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COEFFICIENT INEQUALITIES FOR A UNIFIED CLASS OF BOUNDED TURNING FUNCTIONS ASSOCIATED WITH COSINE HYPERBOLIC FUNCTION

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Abstract. The aim of this paper is to study a new and unified class $\mathcal{R}^{\alpha}_{Cosh}$ of analytic functions associated with cosine hyperbolic function in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$. Some interesting properties of this class such as initial coefficient bounds, Fekete-Szegö inequality, second Hankel determinant, Zalcman inequality and third Hankel determinant have been established. Furthermore, these results have also been studied for two-fold and three-fold symmetric functions.

1. Introduction

Let the class of functions f which are analytic in the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$, is denoted by $\mathcal{A}$ and is defined as

$$\mathcal{A} = \left\{ f : f(z) = z + \sum_{k=2}^{\infty} a_k z^k, z \in E \right\}.$$

The subclass of $\mathcal{A}$ which consists of univalent functions in E, is denoted by $\mathcal{S}$. In the theory of univalent functions, the most famous result is Bieberbach's conjecture which was established by L. Bieberbach [6] in 1916. It states that, if $f \in \mathcal{S}$ is a univalent function, then $|a_n| \leq n$, $n = 2, 3, ...$ This result remained as a challenge for the mathematicians for a long time. Finally, L. De-Branges [9] proved this conjecture in 1985. During the course of proving this conjecture, various coefficients inequalities were come into existence which helped in defining certain new subclasses of analytic functions. Here we mention only those classes which are relevant to our work.

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The class of starlike functions is denoted by $\mathcal{S}^*$ and is defined as

$$\mathcal{S}^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in E \right\}.$$

Reade [26] introduced the class $\mathcal{CS}^*$ of close-to-star functions which is given by

$$\mathcal{CS}^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{f(z)}{g(z)}\right) > 0, g \in \mathcal{S}^*, z \in E \right\}.$$

For $g(z) = z$, MacGregor [19] studied the following subclass of close-to-star functions:

$$\mathcal{R}' = \left\{ f : f \in \mathcal{A}, Re\left(\frac{f(z)}{z}\right) > 0, z \in E \right\}.$$

Also, MacGregor [18] established the class $\mathcal{R}$ of bounded turning functions which is defined as

$$\mathcal{R} = \left\{ f : f \in \mathcal{A}, Re(f'(z)) > 0, z \in E \right\}.$$

As a generalization, Murugusundaramoorthi and Magesh [20] studied the class $\mathcal{R}(\alpha)$ defined as

$$\mathcal{R}(\alpha) = \left\{ f : f \in \mathcal{A}, Re\left((1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right) > 0, 0 \le \alpha \le 1, z \in E \right\}.$$

Clearly $\mathcal{R}(\alpha)$ is the unification of the classes $\mathcal{R}'$ and $\mathcal{R}$ as $\mathcal{R}(0) \equiv \mathcal{R}'$ and $\mathcal{R}(1) \equiv \mathcal{R}$.

Let f and g be two analytic functions in E. Then f is said to be subordinate to g (denoted as $f \prec g$) if there exists a function w with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. Moreover, if g is univalent in E, then the subordination leads to $f(0) = g(0)$ and $f(E) \subset g(E)$.

Using the concept of subordination, various subclasses of $\mathcal{S}$ were studied by several authors by associating to different superordinating functions $\phi(z)$. Some of the recently studied classes are mentined below:

(i) Janowski [11] studied the class $\mathcal{S}^*(A,B)$ for $\phi(z) = \frac{1+Az}{1+Bz}$.

(ii) For $\phi(z) = 1 + sinz$, Cho et al. [8] studied the class $\mathcal{S}^*_{sin}$.

(iii) Taking $\phi(z) = e^z$, Arif et al. [3] studied the class $\mathcal{S}^*_e$.

(iv) Chosing $\phi(z) = 1 + z - \frac{z^3}{3}$, Wani and Swaminathan [37] studied the class $\mathcal{S}_N$.

(v) Sokol and Stankiewicz [34] studied the class $\mathcal{S}^*_L$ associated with $\phi(z) = \sqrt{1+z}$.

