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# UNSTEADY FLOW OF BINGHAM FLUID IN A TWO DIMENSIONAL ELASTIC DOMAIN

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Abstract. The main goal of this work is to study an initial boundary value problem relating to the unsteady flow of a rigid, viscoplastic, and incompressible Bingham fluid in an elastic bounded domain of $\mathbb{R}^2$. By using the approximation sequences of the Faedo-Galerkin method together with the regularization techniques, we obtain the results of the existence and uniqueness of local solutions.

## 1. Introduction

This work is devoted to the study of the existence of the solution to some boundary-value problems relating to the unsteady flow of a rigid, viscoplastic and incompressible Bingham fluid in an elastic domain. A Bingham fluid is a non-Newtonian viscoplastic fluid that possesses a yield stress that must be outstripped before the fluid will flow. In many geological and industry materials, Bingham fluids are used as a general mathematical basis of flow in drilling engineering, including in the handling of slurries, granular suspensions, etc. Bingham fluid is named after Eugene C. Bingham, who declared its mathematical explanation. It is well known that Bingham first investigated the Bingham-plastic constitutive equation [2, 10], which is the most often used model for a viscoplastic material. That regions of rigid-solid and inelastic-fluid behavior are separated in terms of von Mises' yield condition. The constitutive equation relating the deviators $\tau$ stress and rate-of-strain $\dot{\gamma}$ tensors is given as [3, 10]

$$\tau = \left(\eta_0 + \frac{\tau_\eta}{\dot{\gamma}}\right)\dot{\gamma},\ \left(\tau > \tau_\eta\right), \tag{1.1}$$

$$\dot{\gamma} = 0 \qquad \left(\tau \leq \tau_\eta\right), \tag{1.2}$$

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where $\tau$ and $\gamma$ are the second invariants of the tensors, defined as

$$\tau = \left(\frac{1}{2}\tau, \tau\right)^{\frac{1}{2}}, \quad \dot{\gamma} = \left(\frac{1}{2}\dot{\gamma}, \dot{\gamma}\right)^{\frac{1}{2}}, \tag{1.3}$$

and $\eta_0$ and $\tau_\eta$ are called the plastic viscosity and the yield stress, respectively. Equations (1.1)-(1.2) define two distinct regions of flow. In the first the invariant $\tau$ exceeds the yield stress and the material flows with a non-Newtonian viscosity function defined as $\eta(\dot{\gamma}) := \eta_0 + \frac{\tau_\eta}{\dot{\gamma}}$. In the second the stress is less than the yield value and the material behaves as a rigid solid. The fluid and solid regions are separated by a distinct yield surface. Recently, an exploration of the laws for plastic flow has been studied by Bingham [1]. Darby and Melson [6] formulated an empirical expression to guess the friction loss factor for the drift of a Bingham fluid. Bird et al. [4] analyzed the rheology and flow phenomena of viscoplastic materials. Convective heat transfer for Bingham plastic inside a circular pipe and the numerical approach for hydro-dynamically emerging flow and the simultaneously emerging flow were studied by Min et al. [9]. Liu and Mei [8] considered the slow spread of a Bingham fluid sheet on an inclined plane. For Bingham fluids, the Couette-Poiseuille flow between two porous plates, considering slip conditions, was investigated by Chen and Zhu [5]. Sreekala and Kesavareddy [13] mentioned the Hall impacts on magnetohydrodynamics (MHD) Bingham plastic flow over a porous medium, including uniform suction and injection. For Bingham fluids, the MHD flow for an unsteady case considering Hall currents was described by Parvin et al. [11]. Rees and Bassom [12] considered Bingham fluids over a porous medium following a rapid modification of surface heat flux. The paper is organized as follows. In Section 2 we present the mechanical problem of the two-dimensional unsteady flow of a rigid, viscoplastic, and incompressible Bingham which occupy a domain $\Omega_0$ in an elastic domain having $\Omega_1 \subset \mathbb{R}^2$ as thickness. We introduce some notation and preliminaries. Moreover, we derive the variational formulation of the problem. In Section 3, we present the mathematical formulation and we prove an existence and uniqueness results.

## 2. Problem statement

Let $\Omega_0$, $\Omega_1 \subset \mathbb{R}^2$, be two bounded domains with a smooth boundary $\Gamma_0 = \partial\Omega_0$ and $\Gamma_1 = \partial\Omega_1 = \Gamma_{11} \cup \Gamma_{12}$. The boundary $\Gamma_1$ of $\Omega_1$ is assumed to be regular and is divided by two closed and disjoint parts $\Gamma_{11}$, $\Gamma_{12}$, here, $\Gamma_0 \neq \emptyset$ and $\Gamma_1 \neq \emptyset$, we consider a mathematical problem modeling the unsteady flow of a rigid, viscoplastic and incompressible Bingham fluid in an elastic domain in which $\Omega_0$ represents the domain occupied by the fluid and $\Omega_1$ is the thickness of the elastic domain where $\Omega_0$, $\Omega_1 \subset \mathbb{R}^2$. Let $\eta$ be the outward normal to $\Gamma_0$ oriented toward the outside of $\Omega_0$, and the inside of $\Omega_1$. We pose $Q_i = \Omega_i \times (0, T)$, $(i = 0, 1)$, where $T$ is a finite positive real. The density of volume forces $f$ act in $Q = \Omega \times (0, T)$ as $f_{\backslash\Omega} = \left(f_{0\backslash\Omega_0}, f_{1\backslash\Omega_1}\right)$, in which

$Div\,(\sigma) = (\sigma_{ij,j})$ denotes the divergence operator for stress tensor $\sigma = (\sigma_{i,j})$, $i, j = 1, 2$. $\varepsilon\,(u) = (\varepsilon_{ij}\,(u)) : \varepsilon_{ij}\,(u) = \frac{1}{2}\,(u_{i,j} + u_{j,i})$ denotes the linearized strain tensor in $\Gamma_1 \times (0, T)$. We denote by $S_2$ the space of symmetric tensors on $\mathbb{R}^2$. We define the inner product and the Euclidean norm on $\mathbb{R}^2$ and $S_2$, respectively, by

$$u.\upsilon = u_i \upsilon_j \text{ for all } u, \upsilon \in \mathbb{R}^2 \text{ and } \sigma.\tau = \sigma_{ij}\tau_{jj} \quad \text{for all } \sigma, \tau \in S_2.$$

$$|u| = (u.u)^{\frac{1}{2}} \text{ for all } u \in \mathbb{R}^2 \text{ and } \sigma = (\sigma.\sigma)^{\frac{1}{2}} \text{ for all } \sigma \in S_2.$$

Here and below, the indices $i$ and $j$ run from 1 to 2 and the summation convention over repeated indices is used, $\mu$, $g$ are the coefficients which characterize the model of Bingham fluid and which represent, respectively: consistency (viscosity) and the threshold of plasticity. We denote by $\widetilde{\sigma}_0$ the deviator of $\sigma_0 = (\sigma_{0ij})$ given by

$$\widetilde{\sigma_0} = \left(\widetilde{\sigma_{0i,j}}\right), \quad \widetilde{\sigma_{0i,j}} = \sigma_{0i,j} - \frac{tra\sigma_0}{2}\delta_{i,j},$$

where $\delta = (\delta_{i,j})$ is the identity tensor and $tra$ the trace operator.

The steady flow of transmission problem to Bingham fluid in the domain $\Omega_i$ is given by the following mechanical problem.

**Problem 2.1.** *Find the velocity field* $u = (u_1, u_2) : \Omega_0 \times (0, T) \to \mathbb{R}^2$, *the stress field* $\sigma_0 = \left(\sigma_{0_{i,j}}\right)_{1 \le i,j \le 2} : \Omega_0 \times (0, T) \to S_2$ *and* $w = (w_1, w_2) : \Omega_1 \times (0, T) \to \mathbb{R}^2$ *such that*

$$\frac{Du}{Dt} = \left(\frac{\partial u}{\partial t} + u\nabla u\right) = Div\,(\sigma_0) + f_0 \ \text{on } Q_0, \tag{2.1}$$

$$\operatorname{div} u = \sum_{i=1}^{2} D_i u_i = 0 \ \text{on } Q_0, \tag{2.2}$$

$$\frac{\partial^2 w}{\partial t^2} = \operatorname{div}\sigma_1 + f_1 \ \text{on } Q_1, \tag{2.3}$$

$$\begin{cases} \widetilde{\sigma}_0 = 2\mu\varepsilon\,(u) + g\frac{\varepsilon(u)}{|\varepsilon(u)|} & \text{if } |\varepsilon\,(u)| \neq 0 \\ |\widetilde{\sigma_0}| \le g & \text{if } |\varepsilon\,(u)| = 0 \end{cases} \quad \text{on } Q_0, \tag{2.4}$$

$$\sigma_{1i,j} = \sum_{k,l=1}^{2} A_{i,j,k,l}.\varepsilon^{k,l} \quad \text{on } Q_1, \tag{2.5}$$

*where* $A$ *is the forth order Cauchy tensor and* $\varepsilon^{k,l}$ *is the strain tensor on* $\Omega_1$, *under the boundary and transmission conditions*

$$u = \frac{\partial w}{\partial t} = w' \ \text{in } \Gamma_0 \times (0, T)\,, \tag{2.6}$$

$$\sigma_1 - \sigma_0 = \frac{1}{2}\left(\sum_{i=1}^{2} u_i \cos(\eta_i)\right) \times u_i \ \text{in } \Gamma_0 \times (0, T)\,, \tag{2.7}$$

$$\sigma_1 \eta = g \ \text{in } \Gamma_{11} \times (0, T)\,, \tag{2.8}$$

(2.9) $$w = 0 \text{ in } \Gamma_{12} \times (0,T),$$

*and the initial conditions*

(2.10) $$u(0) = u_0 \text{ on } \Omega_0 \times (0,T),$$

(2.11) $$\begin{cases} w(0) = w_0 \text{ on } \Omega_1 \times (0,T), \\ w'(0) = w_1 \text{ on } \Omega_1 \times (0,T). \end{cases}$$

*Here, the flow is given by the equation* (2.1)-(2.3). *Equation* (2.4) *and* (2.5) *represents the constitutive law of Bingham fluid.* (2.6)-(2.9) *represents the acquisition condition on the boundary* $\Gamma_{11}$, $\Gamma_{12}$ *and* $\Gamma_0$. *The equation* (2.10) *and* (2.11) *represents the initial conditions.*

For the rest of this paper, we will denote by $c$ or $C$ possibly different positive constants depending only on the data of the problem.

