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THE UNIMODALITY OF THE r_3-CRANK OF 3-REGULAR OVERPARTITIONS

Robert XiaoJian Hao and Erin YiYing Shen

Abstract. An l-regular overpartition of n is an overpartition of n with no parts divisible by l. Recently, the authors introduced a partition statistic called r_l-crank of l-regular overpartitions. Let $M_{r_l}(m,n)$ denote the number of l-regular overpartitions of n with r_l-crank m. In this paper, we investigate the monotonicity property and the unimodality of $M_{r_3}(m,n)$. We prove that $M_{r_3}(m,n) \geq M_{r_3}(m,n-1)$ for any integers m and $n \geq 6$ and the sequence $\{M_{r_3}(m,n)\}_{|m|\leq n}$ is unimodal for all $n \geq 14$.

1. Introduction

A partition λ of a positive integer n is a weakly-decreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$ such that $|\lambda| = \sum_{i=1}^{l} \lambda_i = n$. Let $p(n)$ denote the number of partitions of n. The partition statistic crank introduced by Andrews and Garvan [2] can be used to provide combinatorial interpretations for Ramanujan's famous congruences [7] as given by

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Recall that the crank [2] of λ is defined as

$$\operatorname{crank}(\lambda) = \begin{cases} \lambda_1, & \text{if } n_1(\lambda) = 0, \\ \mu(\lambda) - n_1(\lambda), & \text{if } n_1(\lambda) > 0, \end{cases}$$

where $n_1(\lambda)$ is the number of ones in λ and $\mu(\lambda)$ is the number of parts larger than $n_1(\lambda)$. Let $M(m,n)$ denote the number of partitions of n with crank m. Andrews and Garvan [2] gave the following generating function of $M(m,n)$

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} M(m,n) z^m q^n = \frac{(q;q)_\infty}{(zq;q)_\infty (z^{-1}q;q)_\infty}. \tag{1.1}$$

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Here and throughout the rest of this paper, we adopt the common q-series notation

$$(a;q)_\infty = \prod_{n=1}^{\infty}(1-aq^{n-1}),$$

and

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}.$$

Recently, Ji and Zang [5] discovered the following monotonicity property and unimodality of $M(m,n)$.

Theorem 1.1 ([5, Theorem 1.6]). *For $n \geq 14$ and $0 \leq m \leq n-2$,*

$$M(m,n) \geq M(m,n-1).$$

Theorem 1.2 ([5, Theorem 1.7]). *For $n \geq 44$ and $1 \leq m \leq n-1$,*

$$M(m-1,n) \geq M(m,n).$$

Recall that an overpartition [3] is a partition in which the first occurrence of each number may be overlined. For instance, $(9,\overline{6},6,1,1,1)$ is an overpartition of 24. In 2003, Lovejoy [6] considered a special kind of overpartitions which is enumerated by $\overline{A}_l(n)$ with the restriction that no parts of the overpartition can be divisible by l. Later, the second author [8] called the overpartitions counted by $\overline{A}_l(n)$ as l-regular overpartitions and gave the generating function of $\overline{A}_l(n)$ as given by

$$\sum_{n=0}^{\infty}\overline{A}_l(n)q^n = \frac{(-q;q)_\infty(q^l;q^l)_\infty}{(q;q)_\infty(-q^l;q^l)_\infty} = \frac{f_2 f_l^2}{f_1^2 f_{2l}}, \tag{1.2}$$

where f_k is defined by

$$f_k = (q^k;q^k)_\infty$$

with any positive integer k. Andrews [1] introduced (k,i)-singular overpartitions and proved that they are counted by the partition function $\overline{C}_{k,i}(n)$ which denotes the number of k-regular overpartitions of n and only parts $\equiv \pm i \pmod{k}$ may be overlined. Andrews established the generating function of $\overline{C}_{k,i}(n)$ as

$$\sum_{n=0}^{\infty}\overline{C}_{k,i}(n)q^n = \frac{(q^k;q^k)_\infty(-q^i;q^k)_\infty(-q^{k-i};q^k)_\infty}{(q;q)_\infty},\ k\geq 3,\ 1\leq i\leq \lfloor \frac{k}{2}\rfloor, \tag{1.3}$$

and showed that

$$\overline{C}_{3,1}(9n+3) \equiv \overline{C}_{3,1}(9n+6) \equiv 0 \pmod{3}. \tag{1.4}$$

By (1.2) and (1.3), we have $\overline{A}_3(n) = \overline{C}_{3,1}(n)$. In light of the fact, the authors [4] introduced the r_l-crank of l-regular overpartitions based on the following theorem.