(vi) For $\phi(z) = z + \sqrt{1+z^2}$, Raina and Sokol [23] studied the class $\mathcal{S}_C$.

(vii) Considering $\phi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, Sharma et al. [29] studied the class $\mathcal{S}^*_C$.

(viii) For $\phi(z) = 1 + sinh^{-1}z$, Arora and Kumar [4] studied the class $\mathcal{S}^*_p$.

(ix) For $\phi(z) = \frac{2}{1+e^{-z}}$, Goel and Kumar [10] studied the class $\mathcal{S}^*_{SG}$.

(x) Alotaibi et al. [1] studied the class $\mathcal{S}^*_{Cosh}$ related to $\phi(z) = coshz$.

Following the recent trend, now we introduce a unified and generalized subclass of analytic functions associated with the superordinating function $cosh\sqrt{z}$.

Definition 1.1. A function $f \in \mathcal{A}$ is said to be *in the class* $\mathcal{R}^{\alpha}_{Cosh}$ $(0 \leq \alpha \leq 1)$ if it satisfies the condition

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) \prec cosh\sqrt{z}.$$

For $\alpha = 0$ and $\alpha = 1$, the class $\mathcal{R}^{\alpha}_{Cosh}$ reduces to the classes $\mathcal{R}'_{Cosh}$ and $\mathcal{R}_{Cosh}$, respectively.

For $q \geq 1$ and $n \geq 1$, Pommerenke [21] defined the q^{th} Hankel determinant $H_q(n)$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}.$$

For different values of q and n, the Hankel determinant $H_q(n)$ reduces to various coefficient functionals. For $q = 2$ and $n = 1$, it redues to $H_2(1) = a_3 - a_2^2$, which is the well known Fekete-Szegö functional. For $q = 2$ and $n = 2$, $H_q(n)$ takes the form of $H_2(2) = a_2a_4 - a_3^2$, which is known as Hankel determinant of second order and for $q = 3$ and $n = 1$, it agrees with $H_3(1)$, which is the Hankel determinant of third order.

The functional $J_{n,m}(f) = a_na_m - a_{m+n-1}$, $n, m \in \mathbb{N} - \{1\}$, is known as generalized Zalcman functional and was introduced by Ma [17]. For $n = 2, m = 3$, it reduces to $J_{2,3}(f) = a_2a_3 - a_4$. The upper bound for the functional $J_{2,3}(f)$ was computed by various authors over different subclasses of analytic functions. It plays very important role in establishing the bounds for the third Hankel determinant.

Now a days, the estimation of Hankel determinants for various subclasses of analytic functions is a topic of great interest. Janteng et al. [12] investigated the second Hankel determinant for the classes of starlike functions, convex functions and the class of functions with bounded boundary rotation. After that second order Hankel determinant was extensively studied by various authors for different classes. Babalola [5] was the first researcher who successfully obtained the upper bound of third Hankel determinant for some fundamental classes. Further a few researchers including Shanmugam et al. [28], Bucur et al. [7], Altinkaya and Yalcin [2] and recently Singh and Singh [30], Singh et al. [31, 32, 33], Sun et al. [35], Riaz et al. [27], Raza et al. [25], Sunthrayuth et al. [36] and many more have been actively engaged in the study of third Hankel determinant for various subclasses of analytic functions.

In this paper, we establish the upper bounds of the third Hankel determinant for the class $\mathcal{R}^{\alpha}_{Cosh}$. Moreover the bounds of $H_3(1)$ are studied for the two-fold and three-fold symmetric functions. Various known results follow as consequences.

2. Preliminary Lemmas

Let $\mathcal{P}$ denote the class of analytic functions p given by

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k,$$

whose real parts are positive in E.

In order to prove our main results, the following lemmas have been used:

Lemma 2.1 ([13, 29])**.** If $p \in \mathcal{P}$, then

$$|p_k| \le 2, k \in \mathbb{N},$$
$$\left|p_2 - \frac{p_1^2}{2}\right| \le 2 - \frac{|p_1|^2}{2},$$
$$|p_{i+j} - \mu p_i p_j| \le 2, 0 \le \mu \le 1,$$

and for complex number ρ, we have

$$|p_2 - \rho p_1^2| \le 2max\{1, |2\rho - 1|\}.$$

Lemma 2.2 ([3])**.** Let $p \in \mathcal{P}$, then

$$|Jp_1^3 - Kp_1p_2 + Lp_3| \le 2|J| + 2|K - 2J| + 2|J - K + L|,$$

where J, K, L are real numbers.