### 2.1. Preliminary

In this section, we present some material that we shall use to present our results. Let

(2.12) $$V_1 = \left\{ v : v \in (H^1(\Omega_0))^2, \ \operatorname{div} v = 0 \text{ on } \Omega_0 \right\}.$$

Since $\Gamma_0$ has nonempty interior and $\Omega$ is a regular domain, we denote by

$$V_2 = \left\{ v : v \in (H^1(\Omega_1))^2, \ v = 0 \text{ in } \Gamma_{12} \right\},$$

the closed subspace of $(H^1(\Omega_0))^2$ and $(H^1(\Omega_1))^2$ with the norm equivalent to the usual norm in $(H^1(\Omega_0))^2$ and $(H^1(\Omega_1))^2$, respectively. Also we define

$$V = \{(v_1, v_2) \in V_1 \times V_2 : v_1 = v_2 \text{ in } \Gamma_0\}.$$

Before presenting the main result of this paper, we introduce some basic notations that will be used in the rest of the paper

$$a_{\Omega_0} : V_1 \times V_1 \to \mathbb{R}, \ a_{\Omega_0}(u,v) = 2\mu \int_{\Omega_0} \varepsilon(u)\, \varepsilon(v)\, dx,$$

$$b_1 : V_1 \times V_1 \times V_1 \to \mathbb{R}, \ b_1(u,v,w) = \sum_{i,j=1}^{2} \int_{\Omega_0} u_i \frac{\partial v_j}{\partial x_i} w_j dx,$$

$$a_{\Omega_1} : V_2 \times V_2 \to \mathbb{R}, \ a_{\Omega_1}(u,\varphi) = \sum_{i,j=1}^{2} \int_{\Omega_1} \frac{\partial u_j}{\partial x_i} \frac{\partial \varphi_j}{\partial x_i} dx,$$

and

$$J : V_1 \to \mathbb{R}, \ J(v) = g \int_{\Omega_0} |\varepsilon(v)|\, dx.$$

For studying problem (2.1), we introduce the new variable

$$\Phi = w' = \frac{\partial w}{\partial t}.$$

Thus, we have a

$$\Phi' = w'' = \frac{\partial^2 w}{\partial t^2}. \tag{2.13}$$

Under hypotheses (2.6)-(2.11), then by multiplying the equation (2.1) by $(v-u) \in H^1(\Omega_0)$ and (2.3) by $(\varphi - \Phi) \in H^1(\Omega_1)$ integrating the results on $\Omega_0$ and on $\Omega_1$, respectively, using the closure of $D(\Omega)$ in $V_i$ $(i = 1, 2)$, and the Green's formula, it is easy to verify that the problem (2.1)-(2.5) is equivalent to the following variational problem.

**Lemma 2.2** (Variational Inequality)**.** *Assume that* (2.6)-(2.11) *hold. Then* (2.1)-(2.5) *is equivalent to the following variational problem*:

*Find* $(u, \Phi) \in L^2(0, T, V_1) \cap L^\infty\left(0, T, L^2(\Omega_0)^2\right) \times L^\infty\left(0, T; L^2(\Omega_1)^2\right)$ *such that*

$$\begin{cases} (u', v-u)_{\Omega_0} + b_1(u, u, v-u) + a_{\Omega_0}(u, v-u) \\ \quad + g\int_{\Omega_0} (|\varepsilon(u)| - |\varepsilon(v)|)\, dx + \int_{\Gamma_0} (\sigma_1 - \sigma_0)(v-u)\, d\Gamma \\ \quad + (\Phi', \varphi - \Phi)_{\Omega_1} + (\sigma_1, \varepsilon(\varphi - \Phi))_{\Omega_1} + \int_{\Gamma_{11}} \sigma_1 \eta_1 (\varphi - \Phi)\, d\Gamma \\ \geq (f_0, v-u)_{\Omega_0} + (f_1, \varphi - \Phi)_{\Omega_1}, \forall (v, \varphi) \in V_1 \times V_2, \\ v = \varphi \ \text{in } \Gamma_0, \\ u(0) = u_0 \ \text{on } \Omega_0 \times (0, T), \\ \Phi(0) = w'(0) = w_1 \ \text{on } \Omega_1 \times (0, T), \\ u = \Phi = \frac{\partial w}{\partial t} = w' \ \text{in } \Gamma_0 \times (0, T). \end{cases} \tag{2.14}$$

### 2.2. Some results

In this subsection, let us first recall the following lemmas, which will be constantly used in the sequel.

**Lemma 2.3.**
(1) *The bilinear function $a_{\Omega_0}(u, v)$ is a continuous and coercive on $V_1 \times V_1$,*
(2) *The bilinear function $a_{\Omega_1}(u, v)$ is a continuous and coercive on $V_2 \times V_2$,*
(3) *The trilinear function $b_1(u, v, w)$ is continuous on $V_1 \times V_1 \times V_1$,*
(4) *The function $J$ is continuous on $V_1$.*

**Lemma 2.4.** $\forall (u, v, w) \in V_1 \times V_1 \times V_1$ *we have*

(1) $|b_1(u, v, w)| \leq C \|u\|^{\frac{1}{2}} |u|^{\frac{1}{2}} \|w\|^{\frac{1}{2}} |w|^{\frac{1}{2}} \|v\|_1$,
(2) $b_1(u, v, w) = -b_1(u, w, v) + \sum_{i,j=1}^{2} \int_{\Gamma_0} u_i w_j v_j \eta_i \ d\Gamma$.

**Lemma 2.5** (see [7, Lemme 9.1.])**.** *If $v \in H^1(\Omega)$ $\left(\Omega \subset \mathbb{R}^2\right)$, $\Gamma$ the boundary of $\Omega$, we have $v_{|\Gamma} \in L^3(\Gamma)$ and*

$$\left\| v_{|\Gamma} \right\|_{L^3(\Gamma)} \leq C \|v\|^{\frac{2}{3}}_{H^1(\Omega)} \|v\|^{\frac{1}{3}}_{L^2(\Omega)}, \ \forall v \in H^1(\Omega).$$

**2.2.1.** *Regularization.* The following result is necessary to regularize the variational problem.

For all $\varepsilon > 0$, let $J_\varepsilon$ be a functional family of regularized function $J$ defined by

$$J_\varepsilon(v) = \frac{g}{1+\varepsilon}\int_{\Omega_0} |\varepsilon(v)|^{1+\varepsilon} dx, \tag{2.15}$$

therefore the following results hold.

**Lemma 2.6.** *Hypothesis of regularization* $(H)$;

1. *For all* $\varepsilon > 0$, *the function* $v \to J_\varepsilon(v)$ *of* $V \to \mathbb{R} \cup \{+\infty\}$ *is differential on* $V$.
2. *For all* $v \in L^2(0,T;V)$, *we have*

$$\int_0^T J_\varepsilon(v(t))\, dt \to \int_0^T J(v(t))\, dt \text{ for } \varepsilon \to 0. \tag{2.16}$$

3. *There exists a constant* $c > 0$ *such that*

$$\|DJ_\varepsilon(v)\|_{V_1'} \le c\|v\|_{V_1}, \ \forall v \in V_1. \tag{2.17}$$

4. *If* $(v_\varepsilon, v_\varepsilon') \to (v, v')$ *weakly in* $L^2(0,T,V_1) \times L^2(0,T,V_1')$ *we have*

$$\lim_{\varepsilon \to 0} \inf \int_0^T D(J_\varepsilon(v_\varepsilon))\, dt \ge \int_0^T D(J(v))\, dt,$$

*which the space* $V_1'$ *is the topological dual of the space* $V_1$.

5. *For all* $\varepsilon > 0$, *the function* $v \to J_\varepsilon(v)$ *is convex and differentiable on* $V$. *Where* $DJ_\varepsilon(v)$ *represents the Gateaux differential of* $J_\varepsilon(v)$.

Our object now is to regularize the problem (2.14), let's replace in (2.14) $v$ by $u + \lambda(v-u)$ with $\lambda > 0$, divided the first inequality of (2.14) by $\lambda$ and make $\lambda \to 0$, we will get the following inequality

$$\begin{aligned}
&(u', v-u)_{\Omega_0} + b_1(u,u,v-u) + a_{\Omega_0}(u, v-u) \\
&+ \lim_{\lambda \to 0} \frac{J(u+\lambda(v-u)) - J(u)}{\lambda} + \int_{\Gamma_0} (\sigma_1 - \sigma_0)(v-u)\, d\Gamma \\
&+ (\Phi', \varphi - \Phi)_{\Omega_1} + (\sigma_1, \varepsilon(\varphi - \Phi))_{\Omega_1} + \int_{\Gamma_{11}} g(\varphi - \Phi)\, d\Gamma \\
&\ge (f_0, v-u)_{\Omega_0} + (f_1, \varphi - \Phi)_{\Omega_1}, \ \forall (v,\varphi) \in V_1 \times V_2,
\end{aligned}$$

and we reproach the functional $J$ by the functional family $(J_\varepsilon)_{\varepsilon>0}$ as in (2.15), then, problem (2.14) gotten the following approached elliptic inequality

$$\begin{aligned}
&(u_\varepsilon', v-u_\varepsilon)_{\Omega_0} + b_1(u_\varepsilon, u_\varepsilon, v-u_\varepsilon) + a_{\Omega_0}(u_\varepsilon, v-u_\varepsilon) \\
&+ (DJ_\varepsilon(u_\varepsilon), v-u_\varepsilon)_{V_1' \times V_1} + \int_{\Gamma_0} (\sigma_1 - \sigma_0)(v-u_\varepsilon)\, d\Gamma \\
&+ (\Phi_\varepsilon', \varphi - \Phi_\varepsilon)_{\Omega_1} + (\sigma_1, \varepsilon(\varphi - \Phi_\varepsilon))_{\Omega_1} + \int_{\Gamma_{11}} g(\varphi - \Phi_\varepsilon)\, d\Gamma
\end{aligned}$$

$$\geq (f_0, v - u_\varepsilon)_{\Omega_0} + (f_1, \varphi - \Phi_\varepsilon)_{\Omega_1}, \ \forall (v, \varphi) \in V_1 \times V_2, \tag{2.18}$$

where $(u_\varepsilon, \Phi_\varepsilon)$ is the approached solution of the regularized inequality (2.18). Consequently, the steady-state flow of Bingham fluid can be also described by the following system (2.19)-(2.23).

