J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2024.31.2.235 ISSN(Online) 2287-6081
Volume 31, Number 2 (May 2024), Pages 235–249

GABOR LIKE STRUCTURED FRAMES IN SEPARABLE HILBERT SPACES

Jineesh Thomas [a,*], N.M.M. Namboothiri [b] and T.C.E. Nambudiri [c]

Abstract. We obtain a structured class of frames in separable Hilbert spaces which are generalizations of Gabor frames in $L^2(\mathbb{R})$ in their construction aspects. For this, the concept of Gabor type unitary systems in [13] is generalized by considering a system of invertible operators in place of unitary systems. Pseudo Gabor like frames and pseudo Gabor frames are introduced and the corresponding frame operators are characterized.

1. Introduction

Frames in separable Hilbert spaces are more flexible tools than orthonormal bases for transforming elements of the space into square summable complex sequences, ensuring the faithful reconstruction. They appear as generalizations of orthonormal bases, but the series expansions using frames are not unique as in the case of an orthonormal basis. The work [10] of Gabor initiated and formulated a fundamental approach to the decomposition of *signals* (elements of $L^2(\mathbb{R})$) in terms of elementary signals. Duffin and Schaeffer absorbed the fundamental notion of Gabor and defined formally the concept of a frame in a Hilbert space [6]. The significant work of Janssen [15] made it an independent topic of mathematical investigation in 1980s. After the innovative work [5] of Daubechies, Grossmann and Meyer in 1986, the theory of frames began to be studied extensively.

Among the different classes of frames, Gabor frames (also called Weyl-Heisenberg frames) in $L^2(\mathbb{R})$ play the key role in the theory. Gabor frames are very special in their nature as they are constructed from a single element of the space using a system of unitary operators on $L^2(\mathbb{R})$. They transform each element f of the space $L^2(\mathbb{R})$

Received by the editors December 12, 2023. Revised March 14, 2024. Accepted April 3, 2024.
2020 *Mathematics Subject Classification.* 42C15, 47B90, 94A12.
Key words and phrases. Gabor like frames, Gabor frame operator, pseudo B-Gabor like frame, pseudo B-Gabor frame, intertwining operator.
*Corresponding author.

into a corresponding square summable complex sequence which is indexed by the *time-frequency* lattice points.

Frame operators are important objects in frame theory [7], [18] since the reconstruction of an element in the Hilbert space using a frame requires the canonical dual frame, which is the image of the frame under the inverse of the frame operator. These operators were completely characterized in the contexts of abstract frames and Gabor frames in [7] and [8] respectively. For a frame in an abstract Hilbert space $\mathcal{H}$, a specific structure similar to that of a Gabor frame in $L^2(\mathbb{R})$ can not be expected in the general setting. Structured frames in separable Hilbert spaces generated by Gabor type unitary systems were discussed in [13] (Chapter 4).

In this article, we attempt for this in a more general way, using invertible operators from $L^2(\mathbb{R})$ into the Hilbert space concerned. Towards this, when Gabor type unitary systems are replaced by a system of invertible operators from $L^2(\mathbb{R})$ into the Hilbert space, we encounter a lot of non trivialities. This is the background of our discussion, but our approach is entirely different from that of [13]. Operators which generate frames with a prescribed frame operator [1] is an interesting theme. Towards this, we provide characterizations for the frame operators of the newly introduced structured frames here.

In Section 2, some basic definitions and results which are inevitable for the present work are provided. Pseudo B-Gabor frames in separable Hilbert spaces, their frame operators and the characterization of these operators are discussed in the next two Sections.

2. Preliminaries

A countable sequence $\{f_k\}_{k=1}^{\infty}$ of elements in a separable Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $\alpha,\ \beta > 0$ such that,

$$\alpha\|f\|^2 \ \leq\ \sum_{k=1}^{\infty} |\ \langle f, f_k\rangle\ |^2 \leq\ \beta\|f\|^2, \ \ \forall\ f \in \mathcal{H}.$$

If $\{f_k\}_{k=1}^{\infty}$ satisfies the upper frame inequality, then it is called a *Bessel sequence.* The numbers α and β are called *lower* and *upper frame bounds* respectively. If a frame has equal frame bounds, then it is called a *tight* frame. In particular, a tight frame with $\alpha = \beta = 1$ is called a *Parseval frame* or *normalized tight frame.*

For a Bessel sequence $\{f_k\}_{k=1}^{\infty} \subset \mathcal{H}$, the operator $T : l^2(\mathbb{N}) \to \mathcal{H}$ defined by $T\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k f_k$ is a bounded linear operator known as the *synthesis operator* or *pre-frame operator*. The adjoint operator $T^* : \mathcal{H} \to l^2(\mathbb{N})$ of T given by $T^* f = \{\langle f, f_k\rangle\}_{k=1}^{\infty}$ is called the *analysis operator*. The operator $S : \mathcal{H} \to \mathcal{H}$ defined by $Sf = TT^* f = \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k$, $f \in \mathcal{H}$ is called the *frame operator* of $\{f_k\}_{k=1}^{\infty}$.

The frame operator of a tight frame is a scalar multiple of the identity operator and that of a normalized tight frame is the identity operator. The frame operator S of a frame $\{f_k\}_{k=1}^{\infty}$ is a bounded, invertible, self-adjoint, positive operator on $\mathcal{H}$. The frame $\{S^{-1} f_k\}_{k=1}^{\infty}$ is called the *canonical dual* frame of $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$. A frame $\{f_k\}_{k=1}^{\infty}$ and its canonical dual frame $\{S^{-1} f_k\}_{k=1}^{\infty}$ together give two different reconstruction formulas:

$$f = \sum_{k=1}^{\infty} \langle f, S^{-1} f_k\rangle f_k \quad \text{and} \quad f = \sum_{k=1}^{\infty} \langle f, f_k\rangle S^{-1} f_k, \text{ for all } f \in \mathcal{H}.$$

Gabor frames in $L^2(\mathbb{R})$ are generated by two classes of operators on $L^2(\mathbb{R})$, namely the translation and modulation operators. For $a, b \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$, the translation operator T_a on $L^2(\mathbb{R})$ is defined by $(T_a f)(x) = f(x - a)$, $x \in \mathbb{R}$ and the modulation operator E_b on $L^2(\mathbb{R})$ by $(E_b f)(x) = e^{2\pi i b x} f(x)$, $x \in \mathbb{R}$. A frame in $L^2(\mathbb{R})$ of the form $\{E_{mb} T_{na} g\}_{m,n\in\mathbb{Z}}$ for some $g \in L^2(\mathbb{R})$ and $a, b > 0$ is called a *Gabor frame or Weyl-Heisenberg frame.* Gabor analysis aims at representing functions $f \in L^2(\mathbb{R})$ as superposition of translated and modulated versions of a fixed *window* function $g \in L^2(\mathbb{R})$. If $f \in L^2(\mathbb{R})$ is expanded as $f = \sum_{m,n\in\mathbb{Z}} c_{m,n}\, E_{mb} T_{na} g$ using a Gabor frame $\{E_{mb} T_{na} g\}_{m,n\in\mathbb{Z}}$ where $c_{m,n} = \langle f, E_{mb} T_{na} h\rangle, h = S^{-1} g$, then the square summable sequence $\{c_{m,n}\}_{(m,n)\in\mathbb{Z}\times\mathbb{Z}}$ is the *representation* of the signal f in the *time-frequency plane*: $a\mathbb{Z} \times b\mathbb{Z}$.

