by curve in C_i , where each f_i contains at least one Dehn twist T_{γ}^t with $|t| > 2M + 3 + d_{\gamma}(C_{i-1}, C_{i+1})$. Then following holds*

$$\sum_{i=1}^n \text{dist}(C_i, C_{i+1}) - 2n \leq l_{\mathcal{C}}(f) \leq \sum_{i=1}^n \text{dist}(C_i, C_{i+1}).$$

In particular f is a pseudo-Anosov map.

Proof. Let $F_i = f_1 \cdots f_i$, and $X_i = F_i N_1(\gamma_i)$, where γ_i is a curve from f_i which satisfies the condition of theorem and Dehn twist's exponent's absolute value is largest. From Lemma 3.3 we get $d_{\gamma_i}(\gamma_{i+1}, f_i(\gamma_{i+1})) > 2M + 3 + d_{\gamma}(C_{i-1}, C_{i+1})$. Then by canceling F_{i-1} and using triangle inequality, we get following:

$$\begin{aligned} d_{F_i \gamma_i}(F_{i-1} \gamma_{i-1}, F_{i+1} \gamma_{i+1}) &= d_{\gamma_i}(\gamma_{i-1}, f_i(\gamma_{i+1})) \\ &= d_{\gamma_i}(\gamma_{i+1}, f_i(\gamma_{i+1})) - d_{\gamma_i}(\gamma_{i-1}, \gamma_{i+1}) \\ &\geq 2M + 4 + d_{\gamma_i}(C_{i-1}, C_{i+1}) - d_{\gamma_i}(\gamma_{i-1}, \gamma_{i+1}) \\ &\geq 2M + 2. \end{aligned}$$

Here, the last inequality comes from that $\text{diam}_{\gamma_i}(C_{i-1}), \text{diam}_{\gamma_i}(C_{i+1}) \leq 1$. Thus letting Y_i as annular subsurface around core $F_i \gamma_i$, our X_i, Y_i satisfy conditions of Lemma 2.5.

$$\sum_{i=1}^n \text{dist}(C_i, C_{i+1}) - 2n \leq d_C(X_1, fX_1) \leq \sum_{i=1}^n \text{dist}(C_i, C_{i+1}).$$

All our condition still holds for f^k , thus changing to f^k and dividing all the sides by k gives us desired result. □

4. Applications

4.1. Minimal word

One direction of understanding translation length is to find an element that gives the smallest translation length among the ones that use the same number of each Dehn twist. We give a partial answer to this question.

Theorem 4.1. *Let α, β be two filling curves. For pseudo-Anosov f that satisfies the condition in Theorem 3.1, let r, s be a total amount of powers of Dehn twist in a, b , respectively. Let $f' = T_a^r T_b^s$. Then*

$$l_C(f) \geq l_C(f').$$

Proof. Let $d(\alpha, \beta) = l \geq 3$ and let $f = T_a^{a_1} T_b^{b_1} \cdots T_a^{a_n} T_b^{b_n}$. Then $l_C(f) = 2n(l - 2) \geq l_C(f') = 2(l - 2)$ by Theorem 3.1. □

Theorem 4.2. *Let $A = \{a_1, \dots, a_m\}, B = \{b_1, \dots, b_n\}$ be filling multicurves with $\text{dist}(A, B) = l \geq 4$. For pseudo-Anosov f that satisfies the condition in Theorem 3.4, let r_i, s_j be a total amount of powers of Dehn twist in a_i, b_j , respectively. Let $f' = T_{a_1}^{r_1} \cdots T_{a_m}^{r_m} T_{b_1}^{s_1} \cdots T_{b_n}^{s_n}$. Then*

$$l_C(f) \geq l_C(f').$$

Proof. Let $\text{dist}(A, B) = l$, and assume $f = f_1 \cdots f_{2k}$, where f_{2i-1}, f_{2i} are products of Dehn twists by curves in A, B , respectively. We can always put f in such form by conjugating, and conjugates have same asymptotic translation length. If $k = 1$, f will be conjugation of f' itself. For $k \geq 2$, by Theorem 3.4, $l_C(f') \leq 2l$ and $2kl - 4k \leq l_C(f)$. Thus $l_C(f) \geq l_C(f')$. □

4.2. Ratio optimizer

Let $l_{\mathcal{T}}(f)$ be an asymptotic translation length of f on Teichmüller space. Gadre-Hironaka-Kent-Leininger proved that minimal value of ratio $\tau(f) = \frac{l_{\mathcal{T}}(f)}{l_{\mathcal{C}}(f)}$ is asymptotic to $\log(|\chi(S)|)$ [6]. Aougab-Taylor used Mangahas’ Lemma 2.5 to find a large class of pseudo-Anosov maps that optimize this ratio [2]. Here we expand the class of ratio optimizers using another result from Aougab-Taylor.

