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INVARIANT NULL RIGGED HYPERSURFACES OF INDEFINITE NEARLY α-SASAKIAN MANIFOLDS

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Abstract. We introduce invariant rigged null hypersurfaces of indefinite almost contact manifolds, by paying attention to those of indefinite nearly α-Sasakian manifolds. We prove that, under some conditions, there exist leaves of the integrable screen distribution of the ambient manifolds admitting nearly α-Sasakian structures.

1. Introduction

Indefinite nearly α-Sasakian manifolds were investigated in [14] as indefinite almost contact metric structure $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ such that the Levi-Civita connection $\overline{\nabla}$ satisfies

$$(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} + (\overline{\nabla}_{\overline{Y}}\overline{\phi})\overline{X} = \alpha\{2\overline{g}(\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{X} - \eta(\overline{X})\overline{Y}\}$$

for any vector fields $\overline{X}$, $\overline{Y}$ on $\overline{M}$. Such a class of manifolds contains the classes of indefinite nearly cosymplectic and indefinite nearly Sasakian manifolds. In [14], the authors studied certain null spaces in indefinite nearly α-Sasakian manifolds, in particular, the quasi-generalized CR-null submanifolds. These null submanifolds were first defined in [13] in nearly Sasakian settings.

The theory of null submanifolds of a semi-Riemannian manifold is one of the most important topics of differential geometry. More precisely, null hypersurfaces appear in general relativity as models of different types of black hole horizons (see [2], [3] and [16]). The study of non-degenerate submanifolds of semi-Riemannian manifolds has many similarities with the Riemannian submanifolds. However, in case the induced metric on the submanifold is degenerate, the study becomes more difficult and it is strikingly different from the study of nondegenerate submanifolds [2]. Some of the pioneering work on null geometry is due to Duggal-Sahin [3], Duggal-Bejancu [2] and Kupeli [5]. Such work motivated many other researchers to invest in the study of null

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submanifolds, for example, [2], [5], [6–12], [13], [14] and many more references therein.

We are also interested in the technique that led to the construction of a Riemannian metric on a null hypersurface. This was developed in [4] in Lorentzian case and it is based on the arbitrary choice of a transverse vector field, called *rigging field*, from which the authors constructed a null section, called *rigged field* and a screen distribution. This technique improves the dependency of the geometric objects only on the choice of a unique object, namely, the rigging field and it also introduces a Riemannian structure coupled with it and useful to study the null hypersurface (see [15], [17] and references therein for more details). The advantage of the rigging technique is not only a lower number of arbitrary elections. An adequate choice of the rigging (for example conformal or closed) gives rise to a screen distribution and a null section with good properties, which allows us to use the possible symmetries of the ambient manifold. The second advantage is that all usual tensors of a null submanifold derived from a rigging are related, unlike if we choose a screen, a transversal screen and a null section independently. The geometry of null rigging hypersurface was studied in [4], [15] and references therein. In this paper, we investigate the effect of rigging in the null hypersurfaces of almost contact structures by paying attention to those of the indefinite nearly α-Sasakian structures.

This paper is organized as follows. In Section 2 we give some basic definitions and properties of null hypersurfaces in semi-Riemannian settings. In Section 3, we introduce the null hypersurfaces in indefinite nearly α-Sasakian manifolds, supported by some examples. Section 4 deals with the invariant rigged null hypersurfaces M of indefinite nearly α-Sasakian manifolds. Among other results, we prove, under some conditions, that there exist leaves of a certain integrable screen distribution of such null hypersurfaces which have indefinite nearly α-Sasakian structures.

2. Preliminaries

Let $(\overline{M}, \overline{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold with index s, $0 < s < 2n+1$, and let (M, g) be a null hypersurface of $\overline{M}$ with $g = \overline{g}_{|M}$. It is well known that the normal bundle $TM^{\perp}$ of the null hypersurface M is a vector subbundle of TM of rank 1. A complementary vector bundle $S(TM)$ of $TM^{\perp}$ in TM is a rank $(2n-1)$ non-degenerate distribution over M, called a *screen distribution* on M, such that

$$TM = S(TM) \perp TM^{\perp}, \tag{2.1}$$

where $\perp$ denotes the orthogonal direct sum. Existence of $S(TM)$ is secured provided M is paracompact.

Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote by $\Gamma(\Xi)$ the set of smooth sections of the vector bundle Ξ.