For more details on this basic discussion, we suggest to refer the survey article of Casazza [2] and the monographs of Christensen [4], Gröchenig [12] and Heil [14].

3. Pseudo Gabor Like Frames in Separable Hilbert Spaces

In our discussions, $\mathcal{H}$ and $\mathcal{K}$ will denote separable Hilbert spaces. We begin with a simple observation, analogous to Corollary 5.3.2 in [4], on the interplay of bounded linear operators between separable Hilbert spaces.

Lemma 3.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. Then every surjective bounded linear operator $A : \mathcal{H} \to \mathcal{K}$ maps frames in $\mathcal{H}$ to frames in $\mathcal{K}$. In particular, every invertible bounded linear operator between two separable Hilbert spaces maps frames in one to frames in the other.*

Proof. Let $\{u_k\}_{k\in\mathbb{N}}$ be a frame in $\mathcal{H}$ with frame bounds $0 < \alpha \leq \beta < \infty$. Then for all $x \in \mathcal{H}$, $\alpha\|x\|^2 \leq \sum_{k\in\mathbb{N}} \mid \langle x, u_k\rangle \mid^2 \leq \beta\|x\|^2$. Obviously, for all $y \in \mathcal{K}$, $\sum\limits_{k\in\mathbb{N}} \mid \langle y, Au_k\rangle \mid^2 = \sum\limits_{k\in\mathbb{N}} \mid \langle A^*y, u_k\rangle \mid^2 \leq \beta\|A^*y\|^2 \leq \beta\|A^*\|^2\|y\|^2 = \beta\|A\|^2\|y\|^2$.

Since $A : \mathcal{H} \to \mathcal{K}$ is surjective, by Lemma 2.5.1 in [4], it has a right inverse $B : \mathcal{K} \to \mathcal{H}$ such that $B \neq 0$ and $AB = I_{\mathcal{K}}$. But then $I_{\mathcal{K}} = (AB)^* = B^*A^*$.

For $y \in \mathcal{K}$, there is $x \in \mathcal{H}$ for which $y = Ax = I_{\mathcal{K}}Ax = B^*A^*Ax$.

Hence $\|y\|^2 = \|B^*A^*Ax\|^2 \leq \|B^*\|^2\|A^*Ax\|^2$

$$\leq \|B^*\|^2(\frac{1}{\alpha})\sum_{k\in\mathbb{N}} \mid \langle A^*Ax, u_k\rangle \mid^2$$

$$= \|B^*\|^2(\frac{1}{\alpha})\sum_{k\in\mathbb{N}} \mid \langle y, Au_k\rangle \mid^2, \text{ where } \|B^*\| > 0.$$

Thus $\alpha\|B^*\|^{-2}\|y\|^2 \leq \sum\limits_{k\in\mathbb{N}} \mid \langle y, Au_k\rangle \mid^2 \leq \beta\|A\|^2\|y\|^2$, as desired. □

In the situation above, the image frame $\{Au_k : k \in \mathbb{N}\}$ has the frame operator S' defined for all $x \in \mathcal{K}$ by

$$S'x = \sum_{k\in\mathbb{N}}\langle x, Au_k\rangle Au_k = A(\sum_{k\in\mathbb{N}}\langle A^*x, u_k\rangle u_k) = ASA^*(x),$$

where S is the frame operator of the frame $\{u_k : k \in \mathbb{N}\}$ in $\mathcal{H}$.

Lemma 3.1 motivates to look at the aspects of the images of Gabor frames under invertible bounded linear operators $B : L^2(\mathbb{R}) \to \mathcal{H}$. Since

$$\begin{aligned} B(\{E_{mb}T_{na}\ g\}_{m,n\in\mathbb{Z}}) &= \{BE_{mb}T_{na}\ g\}_{m,n\in\mathbb{Z}} \\ &= \{BE_{mb}B^{-1}BT_{na}B^{-1}(Bg)\}_{m,n\in\mathbb{Z}} \\ &= \{(BE_bB^{-1})^m(BT_aB^{-1})^n(Bg)\}_{m,n\in\mathbb{Z}}, \end{aligned}$$

they are generated by the action of a group of operators $\{E^B_{mb}T^B_{na}\}_{m,n\in\mathbb{Z}}$ on a single generator Bg, where $E^B_{mb} = BE_{mb}B^{-1}$ and $BT_{na}B^{-1} = T^B_{na}$, $m, n \in \mathbb{Z}$. Thus, such image frames are structured frames in $\mathcal{H}$. For proceeding further, the following definitions will be useful.

Definition 3.2. For an invertible bounded linear operator $B : L^2(\mathbb{R}) \to \mathcal{H}$ and $\alpha \in \mathbb{R}$, we define *B-translation* T^B_α on $\mathcal{H}$ by $T^B_\alpha = BT_\alpha B^{-1}$ and *B-modulation* E^B_α on $\mathcal{H}$ by $E^B_\alpha = BE_\alpha B^{-1}$ where T_α and E_α are respectively the translation and modulation operators on $L^2(\mathbb{R})$. For $a > 0$ and $b > 0$, the family $\{E_{mb}^B T_{na}^B\ x_0\}_{m,n\in\mathbb{Z}}$ generated by $x_0 \in \mathcal{H}$ is called a *Pseudo B-Gabor like system* in $\mathcal{H}$. Such a system is called a *Pseudo B-Gabor like frame (Pseudo B-Gabor like Bessel sequence)* if it forms a frame (Bessel sequence) in $\mathcal{H}$. A frame $\{g_{m,\ n}\}$ in $\mathcal{H}$ is called a *Pseudo Gabor like frame* if $\{g_{m,\ n}\}$ is a Pseudo B-Gabor like frame for some invertible operator $B : L^2(\mathbb{R}) \to \mathcal{H}$.

Gabor type unitary systems discussed in [13] were defined by generalizing the remarkable property $T_a E_b = e^{-2\pi iab} E_b T_a$ of the pair (T_a, E_b) of translation and modulation operators. Interestingly, for the generalization of the system of operators generated by the combination $T_a E_b$, we need not stick on to the unitary system. Instead, a system of invertible operators can be considered, as Proposition 3.3 below suggests.

