J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN(Print) 1226-0657
https://doi.org/10.7468/jksmeb.2024.31.2.217 ISSN(Online) 2287-6081
Volume 31, Number 2 (May 2024), Pages 217–233

EXISTENCE OF A SOLUTION OF THE INTEGRAL EQUATIONS ON TRIPLED QUASI-METRIC SPACES WITH APPLICATIONS

Ghorban Khalilzadeh Ranjbar

Abstract. In this paper we study a tripled quasi-metric with new fixed point theorems around β-implicit contractions in tripled quasi-metric spaces. We give an example on a solution of a integral equations.

1. Introduction and Preliminaries

It is well known that passing from metric spaces to quasi-metric spaces, dropping the requirement that the metric function verifies $d(x,y) = d(y,x)$ carries with it immediate consequences to the general theory. For instance, the topological notions of quasi-metric spaces, such as, limit, continuity, completeness all should be re-considered under the left and right approaches since the quasi-metric is not symmetric. Furthermore, uniqueness of limit of a sequence should be examined carefully since one can easily consider a sequence which has a left limit and right limit which are not equal to each other. Thats why a few results on fixed points in such spaces are considered.

In this paper, we introduce tripled quasi-metric and prove many fixed point results in tripled quasi-metric. We come to the below of the definition of quasi metric space previously defined by a mathematician.

Definition 1.1. Let Y be a non-empty and let $d : Y \times Y \to [0, 1)$ be a function which satisfies:

($d1$) $d(u,v) = 0$ if and only if $u = v$;

($d2$) $d(u,v) \leq d(u,w) + d(w,v)$.

Then d is called a *quasi-metric* and the pair (Y,d) is called a *quasi-metric space.*

Received by the editors December 7, 2023. Revised Dec. 26, 2023. Accepted January 18, 2024.
2020 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. fixed point, implicit contraction, tripled quasi-metric.

Remark 1.2. Any metric space is a quasi-metric space, but the converse is not true in general.

Definition 1.3. Let (Y, d) be a quasi-metric space, $\{y_n\}$ be a sequence in Y, and $y \in Y$. The sequence $\{y_n\}$ converges to y if and only if

$$\lim_{n\to\infty} d(y_n, y) = \lim_{n\to\infty} d(y, y_n) = 0. \tag{1.1}$$

Remark 1.4. A quasi-metric space is Hausdorff, that is, we have the uniqueness of limit of a convergent sequence.

Definition 1.5. Let (Y, d) be a quasi-metric space and $\{y_n\}$ be a sequence in Y. We say that $\{y_n\}$ is *left-Cauchy* if and only if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $d(y_n, y_m) < \varepsilon$ for all $n \geq m > N$.

Definition 1.6. Let (Y, d) be a quasi-metric space and $\{y_n\}$ be a sequence in Y. We say that $\{y_n\}$ is *right-Cauchy* if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(y_n, y_m) < \varepsilon$ for all $m \geq n > N$.

Definition 1.7. Let (Y, d) be a quasi-metric space and $\{y_n\}$ be a sequence in Y. We say that $\{y_n\}$ is *Cauchy* if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(y_n, y_m) < \varepsilon$ for all $m \geq n > N$.

Remark 1.8. A sequence $\{y_n\}$ in a quasi-metric space is Cauchy if and only if it is left-Cauchy and right-Cauchy.

Definition 1.9. Let (Y, d) be a quasi-metric space. We say that

1) (Y, d) is *left-complete* if and only if each left-Cauchy sequence in Y is convergent.
2) (Y, d) is *right-complete* if and only if each right-Cauchy sequence in Y is convergent.
3) (Y, d) is *complete* if and only if each Cauchy sequence in Y is convergent.

Definition 1.10. Let (Y, d) be a quasi-metric space. We say $f : Y \to Y$ be *continuous* if for each sequence $\{y_n\}$ in Y converging to $y \in Y$, the sequence $\{fy_n\}$ converges to fy, that is,

$$\lim_{n\to\infty} d(fy_n, fy) = \lim_{n\to\infty} d(fy, fy_n) = 0. \tag{1.2}$$

On the other hand the study of fixed point for mappings satisfying on implicit relation in initiated and studies by Popa [21, 22]. It leads to interesting known fixed point results. Following Popa approach , many authors proved some fixed point, common fixed point and coincidence point results in various ambient spaces, see [14, 15, 16, 17, 19].

In the literature, there are several types of implicit contraction mappings, where many nice consequences of fixed point theorems could be derived.

First, denote the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying:

(ψ1) ψ is nondecreasing,

(ψ2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t \in \mathbb{R}^+$, where ψ^n is the nth iterate of ψ. We show by Ψ, the set of all function ψ.

Remark 1.11. It is simple to see that if $\psi \in \Psi$, then $\psi(t) < t$ for any $t > 0$.

2. Main Results

Definition 2.1. Let Y be a nonempty set and let $d : Y \times Y \times Y \to [0, \infty)$ be a function which satisfies

(d_1) $d(x, y, z) = 0$ if and only if $x = y = z$;

(d_2) $d(x, y, z) \leq d(x, a_1, a_2) + d(y, a_3, a_4) + d(z, a_2, a_3)$ for all $x, y, z \in Y$ and $a_i \in Y$ for $i = 1, 2, 3, 4$.

Thus d is called a *tripled quasi-metric* and the pair (Y, d) is called a *tripled quasi-metric space.*

Example 2.2. Let $Y = [0, \infty)$ endowed with the tripled quasi metric, $d(x, y, z) = |x| + |y|$ if $x \neq y$, $x \neq z$, $y \neq z$ and $d(x, y, z) = 0$ whenever $x = y = z$.

Definition 2.3. Let (Y, d) de a tripled quasi-metric, $\{y_n\}$ be a sequence in Y, and $x \in Y$. The sequence $\{y_n\}$ converges to x if and only if

$$\lim_{n\to\infty} d(y_n, x, x) = \lim_{n\to\infty} d(x, x, y_n) = \lim_{n\to\infty} d(y_n, y_n, x) = \lim_{n\to\infty} d(x, y_n, y_n) = 0.$$

Definition 2.4. Let (Y, d) be a tripled quasi-metric space and $\{y_n\}$ be a sequence in Y. We say that $\{y_n\}$ is *left-Cauchy* if and only if for every $\varepsilon > 0$ there exists a positive integer N such that $d(y_n, y_m, y_m) < \varepsilon$ for all $n \geq m > n$.