In particular, it is proved in [22] that

$$|p_1^3 - 2p_1p_2 + p_3| \le 2.$$

Lemma 2.3 ([15, 16])**.** If $p \in \mathcal{P}$, then

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$
$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for $|x| \le 1$ and $|z| \le 1$.

Lemma 2.4 ([24])**.** Let m, n, l and r satisfy the inequalities $0 < m < 1$, $0 < r < 1$ and

$$8r(1-r)\left[(mn-2l)^2+(m(r+m)-n)^2\right]+m(1-m)(n-2rm)^2 \leq 4m^2(1-m)^2r(1-r).$$

If $p \in \mathcal{P}$, then

$$\left| lp_1^4 + rp_2^2 + 2mp_1p_3 - \frac{3}{2}np_1^2p_2 - p_4 \right| \leq 2.$$

3. Initial Coefficient Bounds

Theorem 3.1. If $f \in \mathcal{R}^{\alpha}_{Cosh}$, then

$$|a_2| \leq \frac{1}{2(1+\alpha)}, \tag{1}$$

$$|a_3| \leq \frac{1}{2(1+2\alpha)}, \tag{2}$$

$$|a_4| \leq \frac{1}{2(1+3\alpha)}, \tag{3}$$

and

$$|a_5| \leq \frac{1}{2(1+4\alpha)}. \tag{4}$$

The results are sharp.

Proof. Since $f \in \mathcal{R}^{\alpha}_{Cosh}$, so using the concept of subordination in Definition 1.1, we have

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = cosh\sqrt{w(z)}. \tag{5}$$

Taking $p(z) = \frac{1+w(z)}{1-w(z)} = 1 + p_1z + p_2z^2 + p_3z^3 + ...$, which implies $w(z) = \frac{p(z)-1}{p(z)+1}$.
For $f \in \mathcal{A}$, we have

(6)

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = 1 + (1+\alpha)a_2z + (1+2\alpha)a_3z^2 + (1+3\alpha)a_4z^3 + (1+4\alpha)a_5z^4 + ...$$

Also

$$\begin{aligned} cosh\sqrt{w(z)} = 1 + \frac{1}{4}p_1z + \left(\frac{p_2}{4} - \frac{11p_1^2}{96}\right)z^2 + \left(\frac{301p_1^3}{5760} - \frac{11p_1p_2}{48} + \frac{p_3}{4}\right)z^3 \\ + \left(-\frac{91p_1^4}{3840} + \frac{301p_1^2p_2}{1920} - \frac{11p_3p_1}{48} - \frac{11p_2^2}{96} + \frac{p_4}{4}\right)z^4 + ... \end{aligned} \tag{7}$$

Using (6) and (7) in (5), it yields

$1+(1+\alpha)a_2z+(1+2\alpha)a_3z^2+(1+3\alpha)a_4z^3+(1+4\alpha)a_5z^4+...$

$= 1+\frac{1}{4}p_1z+\left(\frac{p_2}{4}-\frac{11p_1^2}{96}\right)z^2+\left(\frac{301p_1^3}{5760}-\frac{11p_1p_2}{48}+\frac{p_3}{4}\right)z^3$

$$+\left(-\frac{91p_1^4}{3840}+\frac{301p_1^2p_2}{1920}-\frac{11p_3p_1}{48}-\frac{11p_2^2}{96}+\frac{p_4}{4}\right)z^4+... \tag{8}$$

Comparing the coefficients of z, z^2, z^3 and z^4 in (8), we obtain

$$a_2=\frac{1}{4(1+\alpha)}p_1, \tag{9}$$

$$a_3=\frac{1}{4(1+2\alpha)}\left[p_2-\frac{11p_1^2}{24}\right], \tag{10}$$

$$a_4=\frac{1}{4(1+3\alpha)}\left[\frac{301}{1440}p_1^3-\frac{11}{12}p_1p_2+p_3\right], \tag{11}$$

and

$$a_5=\frac{1}{4(1+4\alpha)}\left[-\frac{91p_1^4}{960}-\frac{11p_2^2}{24}-\frac{11}{12}p_3p_1+\frac{301p_1^2p_2}{480}+p_4\right]. \tag{12}$$