Proposition 3.3. *Let $B : L^2(\mathbb{R}) \to \mathcal{H}$ be invertible. Then the following statements hold.*

(i) $E_{mb}^B T_{na}^B = e^{2\pi imbna} T_{na}^B E_{mb}^B$ *for all* $m, n \in \mathbb{Z}$.

(ii) $\{E_{mb}^B T_{na}^B x_0\}_{m,n\in\mathbb{Z}}$ *is a Pseudo B-Gabor like frame in* $\mathcal{H}$ *if and only if the family* $\{T_{na}^B E_{mb}^B x_0\}_{m,n\in\mathbb{Z}}$ *is also a frame in* $\mathcal{H}$.

Proof. For all $f \in L^2(\mathbb{R})$, the commutator relation [3]

$$\begin{aligned} T_a E_b f(x) &= T_a(e^{2\pi ibx} f(x)) = e^{2\pi ib(x-a)} f(x-a) = e^{-2\pi iab} e^{2\pi ibx} T_a f(x) \\ &= e^{-2\pi iab} E_b T_a f(x) \text{ holds for all } x \in \mathbb{R}. \end{aligned}$$

Hence for all $f \in L^2(\mathbb{R})$, $T_a E_b f = e^{-2\pi iab} E_b T_a f$

$$\Leftrightarrow B T_a E_b f = B e^{-2\pi iab} E_b T_a f = e^{-2\pi iab} B E_b T_a f$$

$$\Leftrightarrow T_a^B E_b^B B f = e^{-2\pi iab} E_b^B T_a^B B f$$

$$\Leftrightarrow T_a^B E_b^B x = e^{-2\pi iab} E_b^B T_a^B x \text{ for all } x \in \mathcal{H}, \text{ since } B : L^2(\mathbb{R}) \to \mathcal{H} \text{ is invertible.}$$

Hence $E_{mb}^B T_{na}^B = e^{2\pi imbna} T_{na}^B E_{mb}^B$ for all $m, n \in \mathbb{Z}$, proving (i).
Since $\{E_{mb}^B T_{na}^B x_0\}_{m,n\in\mathbb{Z}}$ is a frame in $\mathcal{H}$ and $e^{-2\pi imbna}$ is of absolute value 1, the required frame inequality for the family $\{T_{na}^B E_{mb}^B x_0\}_{m,n\in\mathbb{Z}}$ follows directly from that of $\{E_{mb}^B T_{na}^B x_0\}_{m,n\in\mathbb{Z}}$. The reverse implication follows similarly. □

Upcoming proposition gives a relation between Pseudo B-Gabor like frame in $\mathcal{H}$ and Gabor frame in $L^2(\mathbb{R})$.

Proposition 3.4. *The family* $\{E_{mb}^B T_{na}^B\ x_0\}_{m,n\in\mathbb{Z}}$ *forms a Pseudo B-Gabor like frame in* $\mathcal{H}$ *if and only if* $\{E_{mb} T_{na} B^{-1} x_0\}_{m,n\in\mathbb{Z}}$ *forms a Gabor frame in* $L^2(\mathbb{R})$.

Proof. Since $\{E^B_{mb}T^B_{na}x_0 : m, n \in \mathbb{Z}\} = \{BE_{mb}B^{-1}BT_{na}B^{-1}x_0 : m, n \in \mathbb{Z}\}$
$= B(\{E_{mb}T_{na}B^{-1}x_0 : m, n \in \mathbb{Z}\})$,
the proof follows for both the cases of implications from Lemma 3.1. □

It is well known that if $ab > 1$ for a given pair of frame parameters $a, b > 0$, then no $g \in L^2(\mathbb{R})$ can generate a frame of the form $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ (see Theorem 11.3.1 in [4]). We make an analogous observation here.

Proposition 3.5. *Let $\mathcal{H}$ be a separable Hilbert space and $B : L^2(\mathbb{R}) \to \mathcal{H}$ be a bounded invertible linear map. For $x_0 \in \mathcal{H}$ and $a, b > 0$, $\{E^B_{mb}T^B_{na}x_0\}_{m,n\in\mathbb{Z}}$ does not form a Pseudo B-Gabor like frame for $\mathcal{H}$ whenever $ab > 1$.*

Proof. Let $ab > 1$. If $\{E^B_{mb}T^B_{na}x_0\}_{m,n\in\mathbb{Z}}$ is a Pseudo B-Gabor like frame in $\mathcal{H}$, then Proposition 3.4 yields that $\{E_{mb}T_{na}B^{-1}x_0\}_{m,n\in\mathbb{Z}}$ is a Gabor frame in $L^2(\mathbb{R})$. But then necessarily $ab \le 1$, leading to a contradiction. □

It is also known that, for given $g \in L^2(\mathbb{R})$ and $a, b > 0$, $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence in $L^2(\mathbb{R})$ if and only if $\{E_{m/a}T_{n/b}g\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence in $L^2(\mathbb{R})$ (see Lemma 12.2.2 in [4]). An analogous observation follows from Proposition 3.4.

Lemma 3.6. *Let H be a separable Hilbert space, $B : L^2(\mathbb{R}) \to \mathcal{H}$ be a bounded invertible linear map and $x_0 \in \mathcal{H}$. For $a, b > 0$, the family $\{E^B_{mb}T^B_{na}x_0\}_{m,n\in\mathbb{Z}}$ forms a Pseudo B-Gabor like Bessel sequence in $\mathcal{H}$ if and only if $\{E^B_{m/a}T^B_{n/b}x_0\}_{m,n\in\mathbb{Z}}$ is a Pseudo B-Gabor like Bessel sequence in $\mathcal{H}$.*

Proof. $\{E^B_{mb}T^B_{na}x_0\}_{m,n\in\mathbb{Z}}$ is a Pseudo B-Gabor like Bessel sequence in $\mathcal{H}$
$\Leftrightarrow \{E_{mb}T_{na}B^{-1}x_0\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence in $L^2(\mathbb{R})$, by Proposition 3.4
$\Leftrightarrow \{E_{m/a}T_{n/b}B^{-1}x_0\}_{m,n\in\mathbb{Z}}$ is a Bessel sequence in $L^2(\mathbb{R})$
$\Leftrightarrow \{E^B_{m/a}T^B_{n/b}x_0\}_{m,n\in\mathbb{Z}}$ is a Pseudo B-Gabor like Bessel sequence in $\mathcal{H}$ □

If the family $\{E^B_{mb}T^B_{na}x_0\}_{m,n\in\mathbb{Z}}$, $x_0 \in \mathcal{H}$ is a Pseudo B-Gabor like frame in $\mathcal{H}$ then its frame operator S defined by, $S(x) = \sum\limits_{m,n\in\mathbb{Z}} \langle x, E^B_{mb}T^B_{na}x_0\rangle E^B_{mb}T^B_{na}x_0$, for all $x \in \mathcal{H}$ is called a *Pseudo B-Gabor like frame operator*. These operators are now characterized as follows.