Definition 2.5. Let (Y, d) be a tripled quasi-metric space and $\{y_n\}$ be a sequence in Y. We say that $\{y_n\}$ is *right-Cauchy* if and only if for every $\varepsilon > 0$ there exists a positive integer N such that $d(y_n, y_m, y_m) < \varepsilon$ for all $m \geq n > N$.

Definition 2.6. Let (Y, d) be a tripled quasi-metric space. We say that $\{y_n\}$ is *Cauchy* if and only if for every $\varepsilon > 0$ there exists a positive integer N, such that $d(y_n, y_m, y_m) < \varepsilon$ for all $n, m > N$.

Definition 2.7. Let (Y, d) be a tripled quasi-metric space. We say that

(1) (Y, d) is *left-complete* if and only if each left-Cauchy sequence in Y is convergent;
(2) (Y, d) is *right-complete* if and only if each right-Cauchy sequence in Y is convergent;
(3) (Y, d) is *left-complete* if and only if each Cauchy sequence in Y is convergent.

Definition 2.8. Let (Y, d) be a tripled quasi metric space. The map $f : Y \to Y$ is continuous if for each sequence $\{y_n\}$ in Y converging to $y \in Y$, the sequence $\{fy_n\}$ converges to fy, such that

$$\begin{aligned}\lim_{n\to\infty} d(fy_n, fy, fy) = \lim_{n\to\infty} d(fy, fy, fy_n) &= \lim_{n\to\infty} d(fy_n, fy_n, fy) \\ &= \lim_{n\to\infty} d(fy, fy_n, fy_n) = 0.\end{aligned}$$

Definition 2.9. Let $T : Y \to Y$ and $d : Y \times Y \times Y \to [0, \infty)$ be mappings. We say that the self-mapping T on Y is β *admissible*, if for all $u, v, w \in Y$ we have

$$\beta(u, v, w) \geq 1 \Rightarrow \beta(Tu, Tv, Tw) \geq 1. \tag{2.1}$$

Definition 2.10. Let (Y, d) be a quasi-metric space and $f : Y \to Y$ be a given mapping. We say that f is an *β-implicit contractive mapping* if there exist two functions $\beta : Y \times Y \times Y \to [0, \infty)$ and $\phi \in \Psi$ such that

$$\begin{aligned}\phi\big(\beta(x, y, z)d(fx, fy, fz), d(x, y, z), d(x, fx, f^2x), d(y, fy, f^2y), d(z, fz, f^2z),\\ d(x, fx, z), d(y, fx, y), d(z, fy, z)\big) \leq 0\end{aligned}$$

for all $x, y, z \in Y$.

Definition 2.11. Let Φ be the set of all continuous functions $\phi(t_1, t_2, \ldots, t_8) : \mathbb{R}^8_+ \to \mathbb{R}$ such that

(Φ_1) ϕ is nondecreasing in variable t_1;
(Φ_2) There exists $f_1 \in \Psi$ such that for all $u, v, w \geq 0$, $\phi(u, v, v, u, w, v, 0, 0) \leq 0$ implies that $u \leq f_1(v)$;
(Φ_3) There exists $f_2 \in \Psi$ such that for all $t, t_1, t_2, t_3 > 0$ $\phi(t, t, 0, 0, 0, t_1, t_2, t_3) \leq 0$ implies that $t \leq f_2(t_3)$.

Example 2.12. Let

$$\phi(t_1, t_2, \dots, t_8) = t_1 - a_1 t_2 - a_2 t_3 - a_3 t_4 - a_4 t_5 - a_5 t_6 - a_6 t_7 - a_7 t_8,$$

where $a_i \geq 0$ for $i = 1, 2, \dots, 7$ and $\sum_{i=1}^{7} a_i < 1$.

Example 2.13. Let

$$\phi(t_1, t_2, \dots, t_8) = t_1 - k \max\{t_2, \dots, t_8\},$$

where $k \in [0, 1)$.

Theorem 2.14. *Let (Y, d) be a complete tripled quasi-metric space and $g : Y \to Y$ be an β-implicit contractive mapping. Let that*

(i) *g is β-admissible;*
(ii) *There exists $x_0 \in Y$ such that $\beta(x_0, gx_0, g^2 x_0) \geq 1$ and $\beta(g^2 x_0, gx_0, x_0) \geq 1$;*
(iii) *g is continuous.*

Then there exists $\lambda \in Y$ such that $g\lambda = \lambda$.

Proof. By assumption (ii), exists $y_0 \in Y$ such that

$$\beta(y_0, gy_0, g^2 y_0) \geq 1 \text{ and } \beta(g^2 y_0, gy_0, y_0) \geq 1.$$

We define a sequence $\{y_n\}$ in Y by $y_{n+1} = gy_n = g^{n+1} y_0$ for all $n \geq 0$. Let that $x_{n_0} = x_{n_0+1}$ for some n_0. So the proof is complete, because,

$$u = x_{n_0} = x_{n_0+1} = gx_{n_0} = gu.$$

Consequently, throughout the proof, we assume that $y_n \neq y_{n+1}$ for any n. Since g is β-admissible and $\beta(y_0, y_1, y_2) = \beta(y_0, gy_0, g^2 y_0) \geq 1$, so observe that $\beta(gy_0, gy_1, gy_2) \geq 1$. By repeating the process above, we obtain that

$$\beta\left(y_n, y_{n+1}, y_{n+2}\right) \geq 1 \tag{2.2}$$

for any $n \in \mathbb{N} \cup \{0\}$. Now, consider the case where $\beta(g^2 y_0, gy_0, y_0) \geq 1$. By using the same way above, we get that