$$\begin{aligned}
&(u_\varepsilon', v - u_\varepsilon)_{\Omega_0} + b_1 (u_\varepsilon, u_\varepsilon, v - u_\varepsilon) + a_{\Omega_0} (u_\varepsilon, v - u_\varepsilon) \\
&+ (DJ_\varepsilon (u_\varepsilon), v - u_\varepsilon)_{V_1' \times V_1} + \int_{\Gamma_0} (\sigma_1 - \sigma_0)(v - u_\varepsilon) \, d\Gamma \\
&+ (\Phi_\varepsilon', \varphi - \Phi_\varepsilon)_{\Omega_1} + (\sigma_1, \varepsilon (\varphi - \Phi_\varepsilon))_{\Omega_1} + \int_{\Gamma_{11}} g (\varphi - \Phi_\varepsilon) \, d\Gamma \\
&= (f_0, v - u_\varepsilon)_{\Omega_0} + (f_1, \varphi - \Phi_\varepsilon)_{\Omega_1}, \ \forall (v, \varphi) \in V_1 \times V_2,
\end{aligned} \tag{2.19}$$

$$v = \varphi \text{ in } \Gamma_0, \tag{2.20}$$

$$u_\varepsilon (0) = u_{\varepsilon 0} \text{ on } \Omega_0 \times (0, T), \tag{2.21}$$

$$\Phi_\varepsilon (0) = w' (0) = w_1 \text{ on } \Omega_1 \times (0, T), \tag{2.22}$$

$$u_\varepsilon = \Phi_\varepsilon = \frac{\partial w}{\partial t} = w' \text{ in } \Gamma_0 \times (0, T) \tag{2.23}$$

with the following assumptions

$$f_{0|\Omega_0} \in L^2(0, T, V_1'), \ f_{1|\Omega_1} \in L^2(0, T, V_2'), \tag{2.24}$$

$$\begin{aligned}
u_\varepsilon (0) = u_{0\varepsilon} \in \left(L^2 (\Omega_0)\right)^2, \ w_0 \in L^2(0, T, V_2) \cap L^\infty(0, T, V_2), \\
\Phi_\varepsilon (0) = w' (0) = w_1 \in \left(L^2 (\Omega_1)\right)^2.
\end{aligned} \tag{2.25}$$

## 3. Existence and uniqueness

In this section, we shall establish a local existence and uniqueness result of the mathematical problem (2.1)-(2.11)

### 3.1. Existence

Our main goal in this section is, by basing on Faedo–Galerkin approximations and compactness argument, to show the local existence and uniqueness of a weak solution

**Theorem 3.1.** *Assume that the hypotheses* (2.24)-(2.25) *hold. Then, the variational problem* (2.19)-(2.23) *admits at least one solution* $(u, \Phi)$ *for any finished* $T$*, satisfies*

$$u \in L^2(0, T; V_1) \cap L^\infty \left(0, T, \left(L^2 (\Omega_0)^2\right)\right), \ u' \in L^2(0, T, V_1^{'}),$$

$$\Phi \in L^2(0, T; V_2) \cap L^\infty \left(0, T; \left(L^2 (\Omega_1)^2\right)\right), \ \Phi' \in L^2(0, T, V_2^{'}).$$

*Proof.* We construct approximations of the solutions $(u, \Phi)$ by the Faedo–Galerkin method as follows. Let $\{y_k\}_{1 \leq k \leq m}$ and $\{z_k\}_{1 \leq k \leq m}$ be orthonormal bases of $V_1$ and $V_2$, respectively. For each $\varepsilon \in (0,1)$ and $k \in \mathbb{N}$, according to (2.19), we consider

$$u_{m\varepsilon}(t) = \sum_{k=1}^{m} \lambda_{\varepsilon m,k}(t) y_k,\ x \in \Omega_0,\ t \in (0,T)\,,$$
$$\Phi_{m\varepsilon}(t) = \sum_{k=1}^{m} \mu_{\varepsilon m,k}(t) z_k,\ x \in \Omega_1,\ t \in (0,T)\,,$$

local solutions to the approximate perturbed problem

$$(3.1)\quad \begin{cases} (u'_{m\varepsilon}, y_k - u_{m\varepsilon})_{\Omega_0} + b_1\left(u_{m\varepsilon}, u_{m\varepsilon}, y_k - u_{m\varepsilon}\right) + a_{\Omega_0}\left(u_{m\varepsilon}, y_k - u_{m\varepsilon}\right) \\ \quad + \left(DJ_\varepsilon\left(u_{m\varepsilon}\right), y_k - u_{m\varepsilon}\right)_{V_1' \times V_1} + \displaystyle\int_{\Gamma_0} \left(\sigma_1 - \sigma_0\right)\left(y_k - u_{m\varepsilon}\right) d\Gamma \\ \quad + (\Phi'_{m\varepsilon}, z_k - \Phi_{m\varepsilon})_{\Omega_1} + (\sigma_1, \varepsilon\left(z_k - \Phi_{m\varepsilon}\right))_{\Omega_1} + \displaystyle\int_{\Gamma_{11}} g\left(z_k - \Phi_{m\varepsilon}\right) d\Gamma \\ = (f_0, y_k - u_{m\varepsilon})_{\Omega_0} + (f_1, z_k - \Phi_{m\varepsilon})_{\Omega_1}, \\ u_{m\varepsilon}(0) = u_{0\varepsilon} = \displaystyle\sum_{k=1}^{m} \left(u_0, y_k\right) y_k, \\ \Phi_{m\varepsilon}(0) = \Phi_{0\varepsilon} = \displaystyle\sum_{k=1}^{m} \left(w_1, z_k\right) z_k. \end{cases}$$

Since (3.1) is a normal system of ordinary differential equations, then there exist $(u_{m\varepsilon}, \Phi_{m\varepsilon})$, solutions to the problem (3.1). A solution $(u, \Phi)$ to the problem (2.19)-(2.23) will be obtained as the limit of $(u_{m\varepsilon}, \Phi_{m\varepsilon})$ as $m \to \infty$ and $\varepsilon \to 0$. Therefore, uniform estimates with respect to $m$ and $\varepsilon$ are needed. Indeed, from the first and the second equations in (3.1), we have the approximate equations.

$$(3.2)\quad \begin{cases} (u'_{m\varepsilon}, v - u_{m\varepsilon})_{\Omega_0} + b_1\left(u_{m\varepsilon}, u_{m\varepsilon}, v - u_{m\varepsilon}\right) + a_{\Omega_0}\left(u_{m\varepsilon}, v - u_{m\varepsilon}\right) \\ \quad + \left(DJ_\varepsilon\left(u_{m\varepsilon}\right), v - u_{m\varepsilon}\right)_{V_1' \times V_1} + \displaystyle\int_{\Gamma_0} \left(\sigma_1 - \sigma_0\right)\left(v - u_{m\varepsilon}\right) d\Gamma \\ = (f_0, v - u_{m\varepsilon})_{\Omega_0}, \\ u_{m\varepsilon}(0) = u_{0\varepsilon} = \displaystyle\sum_{k=1}^{m} \left(u_0, v\right) v, \end{cases}$$

and

$$(3.3)\quad \begin{cases} (\Phi'_{m\varepsilon}, \varphi - \Phi_{m\varepsilon})_{\Omega_1} + (\sigma_1, \varepsilon\left(\varphi - \Phi_{m\varepsilon}\right))_{\Omega_1} + \displaystyle\int_{\Gamma_{11}} g\left(\varphi - \Phi_{m\varepsilon}\right) d\Gamma \\ = (f_1, \varphi - \Phi_{m\varepsilon})_{\Omega_1}, \\ \Phi_{m\varepsilon}(0) = \Phi_{0\varepsilon} = \displaystyle\sum_{k=1}^{m} \left(w_1, \varphi\right) \varphi, \end{cases}$$

which hold for all $v \in span\left\{y_1, y_2, \ldots, y_m\right\}$ and $\varphi \in span\left\{z_1, z_2, \ldots, z_m\right\}$.

In the next estimate, we need to show that $\forall m \in \mathbb{N}$, then $t_m = T$. Thus, we have to show some a priori uniform estimates with respect to $m$.

*Estimate I.* Taking $v = 2u_{m\varepsilon}$ in (3.2), $\varphi = 2\Phi_{m\varepsilon}$ in (3.3), and adding the second equation with the first, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\|u_{m\varepsilon}(t)\right\|^2_{L^2(\Omega_0)} + b_1(u_{m\varepsilon}(t), u_{m\varepsilon}(t), u_m(t)) + a_{\Omega_0}(u_{m\varepsilon}(t), u_{m\varepsilon}(t)) \\ &+ \left(DJ_\varepsilon\left(u_{m\varepsilon}(t)\right), u_{m\varepsilon}(t)\right)_{V_1'\times V_1} + \int_{\Gamma_0}\left(\sigma_1-\sigma_0\right)u_{m\varepsilon}(t)d\Gamma \\ &+ \frac{1}{2}\frac{d}{dt}\left\|\Phi_{m\varepsilon}(t)\right\|^2_{L^2(\Omega_1)} + \left(\sigma_1, \varepsilon\left(\Phi_{m\varepsilon}(t)\right)\right)_{\Omega_1} + \int_{\Gamma_{11}} g\Phi_{m\varepsilon}(t)d\Gamma \\ &= (f_0, u_{m\varepsilon}(t))_{\Omega_0} + (f_1, \Phi_{m\varepsilon}(t))_{\Omega_1}, \forall\left(u_m, \Phi_m\right) \in V_{1,m}\times V_{2,m}. \end{aligned} \tag{3.4}$$

Using the Cauchy–Schwartz and Young's inequalities, we have

$$\begin{aligned} \left|(f_0(t), u_{m\varepsilon}(t))_{\Omega_0}\right| &\le c\left\|f_0(t)\right\|_{L^2(\Omega_0)}\left\|u_{m\varepsilon}(t)\right\|_{L^2(\Omega_0)} \\ &\le c\left\|f_0(t)\right\|_{L^2(\Omega_0)}\left\|u_{m\varepsilon}(t)\right\|_{V_1} \\ &\le \frac{c_1}{2}\left\|f_0(t)\right\|^2_{L^2(\Omega_0)} + \frac{c_1}{2}\left\|u_{m\varepsilon}(t)\right\|^2_{V_1}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \left|(f_1(t), \Phi_{m\varepsilon}(t))_{\Omega_1}\right| &\le c\left\|f_1(t)\right\|_{L^2(\Omega_1)}\left\|\Phi_{m\varepsilon}(t)\right\|_{L^2(\Omega_1)} \\ &\le c\left\|f_1(t)\right\|_{L^2(\Omega_1)}\left\|\Phi_{m\varepsilon}(t)\right\|_{V_2} \\ &\le \frac{c_2}{2}\left\|f_1(t)\right\|^2_{L^2(\Omega_1)} + \frac{c_2}{2}\left\|\Phi_{m\varepsilon}(t)\right\|^2_{V_2}. \end{aligned} \tag{3.6}$$