Theorem 3.7. *Let $B : L^2(\mathbb{R}) \to \mathcal{H}$ be an invertible map. A bounded linear operator S on $\mathcal{H}$ is a Pseudo B-Gabor like frame operator if and only if $S = TT^*$, where*

$T = BA$ and A is a surjective operator on $L^2(\mathbb{R})$ commuting with some translation T_a and some modulation E_b for some $a, b > 0$ with $ab \leq 1$.

Proof. Assume that S is a Pseudo B-Gabor like frame operator on $\mathcal{H}$

$\Leftrightarrow B^{-1}S(B^{-1})^*$ is a Gabor frame operator on $L^2(\mathbb{R})$

$\Leftrightarrow B^{-1}S(B^{-1})^* = AA^*$, where A is a surjective operator on $L^2(\mathbb{R})$ commuting with some T_a and E_b for some $a, b > 0$ with $ab \leq 1$, by Theorem 7 in [9].

$\Leftrightarrow S = BAA^*B^* = (BA)(BA)^* = TT^*$ with $T = BA$, where A is a surjective operator on $L^2(\mathbb{R})$ commuting with some T_a and E_b for $a, b > 0$ with $ab \leq 1$. $\square$

The observation made above is very specific for the operator $B : L^2(\mathbb{R}) \to \mathcal{H}$ since the frame under consideration in $\mathcal{H}$ is a Pseudo B-Gabor like frame and not a Pseudo T-Gabor like frame, even in the case when A is also invertible. It may be interesting to look at the Gabor frames contributing to this context. Assume that $S = TT^*$ where $T = BA$, $B : L^2(\mathbb{R}) \to \mathcal{H}$ is invertible and A is a surjective operator on $L^2(\mathbb{R})$ which commutes with some translation T_a and some modulation E_b for some $a, b > 0$ with $ab \leq 1$. The existence of a Parseval Gabor frame of the form $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ together with the surjectivity and commutativity of A ensure that $A(\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}) = \{AE_{mb}T_{na}g = E_{mb}T_{na}Ag\}_{m,n\in\mathbb{Z}}$ is also a Gabor frame in $L^2(\mathbb{R})$. This frame is used for generating a Pseudo B-Gabor like frame $B(\{E_{mb}T_{na}Ag\}_{m,n\in\mathbb{Z}})$ in $\mathcal{H}$.

Example 3.8. For $\beta, \gamma \in \mathbb{C} - \{0\}$, with $|\beta| \neq |\gamma|$ define

$$\psi_{\beta,\gamma}(t) = \begin{cases} \beta & \text{if } t \geq 0 \\ \\ \gamma & \text{if } t < 0. \end{cases}$$

Also for $a > 0$, define

$$\phi_{\beta,\gamma}(t) = \begin{cases} \beta & \text{if } 0 < t \leq a/2 \\ \\ \gamma & \text{if } a/2 < t \leq a \end{cases}$$

and extend $\phi_{\beta,\gamma}$ to $\mathbb{R}$ as an a-periodic function.

The multiplication operators $M_{\psi_{\beta,\gamma}}$ and $M_{\phi_{\beta,\gamma}}$ on $L^2(\mathbb{R})$ defined by $M_{\psi_{\beta,\gamma}}(f) = \phi_{\beta,\gamma}.f$ and $M_{\phi_{\beta,\gamma}}(f) = \phi_{\beta,\gamma}.f$, are invertible with inverses $M^{-1}_{\psi_{\beta,\gamma}} = M_{\psi_{\frac{1}{\beta},\frac{1}{\gamma}}}$ and $M^{-1}_{\phi_{\beta,\gamma}} = M_{\phi_{\frac{1}{\beta},\frac{1}{\gamma}}}$ respectively with adjoints $M^*_{\psi_{\beta,\gamma}} = M_{\psi_{\overline{\beta},\overline{\gamma}}}$ and $M^*_{\phi_{\beta,\gamma}} = M_{\phi_{\overline{\beta},\overline{\gamma}}}$. Hence $M^*_{\psi_{\beta,\gamma}}M_{\psi_{\beta,\gamma}} = M_{\psi_{|\beta|^2,|\gamma|^2}}$ and $M^*_{\phi_{\beta,\gamma}}M_{\phi_{\beta,\gamma}} = M_{\phi_{|\beta|^2,|\gamma|^2}}$.

Define $T : L^2(\mathbb{R}) \to \mathcal{H} = L^2(\mathbb{R})$ by $T(f) = BA(f)$, where $B(f) = M_{\psi_{\beta,\gamma}}(f)$ and $A(f) = M_{\phi_{\beta,\gamma}}(f)$ for all $f \in L^2(\mathbb{R})$. Then $TT^* = M_{\psi_{|\beta|^2,|\gamma|^2}}M_{\phi_{|\beta|^2,|\gamma|^2}}$.

Since $\phi_{\beta,\gamma}$ is a-periodic, $A = M_{\phi_{\beta,\gamma}}$ commutes with the translation T_a for the given $a > 0$ and modulation E_b for every $b > 0$. However $B = M_{\psi_{\beta,\gamma}}$ does not have this property. Now, in view of Theorem 3.7, TT^* is a Pseudo B-Gabor like frame operator on $\mathcal{H} = L^2(\mathbb{R})$. For the forthcoming discussions, it is significant to note that $T^*T = TT^* = M_{\psi_{|\beta|^2,|\gamma|^2}} M_{\phi_{|\beta|^2,|\gamma|^2}}$ is not a Gabor frame operator on $L^2(\mathbb{R})$.

For Bessel sequences $\{u_k\}_{k\in\mathbb{Z}}$ and $\{v_k\}_{k\in\mathbb{Z}}$ in $\mathcal{H}$ and $\mathcal{K}$ respectively, we can have a bounded linear operator $M : \mathcal{H} \to \mathcal{K}$ given by $M(x) = \sum_{k\in\mathbb{Z}} \langle x, v_k\rangle u_k$, where the series defining M converges for all $x \in \mathcal{H}$. The operator M is called the *mixed frame operator* associated with the Bessel sequences $\{v_k\}$ and $\{u_k\}$ [4].

Here is an interesting connection between mixed frame operators and invertible operators from $L^2(\mathbb{R})$ into $\mathcal{H}$.