$$\beta\left(y_{n+2}, y_{n+1}, y_n\right) \geq 1 \tag{2.3}$$

for all $n \in \mathbb{N} \cup \{0\}$. By using (1.2) we get

$$\begin{aligned} \phi\big(\beta\left(y_{n-1}, y_n, y_{n+1}\right) d\left(gy_{n-1}, gy_n, gy_{n+1}\right), d\left(y_{n-1}, y_n, y_{n+1}\right), d\left(y_{n-1}, gy_{n-1}, g^2 y_{n-1}\right), \\ d\left(y_n, gy_n, g^2 y_n\right), d\left(y_{n+1}, gy_{n+1}, g^2 y_{n+1}\right), \\ d\left(y_{n-1}, gy_{n-1}, y_{n+1}\right), d\left(y_n, gy_{n-1}, y_n\right), \\ d\left(y_{n+1}, gy_n, y_{n+1}\right)\big) \leq 0, \end{aligned}$$

that is

$$\begin{aligned}\phi\big(\beta\left(y_{n-1},y_n,y_{n+1}\right)d\left(y_n,y_{n+1},y_{n+2}\right),&\,d\left(y_{n-1},y_n,y_{n+1}\right),d\left(y_{n-1},y_n,y_{n+1}\right),\\&d\left(y_n,y_{n+1},y_{n+2}\right),d\left(y_{n+1},y_{n+2},y_{n+3}\right),\\&d\left(y_{n-1},y_n,y_{n+1}\right),d\left(y_n,y_n,y_n\right),\\&d\left(y_{n+1},y_{n+1},y_{n+1}\right)\big)\leq 0,\end{aligned}$$

and

$$\begin{aligned}\phi\big(\beta\left(y_{n-1},y_n,y_{n+1}\right)d\left(y_n,y_{n+1},y_{n+2}\right),&\,d\left(y_{n-1},y_n,y_{n+1}\right),d\left(y_{n-1},y_n,y_{n+1}\right),\\&d\left(y_n,y_{n+1},y_{n+2}\right),d\left(y_{n+1},y_{n+2},y_{n+3}\right),\\&d\left(y_{n-1},y_n,y_{n+1}\right),0,0\big)\leq 0.\end{aligned}$$

By (2.2) and from (Φ_1) in the first variable, we have

$$\begin{aligned}\phi\big(d\left(y_n,y_{n+1},y_{n+2}\right),d\left(y_{n-1},y_n,y_{n+1}\right),d\left(y_{n-1},y_n,y_{n+1}\right),d\left(y_n,y_{n+1},y_{n+2}\right),\\d\left(y_{n+1},y_{n+2},y_{n+3}\right),d\left(y_{n-1},y_n,y_{n+1}\right),0,0\big)\leq 0.\end{aligned}$$

Due to (Φ_2), we obtain $d\left(y_n,y_{n+1},y_{n+2}\right)\leq f_1\left(d\left(y_{n-1},y_n,y_{n+1}\right)\right)$. If we go on like this, we get

$$d\left(y_n,y_{n+1},y_{n+2}\right)\leq f_1^n\left(d\left(y_0,y_1,y_2\right)\right). \tag{2.4}$$

We prove that $\{y_n\}$ is a Cauchy sequence in the tripled quasi-metric space (Y,d). Take $m>n$ from (d_2), we have

$$\begin{aligned}d\left(y_n,y_m,y_m\right)&\leq d\left(y_n,y_{n+1},y_{n+2}\right)+d\left(y_m,y_m,y_m\right)+d\left(y_m,y_{n+2},y_m\right)\\&\leq f_1^n\big(d\left(y_0,y_1,y_2\right)\big)+d\left(y_m,y_{n+2},y_m\right)\\&\leq f_1^n\big(d\left(y_0,y_1,y_2\right)\big)\\&\quad+\big[d\left(y_m,y_m,y_m\right)+d\left(y_{n+2},y_{n+3},y_{n+4}\right)+d\left(y_m,y_m,y_{n+3}\right)\big]\\&\leq f_1^n\big(d\left(y_0,y_1,y_2\right)\big)+f_1^{n+2}\big(d\left(y_0,y_1,y_2\right)\big)+d\left(y_m,y_m,y_m\right)\\&\quad+d\left(y_m,y_m,y_m\right)+d\left(y_{n+3},y_m,y_m\right)\\&=f_1^n\big(d\left(y_0,y_1,y_2\right)\big)+f_1^{n+2}\big(d\left(y_0,y_1,y_2\right)\big)+d\left(y_{n+3},y_m,y_m\right)\\&\leq f_1^n\big(d\left(y_0,y_1,y_2\right)\big)+f_1^{n+2}\big(d\left(y_0,y_1,y_2\right)\big)\\&\quad+\big[d\left(y_{n+3},y_{n+4},y_{n+5}\right)+d\left(y_m,y_m,y_m\right)+d\left(y_m,y_{n+4},y_m\right)\big]\end{aligned}$$

$$\begin{aligned}&\leq f_1^n\big(d(y_0,y_1,y_2)\big)+f_1^{n+2}\big(d(y_0,y_1,y_2)\big)\\&\quad+f_1^{n+3}\big(d(y_0,y_1,y_2)\big)+d(y_m,y_{n+4},y_m)\\&\leq f_1^n\big(d(y_0,y_1,y_2)\big)+f_1^{n+2}\big(d(y_0,y_1,y_2)\big)+f_1^{n+3}\big(d(y_0,y_1,y_2)\big)\\&\quad+d(y_m,y_m,y_m)+d(y_{n+4},y_{n+5},y_{n+6})+d(y_m,y_m,y_{n+5}).\end{aligned}$$