Theorem 1.3 ([4, Theorem 2.1]). *For integers $k_1 \geq -1$, $k_2 \geq 1$ and $l \geq 3$, there is a bijection Δ between the set of l-regular overpartitions of n and the set of vector partitions (α, β, γ) with $|\alpha| + |\beta| + |\gamma|$ equal to n. Here α is an ordinary partition, β is a partition like $(k_1 l + 1, \ldots, 2l + 1, l + 1, 1)$ or $(k_2 l - 1, \ldots, 3l - 1, 2l - 1, l - 1)$ and γ is a distinct partition with all parts $\not\equiv 0, \pm 1 \pmod{l}$.*

The authors gave the definition of the r_l-crank of an l-regular overpartition under the bijection Δ.

Definition 1.4 ([4, Definition 2.2]). Let λ be an l-regular overpartition of n with $l \geq 3$ and let $\Delta(\lambda) = (\alpha, \beta, \gamma)$. The r_l-crank of λ, denoted $c_{r_l}(\lambda)$, is defined by

$$c_{r_l}(\lambda) = crank(\alpha),$$

where $crank(\alpha)$ is the crank of partition α.

In [4], the authors gave combinatorial interpretations for some congruences of $\overline{A}_l(n)$ including (1.4). Let $M_{r_l}(m, n)$ denote the number of l-regular overpartitions of n with r_l-crank m. We deduced the generating functions for the r_l-crank as given by

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} M_{r_l}(m, n) z^m q^n = \frac{(q;q)_\infty (-q;q)_\infty (q^l;q^l)_\infty}{(zq;q)_\infty (z^{-1}q;q)_\infty (-q^l;q^l)_\infty}. \tag{1.5}$$

In this paper, we investigate the distribution of the r_3-cranks of 3-regular overpartitions. We study the monotonicity property and the unimodality of $M_{r_3}(m, n)$. The main results of this paper are presented in the following theorems.

Theorem 1.5. *For any integers m and $n \geq 6$, we have*

$$M_{r_3}(m, n) \geq M_{r_3}(m, n-1).$$

Figure 1 exhibits the sequence $\{M_{r_3}(0, n)\}$ with $0 \leq n \leq 16$.

Theorem 1.6. *The sequence $\{M_{r_3}(m, n)\}_{|m| \leq n}$ is unimodal for all $n \geq 14$.*

Here we present the sequence $\{M_{r_3}(m, 14)\}_{|m| \leq 14}$ in Figure 2.

The rest of this paper is organized as follows. In Section 2, we provide some results that will be used in our proofs. In Section 3, we give a proof of Theorem 1.5 which is concerned with the monotonicity property of $M_{r_3}(m, n)$. We prove the unimodality of $M_{r_3}(m, n)$ presented in Theorem 1.6 in Section 4.

2. Preliminaries

In this section, we present some results that will be employed in our proofs.

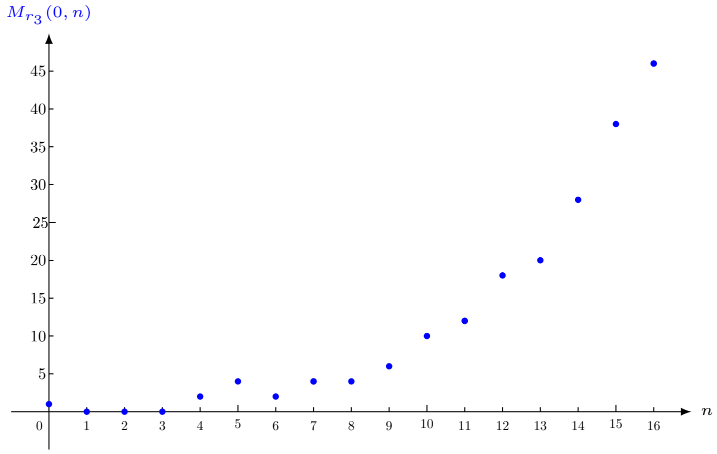

FIGURE 1. The sequence $\{M_{r_3}(0,n)\}_{n\le 16}$.

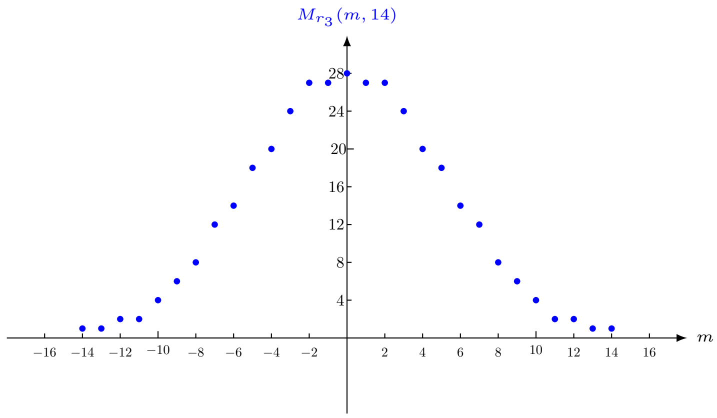

FIGURE 2. The sequence $\{M_{r_3}(m,14)\}_{|m|\le 14}$.