Proposition 3.9. *Every invertible operator $B : L^2(\mathbb{R}) \to \mathcal{H}$ can be identified as a mixed frame operator.*

Proof. Let $B : L^2(\mathbb{R}) \to \mathcal{H}$ is a bounded invertible map. Obviously, B maps any given Gabor frame $\mathcal{G} = \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ to a Pseudo B-Gabor like frame $B(\mathcal{G}) = \{E^B_{mb}T^B_{na}Bg\}_{m,n\in\mathbb{Z}}$ in $\mathcal{H}$. Let M be the mixed frame operator defined by $Mf = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle E^B_{mb}T^B_{na}Bg$, $f \in L^2(\mathbb{R})$. Then for all $f \in L^2(\mathbb{R})$,

$$\begin{aligned} Mf &= \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle BE_{mb}B^{-1}BT_{na}B^{-1}Bg \\ &= B \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle E_{mb}T_{na}g \\ &= BS(f), \text{ where } S \text{ is the frame operator of } \mathcal{G}. \end{aligned}$$

Thus, $M = BS$. Now, by choosing $\mathcal{G}$ as a Parseval Gabor frame in $L^2(\mathbb{R})$, we obtain $S = I_{L^2(\mathbb{R})}$ so that $B : L^2(\mathbb{R}) \to \mathcal{H}$ is precisely a mixed frame operator. □

A bounded linear transformation $S \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ is said to *intertwine* an operator $A \in \mathcal{B}(\mathcal{K})$ to $B \in \mathcal{B}(\mathcal{H})$ if $SA = BS$. In this context, S is called an *intertwining operator* [11, 16]. We observe that Pseudo B-Gabor like frame operators are intertwining operators.

Theorem 3.10. *Let S_B be a Pseudo B-Gabor like frame operator on $\mathcal{H}$ whose involved B-modulations and B-translations are E^B_{mb}, T^B_{na}, $m, n \in \mathbb{Z}$ respectively. Then S_B intertwines $(E^B_{-mb})^*$ to E^B_{mb} and $(T^B_{-na})^*$ to T^B_{na} respectively.*

Proof. Let $\{E^B_{mb}T^B_{na}x_0\}_{m,n\in\mathbb{Z}}$ be a Pseudo B-Gabor like frame in $\mathcal{H}$. Then $\{E^B_{mb}T^B_{na}x_0\}_{m,n\in\mathbb{Z}} = B(\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}})$ where $x_0 = Bg$ and $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a

Gabor frame in $L^2(\mathbb{R})$. If S_B is the frame operator of $\{E^B_{mb}T^B_{na}x_0\}_{m,n\in\mathbb{Z}}$, then for each $k \in \mathbb{Z}$,

$$\begin{aligned}
S_B(E^B_{-kb})^*x &= \sum_{m,n\in\mathbb{Z}} \langle (E^B_{-kb})^*x, E^B_{mb}T^B_{na}x_0\rangle E^B_{mb}T^B_{na}x_0 \\
&= \sum_{m,n\in\mathbb{Z}} \langle x, E^B_{(m-k)b}T^B_{na}x_0\rangle E^B_{mb}T^B_{na}x_0 \\
&= \sum_{l,n\in\mathbb{Z}} \langle x, E^B_{lb}T^B_{na}x_0\rangle E^B_{(l+k)b}T^B_{na}x_0 \\
&= E^B_{kb}\Big(\sum_{l,n\in\mathbb{Z}} \langle x, E^B_{lb}T^B_{na}x_0\rangle E^B_{lb}T^B_{na}x_0\Big) \\
&= E^B_{kb}S_Bx \text{ for all } x \in \mathcal{H}.
\end{aligned}$$

Thus $S_B(E^B_{-kb})^* = E^B_{kb}S_B$ for all $k \in \mathbb{Z}$.

Now by Proposition 3.3 (i), $E^B_{mb}T^B_{na} = e^{2\pi imbna}T^B_{na}E^B_{mb}$ for all $m, n \in \mathbb{Z}$. Hence for each $k \in \mathbb{Z}$,

$$\begin{aligned}
S_B(T^B_{-ka})^*x &= \sum_{m,n\in\mathbb{Z}} \langle (T^B_{-ka})^*x, E^B_{mb}T^B_{na}x_0\rangle E^B_{mb}T^B_{na}x_0 \\
&= \sum_{m,n\in\mathbb{Z}} \langle (T^B_{-ka})^*x, e^{2\pi imbna}T^B_{na}E^B_{mb}x_0\rangle e^{2\pi imbna}T^B_{na}E^B_{mb}x_0 \\
&= \sum_{m,n\in\mathbb{Z}} \langle (T^B_{-ka})^*x, T^B_{na}E^B_{mb}x_0\rangle T^B_{na}E^B_{mb}x_0 \\
&= T^B_{ka}S_Bx \text{ for all } x \in \mathcal{H}, \text{ as computed above.}
\end{aligned}$$

Thus $S_B(T^B_{-ka})^* = T^B_{ka}S_B$ for all $k \in \mathbb{Z}$. □

Writing $U = E^B_b$ and $V = T^B_a$, from Proposition 3.3 (i), we obtain $U^mV^n = E^B_{mb}T^B_{na} = e^{2\pi imbna}T^B_{na}E^B_{mb} = e^{2\pi imbna}V^nU^m$ for all $m, n \in \mathbb{Z}$. Hence $\{U^mV^nx_0\}_{m,n\in\mathbb{Z}}$ $= \{e^{2\pi imnab}V^nU^mx_0\}_{m,n\in\mathbb{Z}}$. This is a remarkable property of Pseudo B-Gabor like frames in $\mathcal{H}$.

Proposition 3.11. *Let $\{U^mV^nx_0\}_{m,n\in\mathbb{Z}}$ be a frame in $\mathcal{H}$ where $U, V \in \mathcal{B}(\mathcal{H})$ are invertible operators. If $U^mV^nx_0 = e^{2\pi imnab}V^nU^mx_0$ for all $m, n \in \mathbb{Z}$ and for some $a, b > 0$ with $ab \le 1$, then the system of equations $U^kX = XE_{kb}$ and $V^kX = XT_{ka}$, $k \in \mathbb{Z}$, has a solution in $\mathcal{B}(L^2(\mathbb{R}), \mathcal{H})$.*

Proof. For the given pair of parameters $a, b > 0$ such that $ab \le 1$, choose a Gabor frame $\mathcal{G} = \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$. Let B be the mixed frame operator defined by

$$Bf = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}g\rangle U^mV^nx_0, f \in L^2(\mathbb{R}).$$

Now, for all $k \in \mathbb{Z}$ and $f \in L^2(\mathbb{R})$,

$$
\begin{aligned}
BE_{kb}f &= \sum_{m,n\in\mathbb{Z}} \langle E_{kb}f, E_{mb}T_{na}g\rangle U^m V^n x_0 \\
&= \sum_{m,n\in\mathbb{Z}} \langle f, E_{(m-k)b}T_{na}g\rangle U^m V^n x_0 \\
&= \sum_{l,n\in\mathbb{Z}} \langle f, E_{lb}T_{na}g\rangle U^{(l+k)} V^n x_0 \\
&= U^k\Big(\sum_{l,n\in\mathbb{Z}} \langle f, E_{lb}T_{na}g\rangle U^l V^n x_0\Big) \\
&= U^k(Bf).
\end{aligned}
$$