Let $n+p=m$, then we have

(2.5)

$$\begin{aligned}d(y_n,y_m,y_m)&\leq f_1^n\big(d(y_0,y_1,y_2)\big)+f_1^{n+2}\big(d(y_0,y_1,y_2)\big)+f_1^{n+3}\big(d(y_0,y_1,y_2)\big)\\&\quad+f_1^{n+4}\big(d(y_0,y_1,y_2)\big)+\ldots+f_1^{n+p}\big(d(y_0,y_1,y_2)\big)\\&\leq\sum_{k=n}^{\infty}f_1^k\big(d(y_0,y_1,y_2)\big)\end{aligned}$$

which implies that $d(y_n,y_m,y_m)\to 0$, when $n,m\to\infty$, but $f_1\in\Psi$. It follows that $\{y_n\}$ is a right-Cauchy sequence. By similarly way we can prove that, $\{y_n\}$ is a left-Cauchy sequence. There fore $\{y_n\}$ is a Cauchy sequence in (Y,d). Since, (Y,d) is tripled quasi-complete, then there exists a point λ in Y, such that $y_n\to\lambda$ as $n\to\infty$, that is

(2.6)

$$\lim_{n\to\infty}d(y_n,y,y)=\lim_{n\to\infty}d(y_n,y_n,y)=\lim_{n\to\infty}d(y,y,y_n)=\lim_{n\to\infty}d(y,y_n,y_n)=0.$$

We shall prove that $g\lambda=\lambda$. Since g is continuous, we verify

$$\lim_{n\to\infty}d(y_{n+1},y_{n+1},g\lambda)=\lim_{n\to\infty}d(gy_n,gy_n,g\lambda)=0,\tag{2.7}$$

and

$$\lim_{n\to\infty}d(g\lambda,y_{n+1},y_{n+1})=\lim_{n\to\infty}d(g\lambda,gy_n,gy_n)=0,$$

that is, $\lim_{n\to\infty}y_{n+1}=g\lambda$, by the uniqueness of limit, we conclude that $g\lambda=\lambda$, that is, λ is a fixed point of g. □

At present, we define a new condition.

(H) If $\{y_n\}$ is a sequence in Y, such that $\beta(y_n,y_{n+1},y_{n+2})\geq 1$ for any n and $y_n\to y\in Y$, until $n\to\infty$, then there exists a subsequence $\{y_{n(k)}\}$ of $\{y_n\}$ such that $\beta\left(y_{n(k)},y,y\right)\geq 1$ for all k.

Theorem 2.15. *Let (Y,d) be a tripled complete quasi-metric space and $g:Y\to Y$ be an β-implicit contractive mapping. Let that*

(i) *g is β-admissible;*

(ii) *there exists* $x_0 \in Y$ *such that* $\beta\left(x_0, gx_0, g^2x_0\right) \geq 1$ *and* $\beta\left(g^2x_0, gx_0, x_0\right) \geq 1$*;*

(iii) *(H) is verified.*

Thus there exists $a, \mu \in Y$ *such that* $g\mu = \mu$*.*

Proof. From the proof of Theorem 2.14, we hnow that the sequence $\{y_n\}$ defined by $y_{n+1} = gy_n$ for all $n \geq 0$ is Cauchy and converges to some $\mu \in Y$. From condition (iii), there exists a subsequence $\{y_{n(k)}\}$ of $\{y_n\}$ such that $\beta\left(y_{n(k)}, \mu, \mu\right) \geq 1$ for all k. We must show that $g\mu = \mu$. By (1.2), we have

$$\begin{aligned}
&F\big(\beta\left(y_{n(k)-1}, \mu, \mu\right) d\left(y_{n(k)-1}, g\mu, g\mu\right), d\left(y_{n(k)-1}, \mu, \mu\right),\\
&\quad d\left(y_{n(k)-1}, gy_{n(k)-1}, g^2y_{n(k)-1}\right), d\left(\mu, g\mu, g^2\mu\right), d\left(y_{n(k)-1}, gy_{n(k)-1}, \mu\right),\\
&\quad d\left(\mu, gy_{n(k)-1}, \mu\right), d\left(\mu, g\mu, \mu\right)\big) \leq 0.
\end{aligned}$$

Using (ϕ_1) and $\beta\left(y_{n(k)-1}, \mu, \mu\right) \geq 1$, we get

$$\begin{aligned}
\phi\big(d\left(y_{n(k)-1}, g\mu, g\mu\right), d\left(y_{n(k)-1}, \mu, \mu\right), d\left(y_{n(k)-1}, y_{n(k)}, y_{n(k)+1}\right),\\
d\left(\mu, g\mu, g^2\mu\right), d\left(\mu, g\mu, g^2\mu\right),\\
d\left(y_{n(k)-1}, y_{n(k)}, \mu\right), d\left(\mu, y_{n(k)}, \mu\right), d\left(\mu, g\mu, \mu\right)\big) \leq 0.
\end{aligned}$$

Letting $k \to \infty$ and by continuing of ϕ, we have

$$\begin{aligned}
\phi\big(d\left(\mu, g\mu, g\mu\right), d\left(\mu, \mu, \mu\right), d\left(\mu, \mu, \mu\right),\\
d\left(\mu, g\mu, g^2\mu\right), d\left(\mu, g\mu, g^2\mu\right),\\
d\left(\mu, \mu, \mu\right), d\left(\mu, \mu, \mu\right), d\left(\mu, g\mu, \mu\right)\big) \leq 0,
\end{aligned}$$

and $\phi(t_1, 0, 0, t_2, t_2, 0, 0, t_3) \leq 0$. By (ϕ_2), $t_1 \leq 0$, that is $d\left(\mu, g\mu, g\mu\right) \leq 0$, which implies $d\left(\mu, g\mu, g\mu\right) = 0$, that is, $\mu = g\mu$. □

For the uniqueness, we need additional condition.

(U) For all $x, y, z \in Fix(g)$, we have $\beta(x, y, z) \geq 1$ where $Fix(g)$ denotes the set of fixed points of g.