Theorem 2.1. *The coefficient of q^n in*

$$\frac{1-q}{(q^2;q)_2}\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}} \tag{2.1}$$

is nonnegative for $n \ge 0$.

Proof. It is obvious that

$$\frac{1-q}{(q^2;q)_2} = \frac{1-q}{(1-q^2)(1-q^3)} = \frac{1}{(1+q)(1-q^3)} = \frac{1-q+q^2}{(1+q^3)(1-q^3)} = \frac{1-q+q^2}{1-q^6}.$$

Let

$$\sum_{n=0}^{\infty} f(n)q^n = \frac{1}{1-q^6} \sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}.$$

Combining (2.1), we see that Theorem 2.1 is equivalent to

$$f(n) - f(n-1) + f(n-2) \geq 0$$

for all $n \geq 0$.

For any nonnegative integer n, we have $f(n) = |S_n|$, where

$$S_n = \Big\{ k \mid \frac{(3k-1)k}{2} \equiv n \pmod 6, \ \frac{(3k-1)k}{2} \leq n \Big\}.$$

For example, let $n = 75$, we have $S_{75} = \{6, -3, -6\}$, thus $f(75) = |S_{75}| = 3$. Denote $\{a_t\} = \{0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, 57, 70, 77, \ldots\}$ the sequence of pentagonal numbers. It is worth noticing that

$$\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}} = \sum_{k=1}^{\infty} q^{\frac{(3k-1)k}{2}} + \sum_{k=0}^{\infty} q^{\frac{(3k+1)k}{2}},$$

and

$$\frac{(3k-1)k}{2} \equiv \frac{\big(3(k+12j)-1\big)(k+12j)}{2} \pmod 6,$$
$$\frac{(3k+1)k}{2} \equiv \frac{\big(3(k+12j)+1\big)(k+12j)}{2} \pmod 6$$

for any nonnegative integers k and j. Moreover, we have that

$$\frac{(3k-1)k}{2} \leq \frac{(3k+1)k}{2} < \frac{\big(3(k+1)-1\big)(k+1)}{2} < \frac{\big(3(k+1)+1\big)(k+1)}{2}.$$

Hence we arrive at

$$a_{t+24} \equiv a_t \pmod 6. \tag{2.2}$$

TABLE 1. The first 24 pentagonal numbers and their residues modulo 6.

Pentagonal number	0	1	2	5	7	12	15	22	26	35	40	51
Residue modulo 6	0	1	2	5	1	0	3	4	2	5	4	3
Pentagonal number	57	70	77	92	100	117	126	145	155	176	187	210
Residue modulo 6	3	4	5	2	4	3	0	1	5	2	1	0

Here we list the first 24 pentagonal numbers and their residues modulo 6 in Table 1. It is clear that these 24 residues contain four 0's, 1's, 2's, 3's, 4's and 5's, respectively. Based on this fact, we conclude that

$$f(n) \geq 4 \left\lfloor \frac{t}{24} \right\rfloor,$$

$$f(n-1) \leq 4\left\lfloor \frac{t}{24} \right\rfloor + 4,$$
$$f(n-2) \geq 4\left\lfloor \frac{t}{24} \right\rfloor - 1,$$

where $a_t < n \leq a_{t+1}$. Thus we obtain that

$$\begin{aligned} f(n) - f(n-1) + f(n-2) &\geq 4\left\lfloor \frac{t}{24} \right\rfloor - \left(4\left\lfloor \frac{t}{24} \right\rfloor + 4\right) + 4\left\lfloor \frac{t}{24} \right\rfloor - 1 \\ &= 4\left\lfloor \frac{t}{24} \right\rfloor - 5. \end{aligned}$$

When $t \geq 48$, we have $4\left\lfloor \frac{t}{24} \right\rfloor - 5 \geq 3$. In view of $a_{48} = 852$, we see that

$$f(n) - f(n-1) + f(n-2) \geq 3$$

for all $n \geq 852$. It can be verified that $f(n) - f(n-1) + f(n-2) \geq 0$ for $0 \leq n \leq 851$. This completes the proof. □

More specifically, the following corollary holds.

Corollary 2.2. *The coefficient of q^n in*

$$\frac{1-q}{(q^2;q)_2} \sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}$$

is positive when $n \neq 1, 3, 6, 8, 16$.

Theorem 2.3. *The absolute value of the coefficient of q^n in*

$$\frac{1-q}{1-q^2} \sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}$$

is no more than 1 for $n \geq 0$.

Proof. Let

$$\sum_{n=0}^{\infty} g(n) q^n = \frac{1}{1-q^2} \sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}.$$

Thus we aim to prove that

$$|g(n) - g(n-1)| \leq 1$$

for all $n \geq 0$.