Hence $U^kB = BE_{kb}$ for all $k \in \mathbb{Z}$. On the other hand,

$$
\begin{aligned}
BT_{ka}f &= \sum_{m,n\in\mathbb{Z}} \langle T_{ka}f, E_{mb}T_{na}g\rangle U^m V^n x_0 \\
&= \sum_{m,n\in\mathbb{Z}} \langle T_{ka}f, e^{2\pi imnab}T_{na}E_{mb}g\rangle e^{2\pi imnab} V^n U^m x_0 \\
&= \sum_{m,n\in\mathbb{Z}} \langle f, T_{(n-k)a}E_{mb}g\rangle V^n U^m x_0 \\
&= \sum_{l,m\in\mathbb{Z}} \langle f, T_{la}E_{mb}g\rangle V^{(l+k)} U^m x_0 \\
&= V^kBf \text{ for all } k \in \mathbb{Z} \text{ and } f \in L^2(\mathbb{R}).
\end{aligned}
$$

Hence $V^kB = BT_{ka}$ for all $k \in \mathbb{Z}$. □

4. Pseudo Gabor Frames in Separable Hilbert Spaces

For each invertible bounded linear map $B : L^2(\mathbb{R}) \to \mathcal{H}$, the bounded linear operator BB^* on $\mathcal{H}$ is positive and invertible. Hence it becomes a frame operator of some frame in $\mathcal{H}$. Interestingly, this frame operator corresponds to a Pseudo B-Gabor like frame in $\mathcal{H}$.

Proposition 4.1. *Every pair of frame parameters $a, b > 0$ satisfying $ab \leq 1$ yields a Pseudo B-Gabor like frame in $\mathcal{H}$ with BB^* as its frame operator.*

Proof. For each pair of translation and modulation parameters $a, b > 0$ satisfying $0 < ab \leq 1$, always there exists a tight Gabor frame $\{E_{mb}T_{na}g_0\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ with identity operator as frame operator (see [8]). If $B : L^2(\mathbb{R}) \to \mathcal{H}$ is a bounded invertible linear map, then $B(\{E_{mb}T_{na}g_0\}_{m,n\in\mathbb{Z}}) = \{E^B_{mb}T^B_{na}Bg_0\}_{m,n\in\mathbb{Z}}$ is a Pseudo B-Gabor like frame in $\mathcal{H}$ with frame operator $BIB^* = BB^*$. □

Apart from the positivity and invertibility of the Gabor frame operators on $L^2(\mathbb{R})$, their commutativity with some specific modulation and translation operators were significant in [8] for characterizing the Gabor frame operators on $L^2(\mathbb{R})$. Here we look at the similar situation in the context of Pseudo B-Gabor like frames.

Theorem 4.2. *The following are equivalent for a given invertible bounded linear operator $B : L^2(\mathbb{R}) \to \mathcal{H}$.*

i) B^*B is a Gabor frame operator on $L^2(\mathbb{R})$.

ii) Every Pseudo B-Gabor like frame operator on $\mathcal{H}$ commutes with its involved B-modulations E^B_{mb} and B-translations T^B_{na}, $m, n \in \mathbb{Z}$.

iii) There exists a Parseval Pseudo B-Gabor like frame in $\mathcal{H}$ for every pair of frame parameters a and b with $ab \leq 1$.

Proof. i) $\Rightarrow$ ii): Assume that S is the frame operator of a Pseudo B-Gabor like frame $\{E^B_{mb}T^B_{na}x_0\}_{m,n\in\mathbb{Z}}$ in $\mathcal{H}$ for some frame parameters $a, b > 0$ with $ab \leq 1$. Then $B^{-1} : \mathcal{H} \to L^2(\mathbb{R})$ maps this frame to the Gabor frame $\{E_{mb}T_{na}B^{-1}x_0\}_{m,n\in\mathbb{Z}}$ whose frame operator is $B^{-1}S(B^{-1})^*$. Hence the operator $B^{-1}S(B^{-1})^*$ commutes with E_{mb} and T_{na} for all $m, n \in \mathbb{Z}$. Now, assuming (i), we obtain

$$\begin{aligned} SE^B_{mb} &= SBE_{mb}B^{-1} = S(B^{-1})^*(B^*B)E_{mb}B^{-1} \\ &= S(B^{-1})^*E_{mb}(B^*B)B^{-1} = S(B^{-1})^*E_{mb}B^* \\ &= BB^{-1}S(B^{-1})^*E_{mb}B^* = BE_{mb}(B^{-1}S(B^{-1})^*)B^* \\ &= BE_{mb}B^{-1}S = E^B_{mb}S \text{ for all } m \in \mathbb{Z}. \end{aligned}$$

Similarly $ST^B_{na} = T^B_{na}S$ for all $n \in \mathbb{Z}$.
ii) $\Rightarrow$ iii): For every pair of frame parameters a and b with $ab \leq 1$, there is always a Gabor frame in $L^2(\mathbb{R})$ and hence there is a Pseudo B-Gabor like frame in $\mathcal{H}$. Since by (ii), the frame operator S of such a Pseudo B-Gabor like frame $\mathcal{P}$ commutes with its involved B-modulations and B-translations, so does the operator $S^{-1/2}$. Hence the image frame $S^{-1/2}(\mathcal{P})$ will be a Parseval Pseudo B-Gabor like frame in $\mathcal{H}$ with frame parameters a and b.
iii) $\Rightarrow$ i): Let $\mathcal{P}$ be a Parseval Pseudo B-Gabor like frame in $\mathcal{H}$ with frame parameters a and b. Then $B^{-1}(\mathcal{P})$ will be a Gabor frame in $L^2(\mathbb{R})$ with frame operator $B^{-1}I(B^{-1})^* = (B^*B)^{-1}$. Thus $(B^*B)^{-1}$ commutes with E_{mb} and T_{na} for all $m, n \in \mathbb{Z}$ and hence its inverse B^*B also has this property. □

Thus, each B as above has a specific control in terms of the bounded linear operator B^*B on $L^2(\mathbb{R})$ for yielding Parseval Pseudo B-Gabor like frames in $\mathcal{H}$ as well as Pseudo B-Gabor like frames having canonical dual frames with same structure. Such frames are more similar to Gabor frames in $L^2(\mathbb{R})$.

Example 4.3. For $\alpha, \beta \in \mathbb{C} - \{0\}$ with $\mid \alpha \mid \neq \mid \beta \mid$, let

$$\phi_{\alpha,\beta}(x) = \begin{cases} \alpha & \text{if } x \leq 0 \\ \beta & \text{if } x > 0 \end{cases}$$

The multiplication operator $M_{\phi_{\alpha,\beta}}$ on $L^2(\mathbb{R})$ defined by $M_{\phi_{\alpha,\beta}}(f) = \phi_{\alpha,\beta}.f$ is invertible with inverse $M_{\phi_{\alpha^{-1},\beta^{-1}}}$.