Theorem 2.16. *Adding condition (U) to the hypothesis of Theorem 2.14 (resp., Theorem 2.15), we obtain that* μ *is the unique fixed point of* g*.*

Proof. We obtain by contradiction, that is, there exist $u, v, w \in Y$ such that $u = gu$, $v = gv$ and $w = gu$ with $u \neq v$, $v \neq w$ and $u \neq w$. By (1.2) we get

$$\begin{aligned}\phi\big(\beta(u,v,w)\,d(gu,gv,gw), d(u,v,w), d(u,u,u),\\ d(v,v,v), d(w,w,w), d(u,u,w),\\ d(v,u,v), d(w,v,w)\big) \le 0,\end{aligned}$$

and

$$\begin{aligned}\phi\big(\beta(u,v,w)\,d(u,v,w), d(u,v,w), 0,0,0,\\ d(u,u,w), d(v,u,v), d(w,v,w)\big) \le 0.\end{aligned}$$

Due to the fact that $\beta(u,v,w) \ge 1$, so by (Φ_1), we argue

$$\phi\big(d(u,v,w), d(u,v,w), 0,0,0, d(u,u,w), d(v,w,v), d(w,v,w)\big) \le 0.$$

Since ϕ satisfies property (Φ_3), so there exists $h_2 \in \Psi$, such that

$$\begin{aligned} d(u,v,w) &\le h_2(d(w,v,w)) \\ &\le h_2^2(d(w,v,w)) \\ &\le \cdots \\ &\le h_2^n(d(w,v,w)). \end{aligned} \tag{2.8}$$

Since $\sum_{n=1}^{\infty} h_2^n(t) < \infty$, for each $t \in \mathbb{R}^+$, then as $n \to \infty$, we have

$$\lim_{n\to\infty} h_2^n(d(u,v,w)) = 0.$$

Thus $d(w,v,w) \le 0$, implies that $d(w,v,w) = 0$, that is, $u = v = w$ a contradiction. □

In the sequel we present the following corollaries consequences of Theorem 2.14 (resp. Theorem 2.15).

Corollary 2.17. *Let (Y,d) be a complete tripled quasi-metric space and $g : Y \to Y$ be such that*

$$\begin{aligned}\beta(x,y,z)\,d(gx,gy,gz) \le\ & a_1 d(x,y,z) + a_2 d\left(x,gx,g^2x\right) + a_3 d\left(y,gy,g^2y\right)\\ &+ a_4 d\left(z,gz,g^2z\right) + a_5 d(x,gx,z) + a_6 d(y,gx,y)\\ &+ a_7 d(z,gy,z),\end{aligned}$$

for all $x,y,z \in Y$, where $a_i \ge 0$ for $i = 1,2,\ldots,7$ and $\sum_{i=1}^{7} a_i < 1$. Let that

(i) *g is β-admissible;*
(ii) *there exists $y_0 \in Y$ such that $\beta\left(y_0, gy_0, g^2y_0\right) \ge 1$ and $\beta\left(g^2y_0, gy_0, y_0\right) \ge 1$;*
(iii) *g is continuous or (H) is verified.*

Then there exists $\lambda \in Y$ such that $g\lambda = \lambda$.

Proof. It suffices to put ϕ in Theorem 2.14 (resp. Theorem 2.15) as given in Example 2.12. □

Corollary 2.18. *Let (Y,d) be a tripled complete quasi-metric space and $g : Y \to Y$ be such that*

$$\beta(x,y,z)\,d(gx,gy,gz) \le k\max\Big\{d(x,y,z), \left(x,gx,g^2x\right), d\left(y,gy,g^2y\right),$$
$$d\left(z,gz,g^2z\right), d(x,gx,z), d(y,gx,y), d(z,gy,z)\Big\},$$

for any $x,y,z \in Y$, where $k \in [0,1)$. Let that

(i) *g is β-admissible;*
(ii) *there exists $x_0 \in Y$ such that $\beta\left(x_0, gx_0, g^2x_0\right) \ge 1$ and $\beta\left(g^2x_0, gx_0, x_0\right) \ge 1$;*
(iii) *g is continuous or (H) is verified.*

Then there exists a $\lambda \in Y$, such that $g\lambda = \lambda$.

Proof. It suffices to take ϕ in Theorem 2.14 (resp. Theorem 2.15) as given in Example 2.12, that is $\phi(t_1, t_2, \cdots, t_8) = t_1 - k\max\{t_2, \ldots, t_8\}$ where $k \in [0,1)$. □

Corollary 2.19. *Let (Y,d) be a complete tripled quasi-metric space and $g : (Y,d) \to (Y,d)$ be a given mapping. Let that*

$$\phi\Big(d(gx,gy,gz) \le d(x,y,z), \left(x,gx,g^2x\right), d\left(y,gy,g^2y\right),$$
$$d\left(z,gz,g^2z\right), d(x,gx,z), d(y,gx,y), d(z,gy,z)\Big) \le 0,$$

for all $x,y,z \in Y$, where $\phi \in \Gamma$. Then g has a unique fixed point.

Proof. It is enough to take $\beta(x,y,z) = 1$ for all $x,y,z \in Y$ in Theorem 2.15. Notice that the hypotheses (U) is satisfied, so we use Theorem 2.14. □

Corollary 2.20. *Let (Y,d) be a complete tripled quasi-metric space and $g : (Y,d) \to (Y,d)$ be a given mapping such that*

$$d(gx,gy,gz) \le k\max\left\{d(x,y,z), \left(x,gx,g^2x\right), d\left(y,gy,g^2y\right),\right.$$
$$\left.d\left(z,gz,g^2z\right), d(x,gx,z), d(y,gx,y), d(z,gy,z)\right\} \le 0,$$

for all $x,y,z \in X$, where $k \in [0,1)$. Then g has a unique fixed point.

Proof. It suffices to take ϕ as given in Example 2.12. Then we apply Corollary 2.17. □

Now we show the following example establishing Corollary 2.18.