For any nonnegative integer n, we have $g(n) = |H_n|$, where

$$H_n = \left\{ k \mid \frac{(3k-1)k}{2} \equiv n \pmod 2, \ \frac{(3k-1)k}{2} \leq n \right\}. \tag{2.3}$$

Similar to (2.2), the congruence

$$a_{t+8} \equiv a_t \pmod 2$$

is true. Here we list the first 8 pentagonal numbers and their residues modulo 2 in Table 2.

TABLE 2. The first 8 pentagonal numbers and their residues modulo 2.

Pentagonal number	0	1	2	5	7	12	15	22
Residue modulo 2	0	1	0	1	1	0	1	0

Let $\omega = \omega_1\omega_2\cdots\omega_7 = 0101101$ and $|\omega_1\omega_2\cdots\omega_i|_j$ be the number of j in the first i elements of ω with $0 \le i \le 7$ and $j = 0, 1$. Here we set $|\omega_1\omega_2\cdots\omega_i|_j = 0$ if $i = 0$.

Suppose that $a_t < n \le a_{t+1}$, $t \equiv i \pmod 8$ and $n \equiv j \pmod 2$, we have

$$g(n) = 4\left\lfloor \frac{t}{8} \right\rfloor + |\omega_1\omega_2\cdots\omega_i|_j,$$

$$g(n-1) = 4\left\lfloor \frac{t}{8} \right\rfloor + |\omega_1\omega_2\cdots\omega_i|_{|j-1|}.$$

Since

$$\big||\omega_1\omega_2\cdots\omega_i|_j - |\omega_1\omega_2\cdots\omega_i|_{|j-1|}\big| \le 1$$

for each $0 \le i \le 7$ and $j = 0, 1$, we arrive at

$$|g(n) - g(n-1)| \le 1$$

with $n \ge 0$. This completes the proof. □

The following theorem is proved by Ji and Zang in [5].

Theorem 2.4 ([5, Theorem 6.5]). *For $m \ge 3$,*

$$\begin{aligned}
&\sum_{n=0}^{\infty}\big(M(m-1,n) - M(m,n)\big)q^n \\
&= -q^{2m} + q^{2m+1} + q^{3m+1} + \frac{q^{m-1}}{(q^2;q)_{m-2}} - \frac{q^m}{(q^2;q)_{m-2}} \\
&\quad + \frac{q^{2m+5}}{(q^2;q)_{m-3}(1-q^m)} + \sum_{k=3}^{m}\frac{q^{2k+2m+1}}{(q^k;q)_{m-k+1}} + \sum_{k=2}^{\infty}\frac{q^{k(k+m)+3k+2m-2}}{(q^3;q)_{k-2}(q^2;q)_{k+m-2}} \\
&\quad + \sum_{k=1}^{\infty}\frac{q^{k(k+m)+4k+2m+2}(1-q^{m-2})}{(q^2;q)_k(q^2;q)_{k+m-2}} + \sum_{k=1}^{\infty}\frac{q^{k(k+m)+5k+3m+1}}{(q^2;q)_k(q^2;q)_{m-3}(q^m;q)_{k+1}}.
\end{aligned} \tag{2.4}$$

3. A proof of Theorem 1.5

In this section, we give a proof of Theorem 1.5.

Proof. Setting $l = 3$ in (1.5), we have

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M_{r_3}(m,n)z^m q^n = \frac{(q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}\frac{(-q;q)_\infty(q^3;q^3)_\infty}{(-q^3;q^3)_\infty}$$

$$= \frac{(q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} M(m,n)z^m q^n \sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}. \tag{3.1}$$

The last equality follows by (1.1) and the Jacobi triple product identity

$$\sum_{n=-\infty}^{\infty} z^n q^{\binom{n}{2}} = (-z;q)_\infty(-q/z;q)_\infty(q;q)_\infty$$

with q replaced by q^3 and z replaced by q.

Using the equation proved by Ji and Zang [5, Eq. (2.2)] as given by

$$\sum_{n=0}^{\infty}\big(M(m,n)-M(m,n-1)\big)q^n$$
$$= \frac{(1-q)^2q^m}{(q;q)_m} + \frac{q^{2m+3}}{(q^2;q)_m} + \sum_{k=2}^{\infty}\frac{q^{k(k+m)+2k+m}}{(q^2;q)_{k-1}(q^2;q)_{k+m-1}},\ m\ge 0,$$

and (3.1), we obtain the generating function of $M_{r_3}(m,n)-M_{r_3}(m,n-1)$ as

$$\sum_{n=0}^{\infty}\big(M_{r_3}(m,n)-M_{r_3}(m,n-1)\big)q^n$$
$$= \left(\frac{(1-q)^2q^m}{(q;q)_m} + \frac{q^{2m+3}}{(q^2;q)_m} + \sum_{k=2}^{\infty}\frac{q^{k(k+m)+2k+m}}{(q^2;q)_{k-1}(q^2;q)_{k+m-1}}\right)\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}. \tag{3.2}$$