Note that

$$\left|a_{\Omega_0}(u_{m\varepsilon}(t), u_{m\varepsilon}(t))\right| \le c_3\left\|u_{m\varepsilon}(t)\right\|_{V_1}\left\|u_{m\varepsilon}(t)\right\|_{V_1} = c_3\left\|u_{m\varepsilon}(t)\right\|^2_{V_1}, \tag{3.7}$$

and

$$b_1(u_{m\varepsilon}(t), u_{m\varepsilon}(t), u_{m\varepsilon}(t)) = 0,\ \forall u_{m\varepsilon} \in V_{1,m}. \tag{3.8}$$

According to the Hypothesis of regularization $(H)$ (case (3)), we have

$$\begin{aligned} \left|\left(DJ_\varepsilon\left(u_{m\varepsilon}(t)\right), u_{m\varepsilon}(t)\right)_{V_1'\times V_1}\right| &\le \left\|DJ_\varepsilon\left(u_{m\varepsilon}(t)\right)\right\|_{V_1'}\left\|u_{m\varepsilon}(t)\right\|_{V_1} \\ &\le C\left(\varepsilon, \delta\right)\left\|u_{m\varepsilon}(t)\right\|_{V_1}\left\|u_m(t)\right\|_{V_1} \\ &\le C\left\|u_{m\varepsilon}(t)\right\|^2_{V_1}. \end{aligned} \tag{3.9}$$

In the other hand, we have

$$\begin{aligned} &\left|\int_{\Gamma_0}\left(\sigma_1-\sigma_0\right)u_{m\varepsilon}(t)d\Gamma\right| \\ \le\ &\int_{\Gamma_0}\left|\left(\sigma_1-\sigma_0\right)u_{m\varepsilon}(t)\right|d\Gamma \\ \le\ &\int_{\Gamma_0}\left|\left(\frac{1}{2}\left(\sum_{i=1}^{2}u_{im\varepsilon}(t)\cos\left(\eta_i\right)\right)\times u_{im\varepsilon}(t)\right)u_{m\varepsilon}(t)\right|d\Gamma \end{aligned}$$

$$\leq C\sum_{i=1}^{2}\left(\int_{\Gamma_0}|u_{im\varepsilon}(t)|^3\,d\Gamma\right)^{\frac{1}{3}}\left(\int_{\Gamma_0}|u_{im\varepsilon}(t)|^3\,d\Gamma\right)^{\frac{1}{3}}\left(\int_{\Gamma_0}|u_{m\varepsilon}(t)|^3\,d\Gamma\right)^{\frac{1}{3}}$$
$$\leq C\left\|u_{m\varepsilon}(t)\right\|^3_{L^3(\Gamma_0)},$$

and according to Lemma 2.5, we have

$$\left\|v_{|\Gamma_0}\right\|_{L^3(\Gamma_0)}\leq C\left\|v\right\|^{\frac{2}{3}}_{H^1(\Omega_0)}\left\|v\right\|^{\frac{1}{3}}_{L^2(\Omega_0)},\ \forall v\in V_1\subset H^1(\Omega_0).$$

Consequently, we deduce

$$\left|\int_{\Gamma_0}(\sigma_1-\sigma_0)\,u_{m\varepsilon}(t)d\Gamma\right|$$
$$\leq C\left\|u_{m\varepsilon}(t)\right\|_{L^3(\Gamma_0)}\left\|u_{m\varepsilon}(t)\right\|_{L^3(\Gamma_0)}\left\|u_{m\varepsilon}(t)\right\|_{L^3(\Gamma_0)}$$
$$\leq c\left(\left\|u_{m\varepsilon}(t)\right\|^{\frac{2}{3}}_{H^1(\Omega_0)}\left\|u_{m\varepsilon}(t)\right\|^{\frac{1}{3}}_{L^2(\Omega_0)}\right)\left(\left\|u_{m\varepsilon}(t)\right\|^{\frac{2}{3}}_{H^1(\Omega_0)}\left\|u_{m\varepsilon}(t)\right\|^{\frac{1}{3}}_{L^2(\Omega_0)}\right)$$
$$\times\left(\left\|u_{m\varepsilon}(t)\right\|^{\frac{2}{3}}_{H^1(\Omega_0)}\left\|u_{m\varepsilon}(t)\right\|^{\frac{1}{3}}_{L^2(\Omega_0)}\right)$$
$$\leq c\left(\left\|u_m(t)\right\|^{\frac{2}{3}}_{H^1(\Omega_0)}\left\|u_m(t)\right\|^{\frac{1}{3}}_{L^2(\Omega_0)}\right)^3\leq c_4\left\|u_{m\varepsilon}(t)\right\|^3_{V_1}. \tag{3.10}$$

Using Hölder's and Young's inequalities, we have

$$\left|(\sigma_1,\varepsilon\left(\Phi_{m\varepsilon}(t)\right))_{\Omega_1}\right|$$
$$=\left|\int_{\Omega_1}\sigma_1\varepsilon\left(\Phi_{m\varepsilon}(t)\right)dx\right|$$
$$\leq\int_{\Omega_1}\left|\sigma_1\varepsilon\left(\Phi_{m\varepsilon}(t)\right)\right|dx$$
$$\leq c\left(\int_{\Omega_1}|\sigma_1|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\Omega_1}\left|\varepsilon\left(\Phi_{m\varepsilon}(t)\right)\right|^2dx\right)^{\frac{1}{2}}$$
$$\leq c\left\|\sigma_1\right\|_{L^2(\Omega_1)}\left\|\varepsilon\left(\Phi_{m\varepsilon}(t)\right)\right\|_{L^2(\Omega_1)^{2\times2}}$$
$$\leq c\left\|\sigma_1\right\|_{L^2(\Omega_1)}\left\|\frac{1}{2}\left(\nabla\left(\Phi_{m\varepsilon}(t)\right)+\nabla^T\left(\Phi_{m\varepsilon}(t)\right)\right)\right\|_{L^2(\Omega_1)^{2\times2}}$$
$$\leq c\left\|\sigma_1\right\|_{L^2(\Omega_1)}\left(\left\|\nabla\left(\Phi_{m\varepsilon}(t)\right)\right\|_{L^2(\Omega_1)^{2\times2}}+\left\|\nabla^T\left(\Phi_{m\varepsilon}(t)\right)\right\|_{L^2(\Omega_1)^{2\times2}}\right)$$
$$\leq c\left\|\sigma_1\right\|_{V_2}\left\|\Phi_{m\varepsilon}(t)\right\|_{V_2}\leq c_5\left\|\sigma_1\right\|^2_{V_2}\left\|\Phi_{m\varepsilon}(t)\right\|^2_{V_2}, \tag{3.11}$$

where

$$L^2(\Omega)^{2\times2}=\left\{\sigma=(\sigma_{ij}):\sigma_{ij}=\sigma_{ji}\in L^2(\Omega),\ i,j=1,2\right\}.$$

By the same manner and according to the trace embedding theorem

$$V_2\subset H^1(\Omega_1)^2\hookrightarrow H^{\frac{1}{2}}(\Gamma_1)^2\subset L^2(\Gamma_1)^2\subset H^{-\frac{1}{2}}(\Gamma_1)^2.$$

Using the equation (2.8) and Cauchy-Schwartz inequality and

$$\left\|\sigma_1\eta_1\right\|_{H^{-\frac{1}{2}}(\Gamma_1)}\leq\left\|\sigma_1\eta_1\right\|_{L^2(\Gamma_1)}\leq\left\|\sigma_1\eta_1\right\|_{H^1(\Omega_1)},$$

it results that

$$\begin{aligned}\left|\int_{\Gamma_1} g\Phi_{m\epsilon}(t)d\gamma\right| &= \left|\int_{\Gamma_{11}} \sigma_1\eta_1\Phi_{m\varepsilon}(t)\right| \\ &\le C\left\|\sigma_1\eta_1\right\|_{L^2(\Gamma_1)}\left\|\Phi_{m\varepsilon}(t)\right\|_{V_2},\ \forall\Phi_{m\varepsilon}(t)\in V_{2m} \\ &\le C\left\|\sigma_1\eta_1\right\|^2_{L^2(\Gamma_1)}\left\|\Phi_{m\varepsilon}(t)\right\|^2_{V_2},\ \forall\Phi_{m\varepsilon}(t)\in V_{2m} \\ &\le c_6\left\|\sigma_1\eta_1\right\|^2_{H^1(\Omega_1)}\left\|\Phi_{m\varepsilon}(t)\right\|^2_{V_2},\ \forall\Phi_{m\varepsilon}(t)\in V_{2m}. \end{aligned} \tag{3.12}$$

Substituting those estimates (3.5)-(3.12) into (3.4), integrating the result over $(0,t)$, after using (2.24)-(2.25), we find

$$\begin{aligned}&\left\|u_{m\varepsilon}(t)\right\|^2_{L^2(\Omega_0)} + \int_0^t \left(2C_3+2C+2C_4\left\|u_{m\varepsilon}(s)\right\|_{V_1}-C_1\right)\left\|u_{m\varepsilon}(s)\right\|^2_{V_1} ds \\ &+\left\|\Phi_{m\varepsilon}(t)\right\|^2_{L^2(\Omega_1)} + \left(2c_5\left\|\sigma_1\right\|^2_{V_2}+2c_6\left\|\sigma_1\eta_1\right\|^2_{H^1(\Omega_1)}-C_2\right)\int_0^t \left\|\Phi_{m\varepsilon}(s)\right\|^2_{V_2} ds \\ &\le C, \end{aligned} \tag{3.13}$$

we can get a constant $C = C(T) > 0$, independent of the $m$, $\varepsilon$ and $t \in [0,T]$, such that

$$\left\|\Phi_{m\varepsilon}(t)\right\|_{L^2(\Omega_1)} + \left\|u_{m\varepsilon}(t)\right\|_{L^2(\Omega_0)} + \int_0^t \left\|\Phi_{m\varepsilon}(s)\right\|^2_{V_2} ds + \int_0^t \left\|u_{m\varepsilon}(s)\right\|^2_{V_1} ds \le C. \tag{3.14}$$