Example 2.21. Let $Y=[0,\infty)$ endowed with the ripled quasi-metric $d(x,y,z)=|x|+|y|$, if $x\neq y$, $y\neq z$ and $x\neq z$, also $d(x,y,z)=0$ whenever $x=y=z$. It is obvious that (Y,d) is a complete tripled quasi-metric space. Let the mapping $S:Y\to Y$ defined by

$$Sx=\begin{cases} x^2-5x+6, & x>2,\\ \dfrac{x}{3}, & x\in[0,2].\end{cases}$$

At first we observe that the Banach contraction principle for $d_0(x,y,z)=|x-y|+|x-z|+|y-z|$ can not be used in this case because we have

$$d_0(S0,S4,S8)=d_0(0,2,30)=60>d_0(0,4,8)=16.$$

We define the mapping $\beta:Y\times Y\times Y\to[0,\infty)$ by $\beta(x,y,z)=1$, if $x,y,z\in[0,1]$, otherwise $\beta(x,y,z)=0$. If $x,y,z\in[0,1]$ and $x\neq y$, $y\neq z$ and $z\neq z$, we have

$$\begin{aligned}\beta(x,y,z)d(Sx,Sy,Sz)&=d(Sx,Sy,Sz)\\ &\leq|Sx|+|Sy|\\ &=\frac{x}{3}+\frac{y}{3}\\ &=\frac{1}{3}d(x,y,z)\\ &\leq k\max\big\{d(x,y,z),d(x,Sx,S^2x),d(y,Sy,S^2y),\\ &\quad d(z,Sz,S2^z),d(x,Sx,z),d(y,Sx,y),d(z,Sy,z)\big\},\end{aligned}$$

where $k=\frac{1}{3}$. Now, we shall prove that the hypotheses (H) is satisfied. Let $\{x_n\}$ be a sequence in Y, such that $\beta\left(x_n,x_{n+1},x_{n+2}\right)\geq 1$ for all n and $x_n\to x\in Y$ as $n\to\infty$. Then by definition of β, we get $(x_n,x_{n+1},x_{n+2})\in[0,1]\times[0,1]\times[0,1]$ for any n. Let that $x>1$, then $x_n\neq x$ for any n. Since $x_n\to x\in Y$, so $d(x,x,x_n)=2|x|\to 0$, which is a contradiction. Thus $x\in[0,1]$. We obtain that $(x_n,x,x)\in[0,1]\times[0,1]\times[0,1]$ for all n, that is $\beta(x_n,x,x)=1$, (H) is verified. Put $x_0=1$, we have $\beta(x_0,Sx_0,S^2x_0)=\beta(1,\frac{1}{3},\frac{1}{9})$ and $\beta(S^2x_0,Sx_0,x_0)=\beta(\frac{1}{9},\frac{1}{3},1)=1$. The mapping T is β-admissible. Let $x,y,z\in Y$ such that $\beta(x,y,z)\geq 1$, so $x,y,z\in[0,1]$. Then

$$\beta(Sx,Sy,Sz)=\beta\left(\frac{x}{3},\frac{y}{3},\frac{z}{3}\right)=1.$$

All hypotheses of Corollary 2.18 hold and the mapping S has a fixed point in Y. Note that in this case, we obtain two fixed points of S, that are $\lambda=0$ and $\lambda=3+\sqrt{3}$.

Definition 2.22. Let $(Y, \preceq)$ be a partially ordered set and $g : Y \to Y$ be a given mapping. We say that f is *nondecreasing* with respect to $\preceq$ if $x \preceq y$ then $gx \preceq gy$ for all $x, y, \in Y$.

Definition 2.23. Let $(Y, \preceq)$ be a partially ordered set. A sequence $\{x_n\} \subset Y$ is said to be *nondecreasing* with respect to $\preceq$, if $x_n \preceq x_{n+1}$ for all n.

Definition 2.24. Let $(Y, \preceq)$ be a partially ordered set and d be a tripled quasi-metric on Y. We say that $(Y, \preceq, d)$ is *regular* if for every nondecreasing sequence $\{x_n\} \subset Y$ such that $x_n \to x \in Y$ as $n \to \infty$, there exists a subsequences $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k.

We state the following result.

Theorem 2.25. *Let $(Y, \preceq)$ be a partially ordered set and d be a tripled quasi-metric on Y, such that (Y, d) is complete. Let $g : Y \to Y$ be a nondecreasing mapping with respect to $\preceq$. Let that there exists a function $\phi \in \Gamma$ such that*

$$\phi\big(d\left(gx, gy, gz\right), d\left(x, y, z\right), d\left(x, gx, g^2x\right), d\left(y, gy, g^2y\right), d\left(z, gz, g^2z\right), \\ d\left(x, gx, z\right), d\left(y, gx, y\right), d\left(z, gy, z\right)\big) \leq 0,$$

for all $x, y, z \in Y$ with $x \succeq y \succeq z$ or $x \preceq y \preceq z$. Let that the following conditions hold.

(i) *There exists $x_0 \in Y$ such that $x_0 \preceq gx_0 \preceq g^2x_0$ or $g_2x_0 \preceq gx_0 \preceq x_0$;*

(ii) *g is continuous or $(Y, \preceq, d)$ is regular.*

Then g has a fixed point. Moreover, if $Fix(g)$ is well-ordered, we have uniqueness of the fixed point.

Proof. Define the mapping $\beta : Y \times Y \times Y \to [0, \infty)$ by $\beta(x, y, z) = 1$, if $x \preceq y \preceq z$ or $z \preceq y \preceq x$, otherwise $\beta(x, y, z) = 0$. Obviously, g is an β-implicit contractive mapping, that is

$$\phi\big(\beta(x, y, z)d\left(gx, gy, gz\right), d\left(x, y, z\right), d\left(x, gx, g^2x\right), d\left(y, gy, g^2y\right), d\left(z, gz, g^2z\right), \\ d\left(x, gx, z\right), d\left(y, gx, y\right), d\left(z, gy, z\right)\big) \leq 0.$$

From condition (i) we have $\beta\left(x_0, gx_0, g^2x_0\right) \geq 1$ or $\beta\left(g^2x_0, gx_0, x_0\right) \geq 1$. Moreover, for all $x, y, z \in Y$, from the monotone property of g, we have $\beta(x, y, z) \geq 1$, then $x \succeq y \succeq z$ or $x \preceq y \preceq z$, so $gx \succeq gy \succeq gz$ or $gx \preceq gy \preceq gz$, hence $\beta(gx, gy, gz) \geq 1$. Thus g is β-admissible. Now, if g is continuous the existence of a fixed point follows from Theorem 2.14. Consider now that $(Y, \preceq, d)$ is regular. Let $\{x_n\}$ be a sequence in Y such that $(x_n, x_{n+1}, x_{n+2}) \geq 1$ for any n and $x_n \to x \in Y$ as $n \to \infty$. From the regularity hypotheses, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)} \preceq x$ for all k. This implies from the definition of β that $\beta(x_{n(k)}, x, x) \geq 1$ for all k. In this case, the existence of a fixed point follows from Theorem 2.15. To show the uniqueness. Let $x, y \in Y$, $(x \preceq y$ or $y \preceq x)$. By hypotheses, there exists $z \in Y$ such that $x \preceq y \preceq z$ or $z \preceq y \preceq x$, which implies $\beta(x, y, z) \geq 1$ or $\beta(z, y, x) \geq 1$. This, we deduce the uniqueness of the fixed point by Theorem 2.16. □