It is clear that

$$\left(\frac{q^{2m+3}}{(q^2;q)_m} + \sum_{k=2}^{\infty}\frac{q^{k(k+m)+2k+m}}{(q^2;q)_{k-1}(q^2;q)_{k+m-1}}\right)\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}} \tag{3.3}$$

has only nonnegative coefficients when $m\ge 0$. For $m\ge 3$, we have that

$$\frac{(1-q)^2q^m}{(q;q)_m}\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}} = \frac{q^m}{(q^4;q)_{m-3}}\frac{1-q}{(q^2;q)_2}\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}. \tag{3.4}$$

Applying Theorem 2.1, we find that the coefficient of q^n in (3.4) is nonnegative when $m\ge 3$. By (3.2)–(3.4), we conclude that the coefficient of q^n in (3.2) is nonnegative when $m\ge 3$. Hence Theorem 1.5 is verified for all $m\ge 3$.

Substituting $m=2$ into (3.2), we have

$$\sum_{n=0}^{\infty}\big(M_{r_3}(2,n)-M_{r_3}(2,n-1)\big)q^n$$
$$= \left(\frac{(1-q)q^2}{1-q^2} + \frac{q^7}{(1-q^2)(1-q^3)} + \sum_{k=2}^{\infty}\frac{q^{k^2+4k+2}}{(q^2;q)_{k-1}(q^2;q)_{k+1}}\right)\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}.$$

In view of Theorem 2.3, the coefficient of q^n in

$$\frac{(1-q)q^2}{1-q^2}\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}$$

is no less than -1. Since

$$\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}} = 1+q+\sum_{k\neq 0,1} q^{\frac{(3k-1)k}{2}},$$

it is clear that the coefficient of q^n in

$$\frac{q^7}{(1-q^2)(1-q^3)}\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}$$

is no less than 1 with $n \geq 7$. Noticing that

$$\sum_{k=2}^{\infty} \frac{q^{k^2+4k+2}}{(q^2;q)_{k-1}(q^2;q)_{k+1}} \sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}$$

has only nonnegative coefficients, and the coefficient of q^6 in

$$\frac{(1-q)q^2}{1-q^2}\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}$$

is 1, we conclude that

$$M_{r_3}(2,n) \geq M_{r_3}(2,n-1)$$

when $n \geq 6$.

Substituting $m=1$ into (3.2), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\big(M(1,n)-M(1,n-1)\big)q^n\\ &= \left(q-q^2+\frac{q^5}{1-q^2}+\sum_{k=2}^{\infty}\frac{q^{k^2+3k+1}}{(q^2;q)_{k-1}(q^2;q)_k}\right)\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}.\end{aligned}$$

It is easy to see that the coefficient of q^n in

$$\frac{q^5}{1-q^2}\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}$$

is no less than 1 when $n \geq 5$. Combining the fact that the coefficient of q^n in

$$\begin{aligned}&\left(q-q^2+\sum_{k=2}^{\infty}\frac{q^{k^2+3k+1}}{(q^2;q)_{k-1}(q^2;q)_k}\right)\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}\\ &= -q^2\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}} + \left(q+\sum_{k=2}^{\infty}\frac{q^{k^2+3k+1}}{(q^2;q)_{k-1}(q^2;q)_k}\right)\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}\end{aligned}$$

$$= -\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k+4}{2}} + \left(q + \sum_{k=2}^{\infty} \frac{q^{k^2+3k+1}}{(q^2;q)_{k-1}(q^2;q)_k}\right) \sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}$$

is no less than -1, we arrive at

$$M_{r_3}(1,n) \geq M_{r_3}(1,n-1)$$

for all $n \geq 5$.

The proof of $m = 0$ is similar to that of $m = 2$, hence the details are omitted. Ultimately, by the fact $M_{r_3}(m,n) = M_{r_3}(-m,n)$, we complete the proof of Theorem 1.5. □

4. A proof of Theorem 1.6

We are now in a position to prove Theorem 1.6.