Passing to the limit where $m \to \infty$, from (3.14), we conclude

$$u_{m\varepsilon} \text{ is bounded in } L^2(0,T,V_1)\cap L^\infty(0,T,(L^2(\Omega_0))^2), \tag{3.15}$$

$$\Phi_{m\varepsilon} \text{ is bounded in } L^2(0,T,V_2)\cap L^\infty(0,T,(L^2(\Omega_1))^2). \tag{3.16}$$

*Estimate II.* Let $P^1_m$ and $P^2_m$ be the orthogonal projects to $V_1$ in $V_{1,m}$ and to $V_2$ in $V_{2,m}$, respectively, having the following proprieties

$$\begin{cases} V_1 \to V_{1,m},\ h \to P^1_m(h) = \sum_{i=1}^m (h, y_i)_{\Omega_0} y_i, \\ P^1_m \text{ is bounded in } \mathcal{L}(V_1,V_1), \\ \left\|P^1_m\right\|_{\mathcal{L}(V_1,V_1)} \le 1, \end{cases}$$

and

$$\begin{cases} V_2 \to V_{2,m},\ h \to P^2_m(h) = \sum_{i=1}^m (h, z_i)_{\Omega_1} z_i, \\ P^2_m \text{ is bounded in } \mathcal{L}(V_2,V_2), \\ \left\|P^2_m\right\|_{\mathcal{L}(V_2,V_2)} \le 1. \end{cases}$$

By the transposition arguments, we have

$$\begin{cases} P^1_m = \overset{*}{P^1_m} \in \mathcal{L}(V_1',V_1') \text{ and } \left\|P^1_m\right\|_{\mathcal{L}(V_1',V_1')} \le 1, \\ P^2_m = \overset{*}{P^2_m} \in \mathcal{L}(V_2',V_2') \text{ and } \left\|P^2_m\right\|_{\mathcal{L}(V_2',V_2')} \le 1. \end{cases}$$

Because $a_{\Omega_0}(u,v)$ is a sesquilinear continuous function on $V_1$, therefore there exists an operator

$$A_0 \in \mathcal{L}(V_1, V_1')$$

such that

$$a_{\Omega_0}(u,v) = \langle A_0(u), v\rangle_{V_1'\times V_1}, \ \forall (u,v) \in V_1 \times V_1.$$

Multiplying the following two equations

$$(3.17) \quad \begin{aligned} &(u'_{m\varepsilon}(t), y_k)_{\Omega_0} + b_1(u_{m\varepsilon}(t), u_{m\varepsilon}(t), y_k) + a_{\Omega_0}(u_{m\varepsilon}(t), y_k) \\ &+ (DJ_\varepsilon(u_{m\varepsilon}(t)), y_k)_{V_1'\times V_1} - \int_{\Gamma_0} \sigma_0 \nu y_k d\Gamma = (f_0, y_k)_{\Omega_0}, \forall y_k \in V_{1,m}, \end{aligned}$$

where $\nu$ denotes the outside normal vector is oriented of $\Omega_1$,

$$(3.18) \quad \begin{aligned} &(\Phi'_{m\varepsilon}(t), z_k)_{\Omega_1} + (\sigma_1, \varepsilon(z_k))_{\Omega_1} + \int_{\Gamma_{11}} g z_k d\Gamma \\ &+ \int_{\Gamma_0} \sigma_1 \eta_1 z_k d\Gamma = (f_1, z_k)_{\Omega_1}, \forall z_k \in V_{2,m}, \end{aligned}$$

by $y_k$ and by $z_k$, respectively, and summing over $k = 1$ to $m$, we get

$$(3.19) \quad \begin{aligned} &P_m^1(u'_{m\varepsilon}(t)) + P_m^1(A_0(u_{m\varepsilon}(t))) + \sum_{k=1}^m (b_1(u_{m\varepsilon}(t), u_{m\varepsilon}(t), y_k)\ y_k) \\ &+ P_m^1(DJ_\varepsilon(u_{m\varepsilon}(t))) - \sum_{k=1}^m \int_{\Gamma_0} \sigma_0 \nu (y_k)^2 d\Gamma_0 = P_m^1(f_0(t)), \end{aligned}$$

and

$$(3.20) \quad \begin{aligned} &P_m^2(\Phi'_{m\varepsilon}(t)) + \sum_{k=1}^m (\sigma_1, \varepsilon(z_k))_{\Omega_1} z_k + \sum_{k=1}^m \int_{\Gamma_{11}} g (z_k)^2 d\Gamma \\ &+ \sum_{k=1}^m \int_{\Gamma_0} \sigma_1 \eta_1 (z_k)^2 d\Gamma = P_m^2(f_1(t)), \end{aligned}$$

according (3.15), we have

$$(3.21) \quad A_0(u_{m\varepsilon}) \in L^2(0,T,V_1').$$

And from assumption (2.24), we have

$$(3.22) \quad P_m^1(f_0) \in L^2(0,T,L^2(\Omega_0)) \ \text{ and } \ P_m^2(f_1) \in L^2(0,T,L^2(\Omega_1)).$$

According to the following corollary, for $\Omega \subset \mathbb{R}^2$, we have

$$\|v\|_{L^4(\Omega)} \leq C\ \|v\|_{L^2(\Omega)}^{\frac{1}{2}} \|v\|_{H^1(\Omega)}^{\frac{1}{2}}\ , \forall v \in H^1(\Omega),$$

then, for $n = 2$, of

$$u_{m\varepsilon} \text{ remains in a bounded of } L^2(0,T,V_1) \cap L^\infty(0,T,(L^2(\Omega_0))^2)$$

it comes

$$u_{m\varepsilon} \in \ L^4(0,T,(L^{\frac{q}{2}}(\Omega_0))^2) = L^4(0,T,(L^2(\Omega_0))^2), \text{where } q = 4,$$

then, implies that

$$u_{m\varepsilon} \in L^4(0,T,(L^2(\Omega_0))^2).$$

And according to Lemma 2.4, we infer

$$\begin{aligned}
&\left|\sum_{k=1}^{m}(b_1(u_{m\varepsilon}(t),u_{m\varepsilon}(t),y_k)\ y_k)\right| \\
&\le C\left(\|u_{m\varepsilon}\|^2_{(L^4(\Omega_0))^2}\|u_{m\varepsilon}\|_{L^2(\Omega_0)}\right)\|u_{m\varepsilon}\|_{V_1}\|y_k\|_{L^4(\Omega_0)^2}\|y_k\|_{V_1} \\
&\le C\left(\left(C\|u_{m\varepsilon}\|^{\frac{1}{2}}_{V_1}\|u_{m\varepsilon}\|^{\frac{1}{2}}_{L^2(\Omega_0)}\right)^2\|u_{m\varepsilon}\|_{L^2(\Omega_0)}\|u_{m\varepsilon}\|_{V_1}\|y_k\|_{L^4(\Omega_0)^2}\right)\|y_k\|_{V_1} \\
&\le C\|u_{m\varepsilon}\|_{V_1}\|u_{m\varepsilon}\|_{L^2(\Omega_0)}\|u_{m\varepsilon}\|_{L^2(\Omega_0)}\|u_{m\varepsilon}\|_{V_1}\|y_k\|^{\frac{1}{2}}_{H^1(\Omega_0)}\|y_k\|^{\frac{1}{2}}_{L^2(\Omega_0)}\|y_k\|_{V_1} \\
&\le C\|u_{m\varepsilon}\|^2_{V_1}\|u_{m\varepsilon}\|^2_{L^2(\Omega_0)}\|y_k\|^{\frac{1}{2}}_{V_1}\|y_k\|^{\frac{1}{2}}_{V_1}\|y_k\|_{V_1} \\
&\le C\|u_{m\varepsilon}\|^2_{V_1}\|u_{m\varepsilon}\|^2_{L^2(\Omega_0)}\|y_k\|^2_{V_1}
\end{aligned}$$

from where, it results

$$\sum_{k=1}^{m}(b_1(u_{m\varepsilon}(t),u_{m\varepsilon}(t),y_k)\ y_k)\in L^2(0,T,V_1^{'}). \tag{3.23}$$

According to estimates (3.12), we have

$$\left|\sum_{k=1}^{m}\int_{\Gamma_0}(\sigma_1-\sigma_0)\,y_k^2 d\Gamma_0\right|\le C\left\|y_k^2\right\|^3_{V_1}\le C\left\|y_k^2\right\|_{V_1}\left\|y_k^2\right\|^2_{V_1}\le C\|y_k\|^4_{V_1}.$$

Also

$$\begin{aligned}
\left|\sum_{k=1}^{m}(\sigma_1,\varepsilon(z_k))_{\Omega_1}z_k\right| &= \left|\sum_{k=1}^{m}\int_{\Omega_1}\sigma_1\varepsilon(z_k)\,z_k dx\right|\le\sum_{k=1}^{m}\int_{\Omega_1}|\sigma_1\varepsilon(z_k)\,z_k|\,dx \\
&\le\sum_{k=1}^{m}\int_{\Omega_1}|\sigma_1\varepsilon(z_k)|\,|z_k|\,dx \\
&\le\left(\sum_{k=1}^{m}\int_{\Omega_1}|\sigma_1\varepsilon(z_k)|\,dx\right)\left(\sum_{k=1}^{m}|z_k|\,dx\right) \\
&\le c\|\sigma_1\|^2_{V_2}\|z_k\|^2_{V_2}\|z_k\|_{V_2}\le c\|\sigma_1\|^2_{V_2}\|z_k\|^3_{V_2}. \qquad (3.24)
\end{aligned}$$

And according to (3.12), we have

$$\begin{aligned}
\left|\int_{\Gamma_{11}}g(z_k)^2\,d\Gamma\right| = \left|\int_{\Gamma_{11}}\sigma_1\eta_1 z_k z_k\right| &\le C\|\sigma_1\eta_1\|^2_{L^2(\Gamma_1)}\left\|z_k^2\right\|^2_{V_2} \\
&\le C\|\sigma_1\eta_1\|^2_{H^1(\Omega_1)}\|z_k\|^4_{V_2}. \qquad (3.25)
\end{aligned}$$