3. Application

Now, we provide an application on the research of a solution of an integral equation. For instance by Corollary 2.20 , we will prove the existence of a solution of the following integral equation, where $E : [0, 1] \times \mathbb{R} \to [0, \infty)$ is a continuous function

$$\begin{aligned} x(t) &= \int_0^1 G(s,t)E(s,x(s))\,ds, \\ y(t) &= \int_0^1 G(s,t)E(s,y(s))\,ds, \\ z(t) &= \int_0^1 G(s,t)E(s,z(s))\,ds. \end{aligned} \tag{3.1}$$

Let $Y = C([0, 1], [0, \infty))$ be the set of nonnegative continuous functions defined on $[0, 1]$, Take the tripled quasi-metric $d : Y \times Y \times Y \to [0, \infty)$ defined by $d(x, y, z) = \|x\|_\infty + \|y\|_\infty$, if $x \neq y$, $x \neq z$ and $y \neq z$, $d(x, y, z) = 0$ whenever $x = y = z$, where $\|x\|_\infty = \sup_{t\in[0,1]} x(t)$. It is easy to show that (Y, d) is a complete tripled quasi-metric. Now, we define the mapping $S : Y \to Y$ as follows

$$Sx(t) = \int_0^1 G(s,t)E(s,x(s))\,ds.$$

Theorem 3.1. *Let the following condition hold. Assume that there exist $\mu_1, \mu_2, \mu_3 \in [0, 1)$ such that $\mu_1 + \mu_2 + \mu_3 < 1$ and for any $s \in [0, 1]$ and $x, y, z \in Y$, $(x \neq y$, $x \neq z$ and $y \neq z)$, we have $E(s, x(s)) \leq \mu_1\|x\|_\infty$, $E(s, y(s)) \leq \mu_2\|y\|_\infty$, and $E(s, z(s)) \leq \mu_3\|z\|_\infty$, where*

$$\begin{aligned} \int_0^1 G(s,t)E(s,x(s))\,ds &\neq \int_0^1 G(s,t)E(s,y(s))\,ds, \\ \int_0^1 G(s,t)E(s,x(s))\,ds &\neq \int_0^1 G(s,t)E(s,z(s))\,ds, \end{aligned}$$

$$\int_0^1 G(s,t)E(s,y(s))\,ds \neq \int_0^1 G(s,t)E(s,z(s))\,ds.$$

Then the integral equation (3.1) *has a unique solution* $x \in C([0,1],[0,\infty))$.

Proof. For all $x, y, z \in Y$, ($x \neq y$, $x \neq z$ and $y \neq z$), we have

$$\|Sx\|_\infty = \sup_{t\in[0,1]} \int_0^1 G(s,t)E(s,x(s))\,ds \leq \frac{1}{8}\mu_1\|x\|_\infty,$$
$$\|Sy\|_\infty = \sup_{t\in[0,1]} \int_0^1 G(s,t)E(s,y(s))\,ds \leq \frac{1}{8}\mu_2\|y\|_\infty,$$
$$\|Sz\|_\infty = \sup_{t\in[0,1]} \int_0^1 G(s,t)E(s,z(s))\,ds \leq \frac{1}{8}\mu_3\|z\|_\infty.$$

It follows that for all $x, y, z \in Y$, ($x \neq y$, $x \neq z$ and $y \neq z$), we obtain

$$\begin{aligned} d(Sx,Sy,Sz) &= \|Sx\|_\infty + \|Sy\|_\infty \\ &\leq \frac{1}{8}\mu_1\|x\|_\infty + \frac{1}{8}\mu_2\|y\|_\infty \\ &\leq \frac{1}{8}\left(\|x\|_\infty + \|y\|_\infty\right) \\ &= \frac{1}{8}d(x,y,z). \end{aligned}$$

Therefore,

$$\begin{aligned} d(Sx,Sy,Sz) \leq \frac{1}{8}\max\{&d(x,y,z), d(x,Sx,S^2x), d(y,Sy,S^2y), d(z,Sz,S^2z), \\ &d(x,Sx,z), d(y,Sx,y), d(z,Sy,z)\}. \end{aligned} \tag{3.2}$$

On the other hand, obviously (3.2) holds. Therefore all condition of Corollary 2.20 are satisfied and so S has a unique fixed point. □

Availability of supporting data

Not applicable.

Competing interests

The authors declare that they has no competing interests.

Funding

Not applicable.

Authors contributions

All authors contributed equally and significantly in this manuscript, and they read and approved the final manuscript.