Proof. In view of (3.1), for any fixed integer m, we have

$$\begin{aligned}&\sum_{n=0}^{\infty} \big(M_{r_3}(m-1,n) - M_{r_3}(m,n)\big)q^n \\ &= \sum_{n=0}^{\infty} \big(M(m-1,n) - M(m,n)\big)q^n \sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}.\end{aligned} \tag{4.1}$$

Applying (2.4) into (4.1), for $m \geq 3$, we obtain that

$$\begin{aligned}&\sum_{n=0}^{\infty} \big(M_{r_3}(m-1,n) - M_{r_3}(m,n)\big)q^n \\ &= \Bigg(- q^{2m} + q^{2m+1} + q^{3m+1} + \frac{q^{m-1}}{(q^2;q)_{m-2}} - \frac{q^m}{(q^2;q)_{m-2}} + \frac{q^{2m+5}}{(q^2;q)_{m-3}(1-q^m)} \\ &\quad + \sum_{k=3}^{m} \frac{q^{2k+2m+1}}{(q^k;q)_{m-k+1}} + \sum_{k=2}^{\infty} \frac{q^{k(k+m)+3k+2m-2}}{(q^3;q)_{k-2}(q^2;q)_{k+m-2}} + \sum_{k=1}^{\infty} \frac{q^{k(k+m)+4k+2m+2}(1-q^{m-2})}{(q^2;q)_k(q^2;q)_{k+m-2}} \\ &\quad + \sum_{k=1}^{\infty} \frac{q^{k(k+m)+5k+3m+1}}{(q^2;q)_k(q^2;q)_{m-3}(q^m;q)_{k+1}}\Bigg) \sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}.\end{aligned} \tag{4.2}$$

Since $m-2 < k+m-1$ when $k \geq 1$, we find that

$$\frac{1-q^{m-2}}{(q^2;q)_{k+m-2}}$$

has only nonnegative coefficients with $m-2 \geq 2$.

Hence we get that

$$\begin{aligned}&\Bigg(q^{3m+1} + \frac{q^{2m+5}}{(q^2;q)_{m-3}(1-q^m)} + \sum_{k=3}^{m} \frac{q^{2k+2m+1}}{(q^k;q)_{m-k+1}} + \sum_{k=2}^{\infty} \frac{q^{k(k+m)+3k+2m-2}}{(q^3;q)_{k-2}(q^2;q)_{k+m-2}} \\ &+ \sum_{k=1}^{\infty} \frac{q^{k(k+m)+4k+2m+2}(1-q^{m-2})}{(q^2;q)_k(q^2;q)_{k+m-2}} + \sum_{k=1}^{\infty} \frac{q^{k(k+m)+5k+3m+1}}{(q^2;q)_k(q^2;q)_{m-3}(q^m;q)_{k+1}}\Bigg) \sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}\end{aligned}$$

has only nonnegative coefficients with $m \geq 4$.

Next, we aim to show that the coefficient of q^n in

$$\left(-q^{2m}+q^{2m+1}+\frac{q^{m-1}}{(q^2;q)_{m-2}}-\frac{q^m}{(q^2;q)_{m-2}}\right)\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}} \tag{4.3}$$

is nonnegative for all $m \geq 4$.

When $m=4$, (4.3) becomes

$$\left(-q^8+q^9+\frac{q^3}{(q^2;q)_2}-\frac{q^4}{(q^2;q)_2}\right)\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}. \tag{4.4}$$

Since

$$\left(\frac{q^3}{(q^2;q)_2}-\frac{q^4}{(q^2;q)_2}\right)\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}} = q^3\frac{1-q}{(q^2;q)_2}\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}, \tag{4.5}$$

by Theorem 2.1 and Corollary 2.2, we obtain that the coefficient of q^n in (4.5) is positive except for $n=4,6,9,11,19$. Noticing that the coefficient of q^n in

$$-q^8\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}} \tag{4.6}$$

is no less than -1, we find that the coefficient of q^n in (4.4) is nonnegative when $n \neq 4,6,9,11,19$. After a simple calculation, we get that (4.4) has only nonnegative coefficients. For $m \geq 5$, the proof is similar to that of $m=4$ and we omit it. Therefore, we conclude that

$$M_{r_3}(m-1,n)-M_{r_3}(m,n) \geq 0$$

for all $m \geq 4$.

Setting $m=3$ in (2.4) and applying it to (4.1), we obtain that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\big(M_{r_3}(2,n)-M_{r_3}(3,n)\big)q^n\\
&=\Bigg(-q^6+q^7+q^{10}+\frac{q^2}{1-q^2}-\frac{q^3}{1-q^2}+\frac{q^{11}}{1-q^3}+\frac{q^{13}}{1-q^3}\\
&\quad+\sum_{k=2}^{\infty}\frac{q^{k^2+6k+4}}{(q^3;q)_{k-2}(q^2;q)_{k+1}}+\sum_{k=1}^{\infty}\frac{q^{k^2+7k+8}(1-q)}{(q^2;q)_k(q^2;q)_{k+1}}\\
&\quad+\sum_{k=1}^{\infty}\frac{q^{k^2+8k+10}}{(q^2;q)_k(q^3;q)_{k+1}}\Bigg)\sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}\\
&=\Bigg(-q^6+q^7+q^{10}+\frac{q^2}{1-q^2}-\frac{q^3}{1-q^2}+\frac{q^{11}}{1-q^3}+\frac{q^{13}}{1-q^3}+\frac{q^{20}}{(q^2;q)_3}\\
&\quad+\sum_{k=3}^{\infty}\frac{q^{k^2+6k+4}}{(q^3;q)_{k-2}(q^2;q)_{k+1}}+\sum_{k=1}^{\infty}\frac{q^{k^2+7k+8}(1-q)}{(q^2;q)_k(q^2;q)_{k+1}}
\end{aligned}$$