Using first inequality of Lemma 2.1 in (9), the result (1) is obvious.
From (10), we have

$$|a_3|=\frac{1}{4(1+2\alpha)}\left|p_2-\frac{11}{24}p_1^2\right|. \tag{13}$$

Using fourth inequality of Lemma 2.1 in (13), the result (2) can be easily obtained.
(11) can be written as

$$|a_4|=\frac{1}{4(1+3\alpha)}\left|\frac{301}{1440}p_1^3-\frac{11}{12}p_1p_2+p_3\right|. \tag{14}$$

Using Lemma 2.2 in (14), the result (3) is obvious.
Further, using Lemma 2.4 in (12), the result (4) is obvious. □

Equality in the results (1), (2), (3) and (4) is attained for the functions f_1, f_2, f_3 and f_4, respectively defined as

$$(1-\alpha)\frac{f_1(z)}{z}+\alpha f_1'(z)=cosh\sqrt{z}, \tag{15}$$

$$(1-\alpha)\frac{f_2(z)}{z}+\alpha f_2'(z)=cosh\sqrt{z^2}, \tag{16}$$

$$(1-\alpha)\frac{f_3(z)}{z}+\alpha f_3'(z)=cosh\sqrt{z^3}, \tag{17}$$

$$(1-\alpha)\frac{f_4(z)}{z}+\alpha f_4'(z)=cosh\sqrt{z^4}. \tag{18}$$

On putting $\alpha = 0$, Theorem 3.1 yields the following result:

Corollary 3.1. If $f \in \mathcal{R}'_{Cosh}$, then

$$|a_2| \le \frac{1}{2}, |a_3| \le \frac{1}{2}, |a_4| \le \frac{1}{2}, |a_5| \le \frac{1}{2}.$$

For $\alpha = 1$, Theorem 3.1 gives the following result due to Khan et al. [14]:

Corollary 3.2. If $f \in \mathcal{R}_{Cosh}$, then

$$|a_2| \le \frac{1}{4}, |a_3| \le \frac{1}{6}, |a_4| \le \frac{1}{8}, |a_5| \le \frac{1}{10}.$$

4. Fekete-Szegö Inequality

Theorem 4.1. If $f \in \mathcal{R}^{\alpha}_{Cosh}$ and μ is any complex number, then

$$|a_3-\mu a_2^2| \le \frac{1}{2(1+2\alpha)} max\left\{1, \frac{|-(1+\alpha)^2+6(1+2\alpha)\mu|}{12(1+\alpha)^2}\right\}. \tag{19}$$

The bound is sharp.

Proof. From (9) and (10), we obtain

$$|a_3-\mu a_2^2| = \frac{1}{4(1+2\alpha)}\left|p_2-\frac{11(1+\alpha)^2+6(1+2\alpha)\mu}{24(1+\alpha)^2}p_1^2\right|. \tag{20}$$

Using fourth inequality of Lemma 2.1, (20) can be expressed as

$$|a_3-\mu a_2^2| \le \frac{1}{2(1+2\alpha)} max\left\{1, \frac{|-(1+\alpha)^2+6(1+2\alpha)\mu|}{12(1+\alpha)^2}\right\}.$$

Equality in the result (19) is attained for the function f_2 defined in (16). □

Substituting for $\alpha = 0$, Theorem 4.1 yields the following result:

Corollary 4.1. If $f \in \mathcal{R}'_{Cosh}$, then

$$|a_3-\mu a_2^2| \le \frac{1}{2} max\left\{1, \frac{|6\mu-1|}{12}\right\}.$$

Putting $\alpha = 1$, Theorem 4.1 yields the following result due to Khan et al. [14]:

Corollary 4.2. If $f \in \mathcal{R}_{Cosh}$, then

$$|a_3-\mu a_2^2| \le \frac{1}{6} max\left\{1, \frac{|9\mu-2|}{24}\right\}.$$

For $\mu = 1$, Theorem 4.1 yields the following result:

Corollary 4.3. If $f \in \mathcal{R}^{\alpha}_{Cosh}$, then

$$|a_3 - a_2^2| \leq \frac{1}{2(1+2\alpha)}.$$

For $\alpha = 0$, Corollary 4.3 yields the following result:

Corollary 4.4. If $f \in \mathcal{R}'_{Cosh}$, then

$$|a_3 - a_2^2| \leq \frac{1}{2}.$$

For $\alpha = 1$, Corollary 4.3 yields the following result due to Khan et al. [14]:

Corollary 4.5. If $f \in \mathcal{R}_{Cosh}$, then

$$|a_3 - a_2^2| \leq \frac{1}{6}.$$

5. Zalcman Inequality

Theorem 5.1. If $f \in \mathcal{R}^{\alpha}_{Cosh}$, then

$$|a_2 a_3 - a_4| \leq \frac{1}{2(1+3\alpha)}. \tag{21}$$

The estimate is sharp.

Proof. Using (9), (10), (11) and after simplification, we obtain

$$\begin{aligned} |a_2 a_3 - a_4| = & \frac{1}{5760(1+\alpha)(1+2\alpha)(1+3\alpha)} \\ & \times \big|(466 + 1398\alpha + 602\alpha^2)p_1^3 - (1680 + 5040\alpha + 2640\alpha^2)p_1 p_2 \\ & + (1440 + 4320\alpha + 2880\alpha^2)p_3\big|. \end{aligned} \tag{22}$$

Applying Lemma 2.2 in (22), (21) can be easily obtained. Equality in (21) is attained for the function f_3 defined in (17). □

Corollary 5.1. If $f \in \mathcal{R}^{'}_{Cosh}$, then

$$|a_2 a_3 - a_4| \leq \frac{1}{2}.$$

On putting $\alpha = 1$ in Theorem 5.1, we can obtain the following result due to Khan et al. [14]:

Corollary 5.2. If $f \in \mathcal{R}_{Cosh}$, then

$$|a_2 a_3 - a_4| \leq \frac{1}{8}.$$

6. Second Hankel Determinant

Theorem 6.1. If $f \in \mathcal{R}^{\alpha}_{Cosh}$, then

$$|a_2 a_4 - a_3^2| \leq \frac{1}{4(1+2\alpha)^2}. \tag{23}$$

Result is sharp.

Proof. Using (9), (10) and (11), we have

$$\begin{aligned} |a_2 a_4 - a_3^2| = {} & \frac{1}{46080(1+\alpha)(1+2\alpha)^2(1+3\alpha)} \\ & \times \big|2880(1+2\alpha)^2 p_1 p_3 - 2640\alpha^2 p_1^2 p_2 + (-3 - 12\alpha + 593\alpha^2) p_1^4 \\ & - 2880(1+4\alpha+3\alpha^2) p_2^2\big|. \end{aligned}$$

Substituting for p_2 and p_3 from Lemma 2.3 and letting $p_1 = p$, we get

$$\begin{aligned} |a_2 a_4 - a_3^2| = {} & \frac{1}{46080(1+\alpha)(1+2\alpha)^2(1+3\alpha)} \\ & \big| - (7\alpha^2 + 12\alpha + 3)p^4 + 120\alpha^2 p^2 (4 - p^2) x \\ & - 720(1+2\alpha)^2 p^2 (4-p^2) x^2 - 720(1+4\alpha+3\alpha^2)(4-p^2)^2 x^2 \\ & + 1440(1+2\alpha)^2 p (4-p^2)(1-|x|^2) z\big|. \end{aligned}$$