Finally, substituting those estimates into (3.19) and in (3.20), according to (3.22) and (3.25), we get

$$u'_{m\varepsilon}(t)=P_m^1(f_0(t))-P_m^1(A_0(u_{m\varepsilon}(t)))-\sum_{k=1}^{m}(b_1(u_{m\varepsilon}(t),u_{m\varepsilon}(t),y_k)\ y_k)$$

$$- P_m^1(DJ_\varepsilon (u_{m\varepsilon}(t))) + \sum_{k=1}^{m} \int_{\Gamma_0} \sigma_0 \nu (y_k)^2 \, d\Gamma, \tag{3.26}$$

and

$$\Phi'_{m\varepsilon}(t) = P_m^2(f_1(t)) - \sum_{k=1}^{m} (\sigma_1, \varepsilon (z_k))_{\Omega_1} z_k$$
$$- \sum_{k=1}^{m} \int_{\Gamma_{11}} g (z_k)^2 \, d\Gamma - \sum_{k=1}^{m} \int_{\Gamma_0} \sigma_1 \eta_1 (z_k)^2 \, d\Gamma. \tag{3.27}$$

Therefore, we deduce

$$u'_{m\varepsilon} \in L^2(0, T; V'_1) \text{ and } \Phi'_{m\varepsilon} \in L^2(0, T, V'_2). \tag{3.28}$$

Consequently, we deduce that there exist subsequences $(u_{m\varepsilon})$ and $(\Phi_{m\varepsilon})$, which we still denote by $(u_{m\varepsilon})$ and $(\Phi_{m\varepsilon})$, respectively, such that

$$(u'_{m\varepsilon}, \Phi'_{m\varepsilon}) \to (u'_\varepsilon, \Phi'_\varepsilon) \text{ weakly in } L^2(0, T, V_1) \times L^2(0, T, V_2), \tag{3.29}$$

and,

$$(u_{m\varepsilon}, \Phi_{m\varepsilon}) \to (u_\varepsilon, \Phi_\varepsilon) \text{ weak star in } L^\infty(0, T; (L^2(\Omega_0))^2) \times L^\infty(0, T; (L^2(\Omega_1))^2). \tag{3.30}$$

From (3.15), (3.16) and (3.29), it results

$$(u_{m\varepsilon}, \Phi_{m\varepsilon}) \to (u_\varepsilon, \Phi_\varepsilon) \text{ weakly in } L^2(0, T, V_1) \times L^2(0, T, V_2), \tag{3.31}$$

it is known that the injection of $H^1(\Omega_0)$ in $L^2(\Omega_0)$ ($H^1(\Omega_1)$ in $L^2(\Omega_1)$) is compact. This permit us to assume that the extracted subsequence $(u_{m\varepsilon}, \Phi_{m\varepsilon})$ verify, in addition to relations (3.31)

$$(u_{m\varepsilon}, \Phi_{m\varepsilon}) \to (u_\varepsilon, \Phi_\varepsilon) \text{ strongly in } L^2(0, T, (L^2(\Omega_0))^2) \times L^2(0, T, (L^2(\Omega_1))^2)$$
$$\text{and a.e in } ((0, T) \times \Omega_0) \times ((0, T) \times \Omega_1), \tag{3.32}$$

from

$$u_{m\varepsilon} \in L^2(0, T, V_1) \cap L^\infty(0, T, (L^2(\Omega_0))^2),$$

we deduce, that

$$u_{m\varepsilon i}.u_{m\varepsilon j} \text{ is bounded in } L^2(0, T, (L^2(\Omega_0))^2),$$

therefore, we can suppose that

$$u_{m\varepsilon i} u_{m\varepsilon j} \to \chi_{i,j} \text{ weakly in } L^2(0, T, (L^2(\Omega_0))^2). \tag{3.33}$$

From (3.32), using [7, Lemme 1.3], we infer that

$$\chi_{i,j} = u_{i\varepsilon} u_{j\varepsilon}. \tag{3.34}$$

Passing to the limit where $m \to \infty$, we have the following convergence:

$$b_1(u_{m\varepsilon}, u_{m\varepsilon}, y_k) \to b_1(u_\varepsilon, u_\varepsilon, y_k) \text{ weakly in } L^2(0, T), \tag{3.35}$$

$$\int_0^T b_1(u_{m\varepsilon}, u_{m\varepsilon}, y_k) v ds \to \int_0^T b_1(u_\varepsilon, u_\varepsilon, y_k) v ds, \forall v \in L^2(0, T), \tag{3.36}$$

$$(u'_{m\varepsilon}, y_k)_{\Omega_0} \to (u'_\varepsilon, y_k)_{\Omega_0} \text{ in } D'(0,T), \tag{3.37}$$

$$a_{\Omega_0}(u_{m\varepsilon}, y_k) \to a_{\Omega_0}(u_\varepsilon, y_k) \text{ in } D'(0,T), \tag{3.38}$$

$$(DJ_\varepsilon(u_{m\varepsilon}), y_k)_{V_1'\times V_1} \to (DJ_\varepsilon(u_\varepsilon), y_k)_{V_1'\times V_1} \text{ in } D'(0,T), \tag{3.39}$$

$$(\Phi'_{m\varepsilon}, z_k)_{\Omega_1} \to (\Phi'_\varepsilon, z_k)_{\Omega_1} \text{ in } D'(0,T). \tag{3.40}$$

The estimates (3.35)-(3.40) are independent of $\varepsilon$. Therefore, by the same precedent argument used to obtain $u_\varepsilon$ and $\Phi_\varepsilon$ from $u_{m\varepsilon}$ and $\Phi_{\mu\varepsilon}$, we can pass to the limit when $\varepsilon \to 0$ in $u_{m\varepsilon}$ and $\Phi_{\mu\varepsilon}$, obtaining $u$ and $\Phi$ such that

$$b_1(u_\varepsilon, u_\varepsilon, y_k) \to b_1(u, u, y_k) \text{ weakly in } L^2(0,T), \tag{3.41}$$

$$\int_0^T b_1(u_\varepsilon, u_\varepsilon, y_k) v ds \to \int_0^T b_1(u, u, y_k) v ds, \forall v \in L^2(0,T), \tag{3.42}$$

$$(u'_\varepsilon, y_k)_{\Omega_0} \to (u', y_k)_{\Omega_0} \text{ in } D'(0,T), \tag{3.43}$$

$$a_{\Omega_0}(u_\varepsilon, y_k) \to a_{\Omega_0}(u, y_k) \text{ in } D'(0,T), \tag{3.44}$$

$$(DJ_\varepsilon(u_\varepsilon), y_k)_{V_1'\times V_1} \to (DJ_\varepsilon(u), y_k)_{V_1'\times V_1} \text{ in } D'(0,T), \tag{3.45}$$

$$(\Phi'_\varepsilon, z_k)_{\Omega_1} \to (\Phi', z_k)_{\Omega_1} \text{ in } D'(0,T). \tag{3.46}$$

Substituting (3.41)-(3.46) into (2.19), it follows

$$\begin{aligned}
&(u', y_k)_{\Omega_0} + b_1(u, u, y_k) + a_{\Omega_0}(u, y_k) + (DJ_\varepsilon(u), y_k)_{V_1'\times V_1} \\
&- \int_{\Gamma_0} \sigma_0 \nu y_k d\Gamma + (\Phi', z_k)_{\Omega_1} + (\sigma_1, \varepsilon(z_k))_{\Omega_1} \\
&+ \int_{\Gamma_{11}} g z_k d\Gamma + \int_{\Gamma_0} \sigma_1 \eta_1 z_k d\Gamma \\
&= (f_0, y_k)_{\Omega_0} + (f_1, z_k)_{\Omega_1}, \ \forall (y_k, z_k) \in V_{1,m} \times V_{2,m}.
\end{aligned} \tag{3.47}$$

Finally, since the space $V_{1,m} \times V_{2,m}$ is dense in $V_1 \times V_2$, we obtain

$$\begin{aligned}
&(u', v-u)_{\Omega_0} + b_1(u, u, v-u) + a_{\Omega_0}(u, v-u) \\
&+ (DJ_\varepsilon(u), v-u)_{V_1'\times V_1} - \int_{\Gamma_0} \sigma_0 \nu (v-u) d\Gamma_0 \\
&+ (\Phi', \varphi - \Phi)_{\Omega_1} + (\sigma_1, \varepsilon(\varphi - \Phi))_{\Omega_1} \\
&+ \int_{\Gamma_{11}} g(\varphi - \Phi) d\Gamma + \int_{\Gamma_0} \sigma_1 \eta_1 (\varphi - \Phi) d\Gamma \\
&= (f_0, v-u)_{\Omega_0} + (f_1, \varphi - \Phi)_{\Omega_1}, \ \forall (v, \varphi) \in V_1 \times V_2.
\end{aligned} \tag{3.48}$$

This shows the existence of a solution to the problem (2.1)-(2.11).

Remains to verify the initial conditions. Using (3.29) and (3.31), we have

$$(u_{m\varepsilon}(0), \Phi_{m\varepsilon}(0)) \to (u(0), \Phi(0)) \text{ weakly in } V_1' \times V_2'.$$

Hence, the results follows, and the proof of Theorem 3.1 is completed. □

### 3.2. Uniqueness

**Theorem 3.2.** *Assume the hypotheses of Theorem 3.1 hold. Then the solution* $(u,\Phi)$ *of the variational problem* $(P.V)$ *is unique.*

*Proof.* Let $(u_1,\Phi_1)$ and $(u_2,\Phi_2)$ be a two pair possible solution to problem (2.19), verifying

$$\begin{cases} u_k \in \ L^2(0,T,V_1)\cap L^\infty(0,T,(L^2(\Omega_0))^2) \text{ for } k=1,2,\\ \Phi_k \in \ L^2(0,T,V_2)\cap L^\infty(0,T,\left(L^2\left(\Omega_1\right)^2\right)) \text{ for } k=1,2.\end{cases}$$