$$+\sum_{k=1}^{\infty}\frac{q^{k^2+8k+10}}{(q^2;q)_k(q^3;q)_{k+1}}\Bigg)\sum_{k=-\infty}^{\infty}q^{\frac{(3k-1)k}{2}}. \tag{4.7}$$

Since

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{q^{k^2+7k+8}(1-q)}{(q^2;q)_k(q^2;q)_{k+1}}\sum_{k=-\infty}^{\infty}q^{\frac{(3k-1)k}{2}}\\&=\sum_{k=1}^{\infty}\frac{q^{k^2+7k+8}}{(q^2;q)_k(q^4;q)_{k-1}}\frac{1-q}{(q^2;q)_2}\sum_{k=-\infty}^{\infty}q^{\frac{(3k-1)k}{2}},\end{aligned}$$

by Theorem 2.1, we arrive at the conclusion that the coefficient of q^n in

$$\begin{aligned}&\Bigg(q^7+q^{10}+\frac{q^{11}}{1-q^3}+\frac{q^{13}}{1-q^3}+\sum_{k=3}^{\infty}\frac{q^{k^2+6k+4}}{(q^3;q)_{k-2}(q^2;q)_{k+1}}\\&+\sum_{k=1}^{\infty}\frac{q^{k^2+7k+8}(1-q)}{(q^2;q)_k(q^2;q)_{k+1}}+\sum_{k=1}^{\infty}\frac{q^{k^2+8k+10}}{(q^2;q)_k(q^3;q)_{k+1}}\Bigg)\sum_{k=-\infty}^{\infty}q^{\frac{(3k-1)k}{2}}\end{aligned}$$

is nonnegative.

Next, we consider the coefficients in

$$\left(-q^6+\frac{q^2}{1-q^2}-\frac{q^3}{1-q^2}+\frac{q^{20}}{(q^2;q)_3}\right)\sum_{k=-\infty}^{\infty}q^{\frac{(3k-1)k}{2}}.$$

Since

$$\begin{aligned}&\left(-q^6+\frac{q^2}{1-q^2}-\frac{q^3}{1-q^2}\right)\sum_{k=-\infty}^{\infty}q^{\frac{(3k-1)k}{2}}\\&=-q^6\sum_{k=-\infty}^{\infty}q^{\frac{(3k-1)k}{2}}+q^2\frac{1-q}{1-q^2}\sum_{k=-\infty}^{\infty}q^{\frac{(3k-1)k}{2}},\end{aligned} \tag{4.8}$$

by Theorem 2.3, we can conclude that the coefficient of q^n in (4.8) is no less than -2. It is clear that the coefficient of q^n in

$$\frac{q^{20}}{(q^2;q)_3}\sum_{k=-\infty}^{\infty}q^{\frac{(3k-1)k}{2}}$$

is no less than 2 for all $n\geq 22$. Hence the coefficient of q^n in (4.7) is nonnegative when $n\geq 22$. It can be checked that $M_{r_3}(2,n)\geq M_{r_3}(3,n)$ for $14\leq n\leq 21$.

For $m=2$, combining (4.1) and [5, Eq. (7.1), (7.2)], we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\big(M_{r_3}(1,n)-M_{r_3}(2,n)\big)q^n\\&=\Bigg(q-q^2-q^4-q^{10}-q^{12}-q^{14}+\frac{q^5}{1-q^2}+\frac{q^{19}}{(1-q^2)(1-q^3)}\end{aligned}$$

$$+\sum_{k=3}^{\infty}\frac{q^{k^2+5k+2}}{(q^3;q)_{k-2}(q^2;q)_k}+\sum_{k=1}^{\infty}\frac{q^{k^2+7k+7}(1-q)}{(q^2;q)_k(q^2;q)_{k+1}}\Bigg)\sum_{k=-\infty}^{\infty}q^{\frac{(3k-1)k}{2}}. \tag{4.9}$$