Since $|p| = |p_1| \leq 2$, we may assume that $p \in [0,2]$. Using the triangle inequality and $|z| \leq 1$ with $|x| = t \in [0,1]$, we obtain

$$\begin{aligned} |a_2 a_4 - a_3^2| \leq {} & \frac{1}{46080(1+\alpha)(1+2\alpha)^2(1+3\alpha)} \\ & \times \big[(7\alpha^2 + 12\alpha + 3)p^4 + 120\alpha^2 p^2 (4-p^2) t + 720(1+2\alpha)^2 p^2 (4-p^2) t^2 \\ & + 720(1+4\alpha+3\alpha^2)(4-p^2)^2 t^2 + 1440(1+2\alpha)^2 p (4-p^2) \\ & - 1440(1+2\alpha)^2 p (4-p^2) t^2\big] = F(p,t). \end{aligned}$$

$$\begin{aligned} \frac{\partial F}{\partial t} = {} & \frac{(4-p^2)}{384(1+\alpha)(1+2\alpha)^2(1+3\alpha)} \big[\alpha^2 p^2 + 12(1+2\alpha)^2 p^2 t \\ & + 12(1+4\alpha+3\alpha^2)(4-p^2)t - 24(1+\alpha)^2 p t\big]. \end{aligned}$$

Clearly $\frac{\partial F}{\partial t} \geq 0$ and so $F(p,t)$ is an increasing function of t.

Therefore,

$$max\{F(p,t)\} = F(p,1) = \frac{1}{46080(1+\alpha)(1+2\alpha)^2(1+3\alpha)}$$
$$\times\big[(7\alpha^2+12\alpha+3)p^4 + 120\alpha^2p^2(4-p^2) + 720(1+2\alpha)^2p^2(4-p^2)$$
$$+ 720(1+\alpha)(1+3\alpha)(4-p^2)^2 + 1440(1+2\alpha)^2p(4-p^2)$$
$$- 1440(1+2\alpha)^2p(4-p^2)\big] = H(p).$$

$H'(p) = 0$ gives $p = 0$. Also $H''(p) < 0$ for $p = 0$.
Therefore $max\{H(p)\} = H(0) = \dfrac{1}{4(1+2\alpha)^2}$, which proves (23).
Equality in (23) is attained for the function f_2 defined in (16). □

Putting $\alpha = 0$, Theorem 6.1 gives the following result:

Corollary 6.1. If $f \in \mathcal{R}'_{Cosh}$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{4}.$$

Substituting for $\alpha = 1$ in Theorem 6.1, the following result due to Khan et al. [14], is obvious:

Corollary 6.2. If $f \in \mathcal{R}_{Cosh}$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{36}.$$

7. Third Order Hankel Determinant $H_3(1)$

On expanding, the third Hankel determinant can be expressed as

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2),$$

and after applying the triangle inequality, it yields

(24) $$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_2a_3 - a_4| + |a_5||a_3 - a_2^2|.$$

Theorem 7.1. If $f \in \mathcal{R}^{\alpha}_{Cosh}$, then

(25) $$|H_3(1)| \leq \frac{5 + 50\alpha + 179\alpha^2 + 268\alpha^3 + 136\alpha^4}{8(1+2\alpha)^3(1+3\alpha)^2(1+4\alpha)}.$$

Proof. By using (3), (4), (5), (21), (23) and Corollary 4.3 in (24), the result (25) can be easily obtained. □

For $\alpha = 0$, Theorem 7.1 yields the following result:

Corollary 7.1. If $f \in \mathcal{R}'_{Cosh}$, then

$$|H_3(1)| \leq \frac{5}{8}.$$

For $\alpha = 1$, Theorem 7.1 yields the following result due to Khan et al. [14]:

Corollary 7.2. If $f \in \mathcal{R}_{Cosh}$, then

$$|H_3(1)| \leq \frac{319}{8640}.$$

8. Bounds of $|H_3(1)|$ for Two-fold and Three-fold Symmetric Functions

A function f is said to be *n-fold symmetric function* if it satisfies the following condition:

$$f(\xi z) = \xi f(z)$$

where $\xi = e^{\frac{2\pi i}{n}}$ and $z \in E$.
By $S^{(n)}$, we denote the set of all n-fold symmetric functions which belong to the class S.
The n-fold univalent function have the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{k=1}^{\infty} a_{nk+1} z^{nk+1}. \tag{26}$$

An analytic function f of the form (26) belongs to the family $\mathcal{R}^{\alpha(n)}_{Cosh}$ if and only if