Also, $(M_{\phi_{\alpha,\beta}})^* = M_{\overline{\phi}_{\alpha,\beta}}$ so that $M^*_{\phi_{\alpha,\beta}} M_{\phi_{\alpha,\beta}} = M_{\phi_{|\alpha|^2,|\beta|^2}}$.
Being a multiplication operator, $M_{\phi_{\alpha,\beta}}$ commutes with all the modulations E_{mb}. However, since $\phi_{\alpha,\beta}$ is not a-periodic for any $a > 0$, $M_{\phi_{\alpha,\beta}}$ does not commute with any translations T_{na}.

Hence $M^*_{\phi_{\alpha,\beta}} M_{\phi_{\alpha,\beta}} = M_{\phi_{|\alpha|^2,|\beta|^2}}$ can not become a Gabor frame operator on $L^2(\mathbb{R})$. Thus, in view of Theorem 4.2, there can not exist a Parseval Pseudo $M_{\phi_{\alpha,\beta}}$-Gabor like frame in $L^2(\mathbb{R})$. But still, being an invertible map, $M_{\phi_{\alpha,\beta}}$ maps Gabor frames into frames which are Pseudo $M_{\phi_{\alpha,\beta}}$-Gabor like frames in $L^2(\mathbb{R})$.

In view of the above discussions we give a new definition which is suitable for identifying the structures more specifically.

Definition 4.4. A Pseudo B-Gabor like frame $\{E^B_{mb} T^B_{na}\ x_0\}_{m,n\in\mathbb{Z}}$ in a separable Hilbert space $\mathcal{H}$ is said to be a *Pseudo B-Gabor frame* if B^*B is a Gabor frame operator on $L^2(\mathbb{R})$. The frame operator of a Pseudo B-Gabor frame is called a *Pseudo B-Gabor frame operator.*

Corollary 4.5. *If $U : L^2(\mathbb{R}) \to \mathcal{H}$ is a unitary operator, then every Pseudo U-Gabor like frame operator on $\mathcal{H}$ commutes with its involved U-modulations and U-translations.*

Proof. If $U : L^2(\mathbb{R}) \to \mathcal{H}$ is a unitary operator, then $U^*U = I_{L^2(\mathbb{R})}$. For each pair of frame parameters $a, b > 0$ (with $ab \leq 1$), there is always a Parseval Gabor frame in $L^2(\mathbb{R})$. Hence $U^*U = I_{L^2(\mathbb{R})}$ is a Gabor frame operator on $L^2(\mathbb{R})$. Now the claim follows from Theorem 4.2. $\square$

Another interesting consequence of Theorem 4.2 is the following.

Theorem 4.6. *Let $B : L^2(\mathbb{R}) \to \mathcal{H}$ be an invertible map such that B^*B is a Gabor frame operator on $L^2(\mathbb{R})$. Then for any given Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ with frame operator S, the canonical dual frame of the Pseudo B-Gabor frame $\{E^B_{mb}T^B_{na}Bg\}_{m,n\in\mathbb{Z}}$ in $\mathcal{H}$ is again a Pseudo B-Gabor frame with generator $CS^{-1}g$*

where $C = (B^*)^{-1}$. Also, this Pseudo B-Gabor frame has a dual Pseudo C-Gabor frame with same generator $CS^{-1}g$.

Proof. Let $B : L^2(\mathbb{R}) \to \mathcal{H}$ be an invertible map. Take $C = (B^*)^{-1}$, then $C : L^2(\mathbb{R}) \to \mathcal{H}$ is an invertible operator and $C^*C = ((B^*)^{-1})^*(B^*)^{-1} = B^{-1}(B^*)^{-1} = (B^*B)^{-1}$. Since B^*B is a Gabor frame operator on $L^2(\mathbb{R})$ so is its inverse $(B^*B)^{-1}$. Thus C^*C is also a Gabor frame operator on $L^2(\mathbb{R})$.

Now, for a given Gabor frame $\mathcal{G} = \{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ with frame operator S, the frame operator of the Pseudo B-Gabor frame $B(\mathcal{G}) = \{E^B_{mb}T^B_{na}Bg\}_{m,n\in\mathbb{Z}}$ is BSB^*. Hence the canonical dual frame of $B(\mathcal{G})$ is $(BSB^*)^{-1}(B(\mathcal{G}))$

$$\begin{aligned}
&= (BSB^*)^{-1}(\{E^B_{mb}T^B_{na}Bg\}_{m,n\in\mathbb{Z}})\\
&= \{E^B_{mb}T^B_{na}(BSB^*)^{-1}Bg\}_{m,n\in\mathbb{Z}}, \text{ by Theorem 4.2 (ii)}\\
&= \{E^B_{mb}T^B_{na}(B^*)^{-1}S^{-1}g\}_{m,n\in\mathbb{Z}}\\
&= \{E^B_{mb}T^B_{na}CS^{-1}g\}_{m,n\in\mathbb{Z}}, \text{ since } C = (B^*)^{-1}.
\end{aligned}$$

Thus, the canonical dual frame of the Pseudo B-Gabor frame $B(\mathcal{G})$ in $\mathcal{H}$ is again a Pseudo B-Gabor frame with generator $CS^{-1}g$ and same frame parameters.

Now, mapping the canonical dual Gabor frame $S^{-1}(\mathcal{G}) = \{E_{mb}T_{na}S^{-1}g\}_{m,n\in\mathbb{Z}}$ of $\mathcal{G}$ by C, we obtain the Pseudo C-Gabor frame $\{E^C_{mb}T^C_{na}CS^{-1}g\}_{m,n\in\mathbb{Z}}$. Frame operator of this frame is $CS^{-1}C^* = (B^*)^{-1}S^{-1}((B^*)^{-1})^* = (B^*)^{-1}S^{-1}(B^{-1}) = (BSB^*)^{-1}$, the canonical dual frame operator of $B(\mathcal{G})$. Thus both the frames $\{E^B_{mb}T^B_{na}CS^{-1}g\}_{m,n\in\mathbb{Z}}$ and $\{E^C_{mb}T^C_{na}CS^{-1}g\}_{m,n\in\mathbb{Z}}$ are dual frames of $B(\mathcal{G})$ with common generator $CS^{-1}g$. □

Obviously, when the map $B : L^2(\mathbb{R}) \to \mathcal{H}$ is unitary, we have $C = (B^*)^{-1} = B$ so that the above frames are precisely the same.

The following version of Example 3.8 is a specific situation of Theorem 4.6. If $B,\ C : L^2(\mathbb{R}) \to \mathcal{H}$ are invertible with $C = (B^*)^{-1}$, then $\{E^B_{mb}T^B_{na}CS^{-1}g\}_{m,n\in\mathbb{Z}}$ and $\{E^C_{mb}T^C_{na}CS^{-1}g\}_{m,n\in\mathbb{Z}}$ are respectively, Pseudo B-Gabor frame and Pseudo C-Gabor frame on $\mathcal{H}$ with same generator $CS^{-1}g$ and same frame operator $(BSB^*)^{-1}$.