References

1. R.P. Agarwal, E. Karapnar & A.F. Roldán-López-de-Hierro: Fixed point theorems in quasimetric spaces and applications to multidimensional fixed point theorems on G-metric spaces. J. Nonlinear Convex Anal. **16** (2015), no. 9, 1787-1816. https://doi.org/10.1007/978-3-319-24082-4-11
2. M.U. Ali, T. Kamran & E. Karapnar: On (α-ψ-η)-contractive multivalued mappings. Fixed Point Theory Appl. **2014**, Article ID 7, 8, pp. 2014. https://doi.org/10.1186/1687-1812-2014-7
3. A. Aliouche & V. Popa: General common fixed point theorems for occasionally weakly compatible hybrid mappings and applications. Novi Sad. J. Math. **39** (2009), no. 1, 89-109.
4. M.A. Alghamdi & E. Karapnar: G-β-ψ-contractive-type mappings and related fixed point theorems. J. Inequal. Appl. **2013**, Article ID 70, 16, pp. 2013. https://doi.org/10.1186/1029-z42x-2013-70
5. M.A. Alghamdi, E. Karapnar: G-β-ψ-contractive type mappings in G-metric spaces. Fixed Point Theory Appl. **2013**, Article ID 123, 17, pp. 2013. https://doi.org/10.1187-1812-2013-123
6. V. Berinde: Approximating fixed points of implicit almost contractions. Hacet. J. Math. Stat. (2012) **41**, no. 1, 93-102.
7. V. Berinde & F. Vetro: Common fixed points of mappings satisfying implicit contractive conditions. Fixed Point Theory Appl. **2012**, Article ID 105, 8, pp. 2012. https://doi.org/10.1186/1687-1812-2012-105
8. L.B. Ćirić: A generalization of Banach's contraction principle. Proc. Amer. Math. Soc. (1974) **45**, no. 2, 267-273. https://doi.org/10.1090/s0002-9939-1974-0356011-2
9. R.C. Dimri & G. Prasad: Coincidence theorems for comparable generalized nonlinear contractions in ordered partial metric spaces. Comm. Korean Math. Soc. **32** (2017), no. 2, 375-387. https://doi.org/10.4134/CKMS.c160127
10. M. Imdad, S. Kumar & M.S. Khan: Remarks on some fixed point theorems satisfying implicit relations. Rad. Math. **11** (2012), no. 1, 135-143. https://doi.org/MR1971330-135-143
11. M. Jleli, E. Karapnar & B. Samet: Best proximity points for generalized alpha-psi-proximal contractive type mappings. J. Appl. Math. **2013**, Article ID 534127, 10, pp. 2013. https://doi.org/10.1155/2013/534127
12. M. Jleli, E. Karapnar & B. Samet: Fixed point results for α-ψ_λ-contractions on gauge spaces and applications. Abstr. Appl. Anal. **2013**, Article ID 730825, 7, pp. 2013. https://doi.org/10.1155/2013/730825

13. M. Jleli & B. Samet: Remarks on G metric spaces and fixed point theorems. Fixed Point Theory Appl. **2012**, Article ID 210, 7, pp. 2012. `https://doi.org/10.1186/1687-1812-2012-210`
14. E. Karapinar & B. Samet: Generalized α-ψ-contractive type mappings and related fixed point theorems with applications. Abstr. Appl. Anal. **2012**, Article ID 793486, 17, pp. 2012. `https://doi.org/10.1155/2012-793486`
15. E. Karapinar: Fixed point theory for cyclic weak ϕ-contraction. Appl. Math. Lett. **24** (2011), 822-825. `https://doi.org/10.1016/j.aml.2010.12.016`
16. V. La Rosa & P. Vetro: Common fixed points for α-ψ-φ-contractions in generalized metric spaces. Nonlinear Anal. Model. Control **19**, no. 1, 43-54. `https://doi.org/10.1186/1029-242x-2014-439`
17. B. Mohammadi, Sh. Rezapour & N. Shahzad: Some results on fixed points of α-ψ-Ciric generalized multifunctions. Fixed Point Theory Appl. **2013**, Article ID 24, 10, pp. 2013. `https://doi.org/10.1186/1687-1812-2013-24`
18. Z. Mustafa & B. Sims: A new approach to generalized metric spaces. J. Nonlinear Convex Anal. **7** (2006), 289-297. `https://ds.doi.org/10.22075/ijnaa.2022.27489.3619`
19. M. Pacurar & I.A. Rus: Fixed point theory for cyclic φ-contractions. Nonlinear Anal., Theory Methods Appl., Ser. A **72** (2010), no. 34, 1181-1187. `https://doi.org/10.1016/j.na.2009.08.002`
20. V. Popa: Fixed point theorems for implicit contractive mappings. Stud. Cercet. Ştiint., Ser. Mat. Univ. Bacău **7** (1997), 129-133.
21. V. Popa: Some fixed point theorems for compatible mappings satisfying an implicit relation. Demonstr. Math. **32** (1999), 157-163. `https://doi.org/10.1515/dema-199-0117`
22. V. Popa: A general fixed point theorem for four weakly compatible mappings satisfying an implicit relation. Filomat **19** (2005), 45-51. `https://doi.org/10.2298/FIL0519045p`
23. V. Popa & A.M. Patriciu: A general fixed point theorem for mappings satisfying an ϕ-implicit relation in complete G-metric spaces. Gazi Univ. J. Science **25** (2012), no. 2, 403-408. `https://doi.org/10.35219/ann-ugal-mah-phys-mee.2018.2004`
24. V. Popa & A.M. Patriciu: A General fixed point theorem for pairs of weakly compatible mappings in G-metric spaces. J. Nonlinear Sci. App. **5** (2012), no. 2, 151-160. `https://dx.doi.org/10.22436/jnsa.005.02.08`
25. G. Prasad & H́useyin Iik: On solution of boundary value problem via week contractions. J. Fun. **2022**, Article ID 6799205. pp. 2022. `https://doi.org/10.1155/2022/6799205`
26. G. Prasad: Coincidence points of relational Ψcontractions and an application. Afr. Mat. **32** (2021), 1475-1490. `https://doi.org/10.1007/s13370-021-00913-6`

27. G. Prasad & R.C. Dimri: Fixed point theorems via comparable mappings in ordered metric spaces. J. Anal. **27** (2019), no. 4, 1139-1150. https://doi.org/10.1007/s41478-019-00165-5
28. S. Reich & A.J. Zaslawski: Well-posedness of fixed point problems. Far East J. Math. Sci., Spec. Vol., Part **III** (2001), 393-401. https://doi.org/10.1007/s11784-018-0538-1
29. B. Samet, C. Vetro & P. Vetro: Fixed point theorems for α-ψ-contractive type mappings. Nonlinear Anal., Theory Methods Appl., Ser. A **75** (2012), no. 4, 2154-2165. https://doi.org/10.1016/j.na.2011.100014

Professor: Department of Mathematics, Faculty of Science, Bu-Ali Sina University, Hamedan 65178, Iran
Email address: gh_khalilzadeh@yahoo.com