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = cosh\sqrt{\left(\frac{p(z)-1}{p(z)+1}\right)}, p \in \mathcal{P}^{(n)},$$

where

$$\mathcal{P}^{(n)} = \left\{ p \in \mathcal{P} : p(z) = 1 + \sum_{k=1}^{\infty} p_{nk} z^{nk}, z \in E \right\}. \tag{27}$$

Theorem 8.1. If $f \in \mathcal{R}^{\alpha(2)}_{Cosh}$, then

$$|H_3(1)| \leq \frac{1}{4(1+2\alpha)(1+4\alpha)}. \tag{28}$$

Proof. If $f \in \mathcal{R}^{\alpha(2)}_{Cosh}$, then there exists a function $p \in \mathcal{P}^{(2)}$ such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = cosh\sqrt{\left(\frac{p(z)-1}{p(z)+1}\right)}. \tag{29}$$

Using (26) and (27) for $n = 2$, (29) yields

$$a_3 = \frac{1}{4(1+2\alpha)} p_2, \tag{30}$$

$$a_5 = \frac{1}{4(1+4\alpha)} \left(p_4 - \frac{11}{24} p_2^2 \right). \tag{31}$$

Also

$$H_3(1) = a_3 a_5 - a_3^3. \tag{32}$$

Using (30) and (31) in (32), it yields

$$H_3(1) = \frac{1}{16(1+2\alpha)(1+4\alpha)} p_2 \left[p_4 - \frac{11(1+2\alpha)^2 + 6(1+4\alpha)}{24(1+2\alpha)^2} p_2^2 \right]. \tag{33}$$

Taking modulus and using third inequality of Lemma 2.1 in (33), we can easily get the result (28). □

Putting $\alpha = 0$, the following result can be easily obtained from Theorem 8.1:

Corollary 8.1. If $f \in \mathcal{R}'^{(2)}_{Cosh}$, then

$$|H_3(1)| \leq \frac{1}{4}.$$

For $\alpha = 1$, Theorem 8.1 agrees with the following result due to Khan et al. [14].

Corollary 8.2. If $f \in \mathcal{R}^{(2)}_{Cosh}$, then

$$|H_3(1)| \leq \frac{1}{60}.$$

Theorem 8.2. If $f \in \mathcal{R}^{\alpha(3)}_{Cosh}$, then

$$|H_3(1)| \leq \frac{1}{4(1+3\alpha)^2}. \tag{34}$$

The bound is sharp.

Proof. If $f \in \mathcal{R}^{\alpha(3)}_{Cosh}$, so there exists a function $p \in \mathcal{P}^{(3)}$ such that

$$(1-\alpha)\frac{f(z)}{z} + \alpha f'(z) = Cosh\sqrt{\left(\frac{p(z)-1}{p(z)+1}\right)}. \tag{35}$$

Using (26) and (27) for $n = 3$ in (35), it gives

$$a_4 = \frac{1}{4(1+3\alpha)} p_3. \tag{36}$$

Also

$$H_3(1) = -a_4^2. \tag{37}$$

Using (36) in (37), it yields

$$H_3(1) = -\frac{1}{16(1+3\alpha)^2}p_3^2. \tag{38}$$

Taking modulus and using first inequality of Lemma 2.1, (34) can be easily obtained from (38).
Equality in (34) is attained for the function f_3 defined in (17). □

Putting $\alpha = 0$ in Theorem 8.2, it gives the following result:

Corollary 8.3. If $f \in \mathcal{R}'^{(3)}_{Cosh}$, then

$$|H_3(1)| \leq \frac{1}{4}.$$

For $\alpha = 1$, Theorem 8.2 yields the following result due to Khan et al. [14].

Corollary 8.4 If $f \in \mathcal{R}^{(3)}_{Cosh}$, then

$$|H_3(1)| \leq \frac{1}{64}.$$

9. Conclusion

In this paper, we have introduced a new and unified class of analytic functions by subordinating to cosine hyperbolic function. We established various coefficient inequalities for this class and also extended the results to two-fold and three-fold symmetric functions. Certain known results follow as the consequences of the results of this paper. Till now, most of the work done on third Hankel determinant is based on some of the standard classes, but here we have investigated the sharp bounds for the third Hankel determinant for a generalized class. So this paper will pave the way for other researchers to investigate some more generalized classes of analytic functions.

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