Using Theorem 2.1, we find that

$$\left(q+\frac{q^5}{1-q^2}+\sum_{k=3}^{\infty}\frac{q^{k^2+5k+2}}{(q^3;q)_{k-2}(q^2;q)_k}+\sum_{k=1}^{\infty}\frac{q^{k^2+7k+7}(1-q)}{(q^2;q)_k(q^2;q)_{k+1}}\right)\sum_{k=-\infty}^{\infty}q^{\frac{(3k-1)k}{2}}$$
$$=\left(q+\frac{q^5}{1-q^2}+\sum_{k=3}^{\infty}\frac{q^{k^2+5k+2}}{(q^3;q)_{k-2}(q^2;q)_k}\right)\sum_{k=-\infty}^{\infty}q^{\frac{(3k-1)k}{2}}$$
$$+\sum_{k=1}^{\infty}\frac{q^{k^2+7k+7}}{(q^2;q)_k(q^4;q)_{k-1}}\frac{1-q}{(q^2;q)_2}\sum_{k=-\infty}^{\infty}q^{\frac{(3k-1)k}{2}}$$

has only nonnegative coefficients. Clearly, the coefficient of q^n in

$$\left(-q^2-q^4-q^{10}-q^{12}-q^{14}\right)\sum_{k=-\infty}^{\infty}q^{\frac{(3k-1)k}{2}}$$

is no less than -5.

Notice that the coefficient of q^n in

$$\frac{1}{(1-q^2)(1-q^3)}(1+q+q^2)=\frac{1}{(1-q)(1-q^2)} \tag{4.10}$$

can be interpreted as the number of partitions of n formed by 1 and 2. We obtain that the coefficient of q^n in (4.10) is no less than 5 for $n\geq 2\times 4$. Hence the coefficient of q^n in

$$\frac{q^{19}}{(1-q^2)(1-q^3)}(1+q+q^2) \tag{4.11}$$

is no less than 5 for $n\geq 27$. It is clear that the coefficient of q^n in

$$\frac{q^{19}}{(1-q^2)(1-q^3)}\sum_{k=-\infty}^{\infty}q^{\frac{(3k-1)k}{2}} \tag{4.12}$$

is no less than the coefficient of q^n in (4.11). Therefore the coefficient of q^n in (4.12) is no less than 5 for $n\geq 27$.

Thus the coefficients of q^n in (4.9) is nonnegative when $n\geq 27$. It can be checked that $M_{r_3}(1,n)\geq M_{r_3}(2,n)$ for $14\leq n\leq 26$.

For $m=1$, according to (4.1) and Theorem 1.2, we obtain that

$$\sum_{n=0}^{\infty}\left(M_{r_3}(0,n)-M_{r_3}(1,n)\right)q^n$$
$$=\Big(1-2q+q^3+q^4-q^7-q^9+q^{10}-q^{11}+2q^{12}-q^{13}+2q^{14}-q^{15}$$
$$+2q^{16}-2q^{17}+3q^{18}-3q^{19}+3q^{20}-2q^{21}+3q^{22}-3q^{23}+6q^{24}$$
$$-4q^{25}+6q^{26}-2q^{27}+7q^{28}-4q^{29}+11q^{30}-5q^{31}+12q^{32}-3q^{33}$$

$$+ 13q^{34} - 4q^{35} + 20q^{36} - 6q^{37} + 22q^{38} - q^{39} + 27q^{40} - 3q^{41}$$

$$+ 37q^{42} - q^{43} + \sum_{n=44}^{\infty} b_n q^n\Big) \sum_{k=-\infty}^{\infty} q^{\frac{(3k-1)k}{2}}, \tag{4.13}$$

where $\{b_n\}_{n=0}^{\infty}$ is a sequence of nonnegative integers. Clearly, the coefficient of q^n in (4.13) is no less than

$$-2-1-1-1-1-1-2-3-2-3-4-2-4-5-3-4-6-1-3-1=-50.$$

Applying the inequalities (9.32) and (9.34) in [5], we have that

$$\begin{aligned} b_n &= M(0,n) - M(1,n) \\ &\geq \frac{n^2}{48} - 2n + 48 - \frac{n-21}{2} - 3 - \frac{n-35}{3} \end{aligned}$$

for $n \geq 106$. When $n \geq 136$, we drive that

$$\begin{aligned} \frac{n^2}{48} - 2n + 48 - \frac{n-21}{2} - 3 - \frac{n-35}{3} &= \frac{n(n-136)}{48} + \frac{403}{6} \\ &> 50. \end{aligned}$$

This yields the positivity of the coefficient of q^n in (4.13) for all $n \geq 136$. For $14 \leq n \leq 135$, it can be checked that $M_{r_3}(0,n) \geq M_{r_3}(1,n)$. This completes the proof. $\square$

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Robert XiaoJian Hao
College of Science and Mathematics
Nanjing Institute of Technology
Nanjing 211167, P. R. China
Email address: haoxj@njit.edu.cn

Erin YiYing Shen
School of Mathematics
Hohai University
Nanjing 210098, P. R. China
Email address: shenyy@hhu.edu.cn