Example 4.7. Define $B : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $B(f) = (M_{\phi_{\beta,\gamma}}D_\alpha)(f)$ for all $f \in L^2(\mathbb{R})$, where D_α is the dilation unitary operator and $M_{\phi_{\beta,\gamma}}$ is the multiplication operator on $L^2(\mathbb{R})$ as we have seen in Example 3.8.
Observe that

$$B^* = D^*_\alpha(M_{\phi_{\beta,\gamma}})^* = D_{\frac{1}{\alpha}}M_{\phi_{\overline{\beta},\overline{\gamma}}} \text{ and } B^*B = D_{\frac{1}{\alpha}}M_{\phi_{|\beta|^2,|\gamma|^2}}D_\alpha.$$

It can be easily verified that B^*B commutes with modulation E_b and translation T_a. Further, B^*B is positive and invertible. Hence B^*B is a Gabor frame operator on $L^2(\mathbb{R})$. Also we see that,

$$\begin{aligned}
&BE_{mb}B^{-1}f(t) = (BE_{mb}D_{\frac{1}{\alpha}}M_{\phi_{\frac{1}{\beta},\frac{1}{\gamma}}})f(t) = e^{(2\pi imbt/\alpha)}f(t) = E_{\frac{mb}{\alpha}}f(t),\\
&BT_{na}B^{-1}f(t) = BT_{na}(\sqrt{|\,\alpha\,|}\phi_{\frac{1}{\beta},\frac{1}{\gamma}}(\alpha t)f(\alpha t))\\
&\qquad\qquad = \phi_{\beta,\gamma}(t)\phi_{\frac{1}{\beta},\frac{1}{\gamma}}(t-na/\alpha)f(t-na/\alpha)
\end{aligned}$$

Taking $C=(B^*)^{-1}$ and by simple computations, we obtain

$$\begin{aligned}
&CE_{mb}C^{-1}f(t) = (B^*)^{-1}E_{mb}B^*f(t) = e^{2\pi imbt/\alpha}f(t) = E_{\frac{mb}{\alpha}}f(t),\\
&CT_{na}C^{-1}f(t) = (B^*)^{-1}T_{na}B^*f(t) = \phi_{\frac{1}{\overline{\beta}},\frac{1}{\overline{\gamma}}}(t)\phi_{\overline{\beta},\overline{\gamma}}(t-na/\alpha)f(t-na/\alpha).
\end{aligned}$$

Thus $BE_{mb}B^{-1} = CE_{mb}C^{-1}$, but $BT_{na}B^{-1} \neq CT_{na}C^{-1}$.

Hence $E^B_{mb}T^B_{na} \neq E^C_{mb}T^C_{na}$ and for a given Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$ with frame operator

$$S, C\{E_{mb}T_{na}S^{-1}g\}_{m,n\in\mathbb{Z}} = \{E^C_{mb}T^C_{na}CS^{-1}g\}_{m,n\in\mathbb{Z}}$$

and

$$\{E^B_{mb}T^B_{na}CS^{-1}g\}_{m,n\in\mathbb{Z}}$$

are different frames with same generator $CS^{-1}g$ and same frame operator $(BSB^*)^{-1}$.

Images of Gabor frames under unitary transformations have received remarkable research attention. Non-unitary approaches also are found to be significant in the context of quantum field theory [17]. This leads to the study about Pseudo B-Gabor like frames.

REFERENCES

1. J. Cahill, P.G. Cazassa & G. Kutyniok: Operators and Frames. J. Oper. Theory. **70** (2013), no. 1, 145-164. `doi:10.7900/jot.2011may10.1973`
2. P.G. Cazassa: Modern tools for Weyl-Heisenberg (Gabor) frame theory. Adv. Imaging Electronic Phys. **115** (2000), 1-127. `doi:10.1016/S1076-5670(01)80094-X`
3. O. Christensen: Functions, Spaces, and Expansions. Second Edition, Birkhäuser, Boston, 2010.
4. O. Christensen: An Introduction to Frames and Riesz Bases. Second Edition, Birkhäuser, Boston, 2016.
5. I. Daubechies, A. Grossmann & Y. Meyer: Painless nonorthogonal expansions. J. Math. Phys. **27** (1986), 1271-1283. `doi:10.1063/1.527388`

6. R.J. Duffin & A.C. Schaeffler: A class of non-harmonic Fourier series. Trans. Amer. Math. Soc. **72** (1952), 341-366 `doi:10.1090/S0002-9947-1952-0047179-6`
7. T.C. Easwaran Nambudiri & K. Parthasarathy: Generalised Weyl-Heisenberg frame operators. Bull. Sci. Math. **136** (2012), 44-53 `doi:10.1016/j.bulsci.2011.09.001`
8. T.C. Easwaran Nambudiri & K. Parthasarathy: Characterization of Weyl-Heisenberg frame operators. Bull. Sci. Math. **137** (2013), 322-324 `doi:org/10.1016/j.bulsci.2012.09.001`
9. T.C. Easwaran Nambudiri & K. Parthasarathy: Wavelet frame operators and admissible frames. Asian-Eur. J. Math. **14** (2021), no. 9, 2150160. `doi:org/10.1142/S1793557121501606`
10. D. Gabor: Theory of communication. Journal of Institution of Electrical Engineers **93** (1946), 429-457.
11. R. Gellar & L. Page: A new look at some familiar spaces of intertwining operators. Pacific J. Math. **47** (1973), no. 2, 435-441.
12. K. Gröchenig: Foundations of Time Frequency Analysis. Birkhäuser, Boston, 2001.
13. D. Han & D.R. Larson: Frames, Bases and Group representations. Mem. Am. Math. Soc. **147** (2000), 697-714.
14. C. Heil: A Basis Theory Primer. Birkhäuser, Boston, 2011.
15. M. Janssen: Gabor representation of generalized functions. J. Math. Anal. Appl. **83** (1981), 377-394. `doi:10.1016/0022-247X(81)90130-X`
16. C.S. Kubrusly: The Elements of Operator Theory, Second Edition, Birkhäuser, Boston, 2011.
17. K.D. Lamb, Gerry, C. Christopher & Grobe Rainer: "Unitary and nonunitary approaches in quantum field theory" (2007), Faculty publications Physics. 40. `https://ir.library.illinoisstate.edu/fpphys/40`
18. G.E. Pfander: Sampling of Operators. J. Fourier Anal. Appl. **19** (2013), 612-650.

[a]Research scholar: St. Thomas College Palai, Arunapuram, Kottayam, Kerala, India
Email address: `jineeshthomas@gmail.com`

[b]Professor: Department of Mathematics, Government College Shanthanpara, Idukki, Kerala, India
Email address: `madhavangck@gmail.com`

[c]Professor: Department of Mathematics, Government Brennen College Kannur, Kerala, India
Email address: `easwarantc@gmail.com`