Theorem 4.3 ([1]). *Let $\omega(g, p) = 3g + p - 4$. Then for any g, p with $\omega(g, p) > 0$, there exists an infinite geodesic ray $\gamma = v_0, v_1, v_2, \dots$ such that*

$$i(v_i, v_j) = O(\omega^{|j-i|-2}).$$

Lemma 4.4 ([2]). *Let $a = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ and let $M = a^{a_1} b^{b_1} \dots a^{a_n} b^{b_n}$, where $a_i, b_j \in \{1, -1\}$ and $t > 1$. Then $\text{trace}(M) \leq (2t)^{2n}$.*

Theorem 4.5. *Let $\gamma = v_0, v_1, v_2, \dots$ be a geodesic ray in Theorem 4.3, and let $T_i = T_{v_i}$ as Dehn twist around v_i . Then for any i, j such that $|i - j| \geq 3$, let $a = T_i^{2M+1}$, $b = T_j^{2M+1}$ and $f = a^{a_1} b^{b_1} \dots a^{a_n} b^{b_n}$, where $a_r, b_s \in \{1, -1\}$. Then we have the following:*

$$\tau(f) = \frac{l_{\mathcal{T}}(f)}{l_{\mathcal{C}}(f)} = O(\log(\omega)).$$

Proof. Since our f is Thurston’s construction made using two filling single curves, we can see that the stretch factor of f coincides with the largest eigenvalue of matrix form by Theorem 2.3. The matrix N is the 1×1 matrix $(i(v_i, v_j))$ here, so $NN^T = (i(v_i, v_j))^2$. The largest eigenvalue is $\mu = i(v_i, v_j)^2$, so $T_i \mapsto \begin{pmatrix} 1 & i(v_i, v_j) \\ 0 & 1 \end{pmatrix}$ and $T_j \mapsto \begin{pmatrix} 1 & 0 \\ i(v_i, v_j) & 1 \end{pmatrix}$. Then

$$a = (T_i)^{2M+1} \mapsto \begin{pmatrix} 1 & i(v_i, v_j) \\ 0 & 1 \end{pmatrix}^{2M+1} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

$$b = (T_j)^{2M+1} \mapsto \begin{pmatrix} 1 & 0 \\ i(v_i, v_j) & 1 \end{pmatrix}^{2M+1} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix},$$

where $t = i(v_i, v_j) \cdot (2M + 1)$. a, b are mapped to matrices with determinant 1, thus the matrix form for f also has determinant 1. Two eigenvalues are the multiplicative inverse of each other, and the trace is a sum of two. Thus largest eigenvalue is smaller than the trace, and by Lemma 4.4, smaller than $(2t)^{2n}$. $l_{\mathcal{T}}(f)$ is a logarithm of the stretch factor, thus $l_{\mathcal{T}}(f) < 2n \log(2t)$. We know that $l_{\mathcal{C}}(f) = 2n(|j - i| - 2)$ by Theorem 3.1 where we exactly calculated the translation length. Combining these with Theorem 4.3 gives us the desired result. □

4.3. Right-angled Artin group

Koberda proved that for a finite set of curves C , there exists K such that for $n > K$, $\{T_\gamma^n\}_{\gamma \in C}$ generates right-angled Artin groups [7]. Koberda suggested a problem, how to decide the value of K .

It is proven that the value of K depends on the choice of C . In the general case of curve sets, answers are given in [12] and [11], using terms of intersection number of pair of curves. We can answer Koberda's question using our theorems on curves with certain relations on the curve graph. Here we give answers in terms of the distance of two curves on subsurface projection on other curves, which is less than or equal to the intersection number of two curves.

Theorem 4.6. *Consider curves $\alpha_1, \alpha_2, \dots, \alpha_t$ which satisfy $d_C(\alpha_i, \alpha_j) \geq 3$ for any distinct i, j . Let $n > 2M + 2 + \max d_{\alpha_k}(\alpha_i, \alpha_j)$. Then the set $\{T_{\alpha_1}^n, \dots, T_{\alpha_t}^n\}$ generates the free group of rank t .*

Proof. Consider element in group generated from $\{T_{\alpha_1}^n, \dots, T_{\alpha_t}^n\}$. If it contains Dehn twists of more than two curves, consider its cyclically reduced form. Our n is chosen to satisfy the condition in our Theorem 3.2. A cyclically reduced element is a pseudo-Anosov map, thus original element is also a pseudo-Anosov map. This means any word written by more than one generator can not be trivial, so there can't exist any relation in the group. \square

Using the same logic, in the case of multicurves, we can find the constant needed to guarantee that any word that contains Dehn twists from at least two multicurves becomes a pseudo-Anosov map. Thus that element can not be trivial, so there is no relation in the group containing more than one multicurve. Each set of Dehn twists on multicurve generates a free Abelian group, thus we get a right-angled Artin group which is a free product of free Abelian groups.

Theorem 4.7. *Consider filling multicurves A, B with $\text{dist}(A, B) \geq 3$. For $n > 2M + 3$, $\{T_\gamma^n \mid \gamma \in A \cup B\}$ generates $\mathbb{Z}^{|A|} * \mathbb{Z}^{|B|}$.*

Theorem 4.8. *Let multicurves $C_1 \cdots C_n$ satisfy $\text{dist}(C_i, C_j) \geq 3$. For $n > 2M + 3 + \max\{d_\gamma(C_i, C_j) \mid \gamma \in \bigcup C_k\}$, $\{T_\gamma^n \mid \gamma \in \bigcup C_k\}$ generates $\mathbb{Z}^{|C_1|} * \cdots * \mathbb{Z}^{|C_n|}$.*

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