We have

$$\begin{aligned} &(u_1',v-u_1)_{\Omega_0}+b_1\left(u_1,u_1,v-u_1\right)+a_{\Omega_0}\left(u_1,v-u_1\right)\\ &+\ \left(DJ_\varepsilon\left(u_1\right),v-u_1\right)_{V_1'\times V_1}+\int_{\Gamma_0}\left(\sigma_1-\sigma_0\right)\left(v-u_1\right)d\Gamma,\\ &(\Phi_1',\varphi-\Phi)_{\Omega_1}+(\sigma_1,\varepsilon\left(\varphi-\Phi\right))_{\Omega_1}+\int_{\Gamma_{11}}g\left(\varphi-\Phi\right)d\Gamma\\ =&\,(f_0,v-u_1)_{\Omega_0}+(f_1,\varphi-\Phi)_{\Omega_1},\forall\left(v,\varphi\right)\in V_1\times V_2,\end{aligned}$$

and

$$\begin{aligned} &(u_2',v-u_2)_{\Omega_0}+b_1\left(u_2,u_2,v-u_2\right)+a_{\Omega_0}\left(u_2,v-u_2\right)\\ &+\ \left(DJ_\varepsilon\left(u_2\right),v-u_2\right)_{V_1'\times V_1}+\int_{\Gamma_0}\left(\sigma_1-\sigma_0\right)\left(v-u_2\right)d\Gamma_0,\\ &(\Phi_2',\varphi-\Phi)_{\Omega_1}+(\sigma_1,\varepsilon\left(\varphi-\Phi\right))_{\Omega_1}+\int_{\Gamma_{11}}g\left(\varphi-\Phi\right)d\Gamma\\ =&\,(f_0,v-u_2)_{\Omega_0}+(f_1,\varphi-\Phi)_{\Omega_1},\forall\left(v,\varphi\right)\in V_1\times V_2,\end{aligned}$$

with the conditions

$$\begin{cases} u_1(x,0)=u_2(x,0)=u_0 \text{ on } \Omega_0,\\ \Phi_1\left(0\right)=\Phi_2\left(0\right)=w'\left(0\right)=w_1 \ \text{ on } \Omega_1\end{cases}$$

from where;

$$\begin{cases} V_1=\left\{v: v\in (H^1(\Omega_0))^2, \operatorname{div} v=0 \text{ on } \Omega_0\right\}\\ \text{and}\\ V_2=\left\{v: v\in (H^1(\Omega_1))^2, v=0 \text{ in } \Gamma_{12}\right\}\\ \text{however,}\\ V=\left\{(v_1,v_2)\in V_1\times V_2: v_1=v_2 \text{ in } \Gamma_0\right\}.\end{cases}$$

Putting $U=u_1-u_2$, $W=\Phi_1-\Phi_2$ for all $(v_1,v_2)\in V_1\times V_2$, then $(U,W)$ satisfy this system

$$\begin{aligned} &(U',v-u)_{\Omega_0}+a_{\Omega_0}\left(U,v-u\right)+\ b_1(u_1,u_1,v-u)-b_1(u_2,u_2,v-u)\\ &+\ \left(DJ_\varepsilon\left(u_1\right),v-u\right)_{V_1'\times V_1}-\left(DJ_\varepsilon\left(u_2\right),v-u\right)_{V_1'\times V_1}\\ &+(W'(t),\varphi-\Phi)_{\Omega_1}=0,\ \forall\left(v,\varphi\right)\in V_1\times V_2. \end{aligned} \tag{3.49}$$

On the other hand, using the proprieties of the trilinear function $b_1(u,u,v)$, we have

$$\begin{aligned} b_1(u_2,u_2,v-u) &= b_1(u_2,u_2-u_1+u_1,v-u) \\ &= b_1(u_2,(u_2-u_1),v-u)+b_1(u_2,u_1,v-u), \end{aligned}$$

therefore

$$\begin{aligned} & b_1(u_1,u_1,v-u)-b_1(u_2,u_2,v-u) \\ &= b_1(u_1,u_1,v-u)-b_1(u_2,u_2-u_1,v-u)-b_1(u_2,u_1,v-u) \end{aligned}$$

gives

$$b_1(u_1,u_1,v-u)-b_1(u_2,u_2,v-u)=b_1(U,u_1,v-u)-b_1(u_2,-U,v-u).$$

In the other hand, we have

$$\begin{aligned} & b_1(u_1,u_1,v-u)-b_1(u_2,u_2,v-u) \\ &= b_1(U,u_1,v-u)-b_1(U,U,v-u)+b_1(u_1,U,v-u),\ \forall v\in V_1. \end{aligned} \tag{3.50}$$

Putting $v-u=U$, $\varphi-\Phi=W$, and substituting (3.50) into (3.49), we get

$$\begin{aligned} & (U',U)_{\Omega_0}+a_{\Omega_0}(U,U)+b_1(U,u_1,U)-b_1(U,U,U) \\ & +b_1(u_1,U,U)+(DJ_\varepsilon(U),U)_{V_1'\times V_1}+(W',W)_{\Omega_1}=0, \end{aligned} \tag{3.51}$$

according to Lemma 2.4, we have

$$\begin{aligned} b_1(U,U,U) &= \frac{1}{2}\sum_{i,j=1}^{2}\int_{\Gamma_0} U_i\,|U_j|^2\,\eta_i d\Gamma \text{ and} \\ b_1(u_1,U,U) &= \frac{1}{2}\sum_{i,j=1}^{2}\int_{\Gamma_0} u_{1i}\,|U_j|^2\,\eta_i d\Gamma, \end{aligned} \tag{3.52}$$

inserting (3.52) into (3.51), it follows

$$\begin{aligned} & (U',U)_{\Omega_0}+a_{\Omega_0}(U,U)+b_1(U,u_1,U)-\frac{1}{2}\sum_{i,j=1}^{2}\int_{\Gamma_0} U_i\,|U_j|^2\,\eta_i d\Gamma \\ & +\frac{1}{2}\sum_{i,j=1}^{2}\int_{\Gamma_0} u_{1i}\,|U_j|^2\,\eta_i d\Gamma+(DJ_\varepsilon(U),U)_{V_1'\times V_1} \\ & +(W^{'},W)_{\Omega_1}=0,\ \forall\,(U,W)\in V_1\times V_2. \end{aligned} \tag{3.53}$$

Note that

$$|a_{\Omega_0}(U(t),U(t))|=\|U(t)\|_{V_1}^2,\ \forall U\in V_1, \tag{3.54}$$

and, according to (3.9),

$$\left|(DJ_\varepsilon(U(t)),U(t))_{V_1'\times V_1}\right|\le C\,\|U(t)\|_{V_1}^2,\ \forall U\in\ V_1. \tag{3.55}$$

Inserting (3.54) and (3.55) into (3.53), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U(t)\|_{V_1}^2+\|U(t)\|_{V_1}^2+C\|U(t)\|_{V_1}^2+\frac{1}{2}\frac{d}{dt}\|W(t)\|_{V_2}^2\\
\leq\ & -b_1(U,u_1,U)+\frac{1}{2}\sum_{i,j=1}^{2}\int_{\Gamma_0}U_i\left|U_j\right|^2\eta_i d\Gamma-\frac{1}{2}\sum_{i,j=1}^{2}\int_{\Gamma_0}u_{1i}\left|U_j\right|^2\eta_i d\Gamma\\
\leq\ & |b_1(U,u_1,U)|+\frac{1}{2}\left|\sum_{i,j=1}^{2}\int_{\Gamma_0}U_i\left|U_j\right|^2\eta_i d\Gamma\right|\\
&+\frac{1}{2}\left|\sum_{i,j=1}^{2}\int_{\Gamma_0}u_{1i}\left|U_j\right|^2\eta_i d\Gamma\right|,\ \forall(U,W)\in V_1\times V_2. \qquad (3.56)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U(t)\|_{V_1}^2+\|U(t)\|_{V_1}^2+C\|U(t)\|_{V_1}^2+\frac{1}{2}\frac{d}{dt}\|W(t)\|_{V_2}^2\\
\leq\ & |b_1(U,u_1,U)|+\frac{1}{2}\left|\sum_{i,j=1}^{2}\int_{\Gamma_0}U_i\left|U_j\right|^2\eta_i d\Gamma\right|\\
&+\frac{1}{2}\left|\sum_{i,j=1}^{2}\int_{\Gamma_0}u_{1i}\left|U_j\right|^2\eta_i d\Gamma\right|,\ \forall(U,W)\in V_1\times V_2. \qquad (3.57)
\end{aligned}$$

According to Lemma 2.3, we have

$$\begin{aligned}
|b_1(U(t),u_1(t),U(t))| &\leq C_1\|U(t)\|_{V_1}^{\frac{1}{2}}\|U(t)\|_{L^2(\Omega_0)}^{\frac{1}{2}}\|U(t)\|_{V_1}^{\frac{1}{2}}\|U(t)\|_{L^2(\Omega_0)}^{\frac{1}{2}}\|u_1(t)\|_{V_1}\\
&\leq C_2\|U(t)\|_{V_1}^2\|u_1(t)\|_{V_1},\ \forall U,u_1\in V_1. \qquad (3.58)
\end{aligned}$$

According to Lemma 2.5 and Hölder inequality, the following estimates hold

$$\left|\frac{1}{2}\sum_{i,j=1}^{2}\int_{\Gamma_0}U_i\left|U_j\right|^2\eta_i d\Gamma\right|\leq C\|U\|_{L^3(\Gamma_0)}^3\leq C\|U\|_{V_1}^3,\ \forall U\in V_1; \qquad (3.59)$$

$$\begin{aligned}
\frac{1}{2}\left|\sum_{i,j=1}^{2}\int_{\Gamma_0}u_{1i}\left|U_j\right|^2\eta_i d\Gamma\right| &\leq C\|u_1\|_{L^3(\Gamma_0)}\|U\|_{L^3(\Gamma_0)}\|U\|_{L^3(\Gamma_0)}\\
&\leq C\|u_1\|_{V_1}\|U\|_{V_1}^2,\ \forall u_1,\ U\in V_1. \qquad (3.60)
\end{aligned}$$

Substituting all above estimates into (3.56), and integrate over $(0,t)$ to obtain

$$\begin{aligned}
&\|U(t)\|_{V_1}^2+\|W(t)\|_{V_2}^2\\
\leq\ &\int_0^t\left(C-C_3\|u_1(s)\|_{V_1}+C\|U(s)\|_{V_1}\right)\|U(s)\|_{V_1}^2\,ds\\
\leq\ &\int_0^t\left(C-C_3\|u_1(s)\|_{V_1}+C\|U(s)\|_{V_1}\right)\left(\|U(s)\|_{V_1}^2+\|W(s)\|_{V_2}^2\right)ds
\end{aligned}$$

$$\leq \max_{t\in]0,T[}\left(C - C_3\left\|u_1(t)\right\|_{V_1} + C\left\|U(t)\right\|_{V_1}\right)\int_0^t\left(\|U(s)\|^2_{V_1} + \|W(s)\|^2_{V_2}\right)ds. \tag{3.61}$$

Using Gronwall's lemma, we get from (3.61) that

$$\|U(t)\|^2_{V_1} + \|W(t)\|^2_{V_2} \leq 0,$$

which implies that

$$(u_1, \Phi_1) = (u_2, \Phi_2). \tag{3.62}$$

Hence, the uniqueness. □

We can now state our result of convergence:

### 3.3. Convergence

**Problem 3.3** ($P_\varepsilon$). *Let us suppose that the assumptions* (2.24), (2.25) *hold. Let* $(u_\varepsilon, \Phi_\varepsilon)$ *be a solution to the following variational inequality problem*

$$\begin{cases} (u'_\varepsilon, v - u_\varepsilon)_{\Omega_0} + b_1(u_\varepsilon, u_\varepsilon, v - u_\varepsilon) + a_{\Omega_0}(u_\varepsilon, v - u_\varepsilon) \\ +g\int_{\Omega_0}(|\varepsilon(v)| - |\varepsilon(u_\varepsilon)|)\,dx + (\Phi'_\varepsilon, \varphi - \Phi_\varepsilon)_{\Omega_1} + (\sigma_1, \varepsilon(\varphi - \Phi_\varepsilon))_{\Omega_1} \\ +\int_{\Gamma_0}(\sigma_1 - \sigma_0)(v - u_\varepsilon)\,d\Gamma + \int_{\Gamma_{11}} g(\varphi - \Phi_\varepsilon)\,d\Gamma \\ \geq (f_0, v - u_\varepsilon)_{\Omega_0} + (f_1, \varphi - \Phi_\varepsilon)_{\Omega_1}, \forall (v, \varphi) \in V_1 \times V_2, \\ v = \varphi \text{ in } \Gamma_0, \\ u_\varepsilon(0) = u_0 \text{ on } \Omega_0 \times (0,T), \\ \Phi_\varepsilon(0) = w'(0) = w_1 \text{ on } \Omega_1 \times (0,T), \\ u_\varepsilon = \Phi_\varepsilon = \frac{\partial w}{\partial t} = w' \text{in } \Gamma_0 \times (0,T). \end{cases} \tag{3.63}$$

Then we have the following theorem:

**Theorem 3.4.** *Assume that the hypotheses* (2.24)-(2.25) *hold for* $\varepsilon \to 0$. *Then the solution* $(u_\varepsilon, \Phi_\varepsilon)$ *of problem* (3.63) *converges to the solution* $(u, \Phi)$ *of problem* (2.14).

*Proof.* Let $(u_\varepsilon, \Phi_\varepsilon)$ be a solution of problem (3.63), for $v \in V_1$ and $\varphi \in V_2$, defined

$$X_\varepsilon = \int_0^T \left[\begin{array}{l} (u'_\varepsilon, v - u_\varepsilon)_{\Omega_0} + b_1(u_\varepsilon, u_\varepsilon, v - u_\varepsilon) + a_{\Omega_0}(u_\varepsilon, v - u_\varepsilon) \\ +\left(DJ_\varepsilon(u_\varepsilon), v - u_\varepsilon\right)_{V_1'\times V_1} + \int_{\Gamma_0}(\sigma_1 - \sigma_0)(v - u_\varepsilon)\,d\Gamma \\ +(\Phi'_\varepsilon, \varphi - \Phi_\varepsilon)_{\Omega_1} + (\sigma_1, \varepsilon(\varphi - \Phi_\varepsilon))_{\Omega_1} \\ +\int_{\Gamma_{11}} g(\varphi - \Phi_\varepsilon)\,d\Gamma - (f_0, v - u_\varepsilon)_{\Omega_0} - (f_1, \varphi - \Phi_\varepsilon)_{\Omega_1} \end{array}\right] dt.$$

It is easy to see that

(1) $\forall\varepsilon\ DJ_\varepsilon(u_\varepsilon)$ is monotony, and consequently $X_\varepsilon \geq 0$ for all $\varepsilon > 0$.
(2) $\liminf_{\varepsilon\to 0} \int_0^T a_{\Omega_0}(u_\varepsilon, v - u_\varepsilon)\,dt \geq \int_0^T a_{\Omega_0}(u, v - u)\,dt$, $\forall v \in V_1$.
(3) $\lim_{\varepsilon\to 0}\left(DJ_\varepsilon(u_\varepsilon), v - u_\varepsilon\right)_{V_1'\times V_1} = g\int_{\Omega_0}(|\varepsilon(v)| - |\varepsilon(u)|)\,dx,\ \forall v \in V_1.$

So, $X_\varepsilon \to X$ as $\varepsilon \to 0$, where

$$X = \int_0^T \left[ \begin{array}{l} (u', v-u)_{\Omega_0} + b_1(u,u,v-u) + a_{\Omega_0}(u,v-u) \\ +g\int_{\Omega_0}(|\varepsilon(v)| - |\varepsilon(u)|)\,dx + (\Phi', \varphi - \Phi)_{\Omega_1} \\ +(\sigma_1, \varepsilon(\varphi - \Phi))_{\Omega_1} + \int_{\Gamma_0}(\sigma_1 - \sigma_0)(v-u)\,d\Gamma \\ +\int_{\Gamma_{11}} g(\varphi - \Phi)\,d\Gamma - (f_0, v-u)_{\Omega_0} - (f_1, \varphi - \Phi)_{\Omega_1} \end{array} \right] dt.$$

Thus $(u, \Phi)$ is the solution of problem (2.14).

Now, let $(u, \Phi)$ be a solution of problem (2.14), and $(u_\varepsilon, \Phi_\varepsilon)$ be a solution of problem (3.63), if we chooses $\left(v = \frac{u_\varepsilon + u}{2}, \varphi = \frac{\Phi_\varepsilon + \Phi}{2}\right)$ as a function test in (2.14) and in (3.63), respectively, adding them up we obtain

$$\begin{aligned} &-((u_\varepsilon - u)', u_\varepsilon - u)_{\Omega_0} - (b_1(u_\varepsilon, u_\varepsilon, u_\varepsilon - u) - b_1(u,u,u_\varepsilon - u)) \\ &- a_{\Omega_0}(u_\varepsilon - u, u_\varepsilon - u) + 2g\textstyle\int_{\Omega_0}\left(\left|\varepsilon\left(\frac{u_\varepsilon+u}{2}\right)\right| - |\varepsilon(u)|\right)dx \\ &- \left((DJ_\varepsilon(u_\varepsilon), u_\varepsilon - u)_{V_1' \times V_1}\right) \\ &- ((\Phi_\varepsilon - \Phi)', \Phi_\varepsilon - \Phi)_{\Omega_1} - 2(\sigma_1, \varepsilon(\Phi_\varepsilon - \Phi))_{\Omega_1} \geq 0, \end{aligned}$$

$$\forall (u, \Phi), (u_\varepsilon, \Phi_\varepsilon) \in V_1 \times V_2, \tag{3.64}$$

using, the previous calculation, and the fact that

$$2g\int_{\Omega_0}\left(\left|\varepsilon\left(\frac{u_\varepsilon + u}{2}\right)\right| - |\varepsilon(u)|\right)dx = 2\left(J\left(\frac{u_\varepsilon+u}{2}\right) - J(u)\right),\ \forall u_\varepsilon, u \in V_1,$$

and the proprieties of trilinear function $b_1$ (see Lemma 2.4), integrating (3.64) over $(0, t)$, we get

$$\begin{aligned} &\tfrac{1}{2}\|(u_\varepsilon - u)(t)\|_{V_1}^2 - \tfrac{1}{2}\|(u_{0\varepsilon} - u_0)\|_{V_1}^2 \\ &+ \left((DJ_\varepsilon(u_\varepsilon), u_\varepsilon - u)_{V_1' \times V_1}\right) - 2\left(J\left(\tfrac{u_\varepsilon+u}{2}\right) - J(u)\right) \\ &+ \textstyle\int_0^t \left(C_2 + C_3\|(u_\varepsilon - u)(s)\|_{V_1} + C_4\|u_\varepsilon(s)\|_{V_1} + 1\right)\|(u_\varepsilon - u)(s)\|_{V_1}^2\,ds \\ &+ \tfrac{1}{2}\|(\Phi_\varepsilon - \Phi)(t)\|_{V_2}^2 - \tfrac{1}{2}\|(\Phi_{0\varepsilon} - \Phi_0)\|_{V_2}^2 \\ &+ 2c_6\textstyle\int_0^t \|\sigma_1\|_{V_2}^2\|(\Phi_\varepsilon(s) - \Phi(s))\|_{V_2}^2\,ds \leq 0,\ \forall (u, \Phi), (u_\varepsilon, \Phi_\varepsilon) \in V_1 \times V_2. \end{aligned}$$

Therefore

$$\begin{aligned} &\|(u_\varepsilon - u)(t)\|_{V_1}^2 + \|(\Phi_\varepsilon - \Phi)(t)\|_{V_2}^2 \\ \leq\ &\|(u_{0\varepsilon} - u_0)\|_{V_1}^2 + \|(\Phi_{0\varepsilon} - \Phi_0)\|_{V_2}^2 + 2\left((DJ_\varepsilon(u_\varepsilon), u_\varepsilon - u)_{V_1' \times V_1}\right) \\ &- 2\left(J\left(\tfrac{u_\varepsilon+u}{2}\right) - J(u)\right),\ \forall (u, \Phi), (u_\varepsilon, \Phi_\varepsilon) \in V_1 \times V_2, \end{aligned}$$

since $u(0) = u_0$ and $u_\varepsilon(0) = u_0$ on $L^2(\Omega_0)$, $u_\varepsilon \in V_1$ and that $\Phi(0) = w_1$ and $\Phi_\varepsilon(0) = w_1$ on $\left(L^2(\Omega_1)\right)^2$, $\Phi_{0\varepsilon} \in V_2$, it result

$$\begin{aligned} &\|(u_\varepsilon - u)(t)\|_{V_1}^2 + \|(\Phi_\varepsilon - \Phi)(t)\|_{V_2}^2 \\ \leq\ &2\left((DJ_\varepsilon(u_\varepsilon), u_\varepsilon - u)_{V_1' \times V_1}\right) - 2\left(J\left(\tfrac{u_\varepsilon+u}{2}\right) - J(u)\right), \end{aligned}$$

$$\forall (u, \Phi), (u_\varepsilon, \Phi_\varepsilon) \in V_1 \times V_2,$$

by passing to the limit when $\varepsilon \to 0$ in the last inequality, yields

$$\|(u_\varepsilon - u)(t)\|_{V_1}^2 + \|(\Phi_\varepsilon - \Phi)(t)\|_{V_2}^2 \leq 0.$$

From where, the solution $(u_\varepsilon, \Phi_\varepsilon)$ of problem (3.63) converges to the solution $(u, \Phi)$ of problem (2.14). □

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