A null hypersurface with a specific screen distribution is denoted by $(M, g, S(TM))$. We know [2] that for such a triplet, there exists a unique rank 1 vector subbundle $\mathrm{tr}(TM)$ of $T\overline{M}$ over M, such that for any non-zero section E of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $\mathrm{tr}(TM)$ on $\mathcal{U}$ satisfying

$$\overline{g}(N, E) = 1, \text{ and } \overline{g}(N, N) = \overline{g}(N, W) = 0, \ \forall\, W \in \Gamma(S(TM)|_{\mathcal{U}}). \tag{2.2}$$

Then, $T\overline{M}$ is decomposed as follows:

$$T\overline{M} = TM \oplus \mathrm{tr}(TM) = (TM^{\perp} \oplus \mathrm{tr}(TM)) \perp S(TM). \tag{2.3}$$

We call $\mathrm{tr}(TM)$ and N the transversal vector bundle and the null transversal vector field of M with respect to $S(TM)$, respectively. The local Gauss and Weingarten formulas are, for any X, $Y \in \Gamma(TM|_{\mathcal{U}})$,

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.4}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.5}$$

$$\text{and } \nabla_X PY = \nabla^*_X PY + C(X, PY)E, \tag{2.6}$$

$$\nabla_X E = -A^*_E X - \tau(X)E, \tag{2.7}$$

where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{M}$ and P is the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (2.1). Also, ∇ and ∇^* are the linear connections, B and C are the local second fundamental forms, A_N and A^*_E are the shape operators on TM and $S(TM)$, respectively, and τ is a 1-form on TM. From the fact that $B(X, Y) = \overline{g}(\overline{\nabla}_X Y, E)$, we know that B is independent of the choice of a screen distribution and satisfies $B(\cdot, E) = 0$. Unfortunately, the induced connection ∇ on TM is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\theta(Z) + B(X, Z)\theta(Y), \tag{2.8}$$

where θ is a differential 1-form locally defined on M by $\theta(\cdot) := \overline{g}(N, \cdot)$. However, the connection ∇^* on $S(TM)$ is metric. The above two local second fundamental forms of M and $S(TM)$ are related to their shape operators by

$$B(X, PY) = g(A^*_E X, PY), \ \ g(A^*_E X, N) = 0,$$
$$C(X, PY) = g(A_N X, PY), \ \ g(A_N X, N) = 0,$$

for any $X, Y \in \Gamma(TM|_{\mathcal{U}})$.

3. Null hypersurfaces of indefinite nearly α-Sasakian manifolds

Let $\overline{M}$ be a $(2n+1)$-dimensional manifold endowed with an almost contact structure $(\overline{\phi}, \xi, \eta)$, i.e., $\overline{\phi}$ is a tensor field of type $(1, 1)$, ξ is a vector field, and η is a 1-form satisfying [1]

$$\overline{\phi}^2 = -\mathbb{I} + \eta \otimes \xi, \ \ \eta(\xi) = 1. \tag{3.1}$$

It follows that $\overline{\phi}\xi = 0$, $\eta\circ\overline{\phi} = 0$ and $\mathrm{rank}(\overline{\phi}) = 2n$. Then $(\overline{\phi}, \xi, \eta, \overline{g})$ is called an indefinite almost contact metric structure on $\overline{M}$ if $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on $\overline{M}$ and $\overline{g}$ is a semi-Riemannian metric on $\overline{M}$ such that [13],

$$\overline{g}(\overline{\phi}\,\overline{X}, \overline{\phi}\,\overline{Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\,\eta(\overline{Y}) \tag{3.2}$$

for any vector fields $\overline{X}$, $\overline{Y}$ on $\overline{M}$. It follows that the $(1,1)$-tensor field ϕ is skew-symmetric and $\eta(\overline{X}) = \overline{g}(\xi, \overline{X})$. If, moreover [14],

$$(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} + (\overline{\nabla}_{\overline{Y}}\overline{\phi})\overline{X} = \alpha\{2\overline{g}(\overline{X}, \overline{Y})\xi - \eta(\overline{Y})\overline{X} - \eta(\overline{X})\overline{Y}\} \tag{3.3}$$

for any vector fields $\overline{X}$, $\overline{Y}$ on $\overline{M}$, where $\overline{\nabla}$ is the Levi-Civita connection for the semi-Riemannian metric $\overline{g}$, we call $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ an *indefinite nearly α-Sasakian manifold*. More precisely, $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ is called nearly cosymplectic (resp. nearly Sasakian [13] and references therein) manifold if $\alpha = 0$ (resp. $\alpha = 1$).

Example 3.1. We consider the three dimensional manifold $\overline{M} = \{(x, y, z) \in \mathbb{R}^3 : y \neq 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. We choose the vector fields

$$e_1 = e^x\frac{\partial}{\partial y},\ \ e_2 = e^x\left\{\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}\right\},\ \ e_3 = \frac{\partial}{\partial z},$$

which are linearly independent at each point of $\overline{M}$. Let $\overline{g}$ be the semi-Riemannian metric defined by $\overline{g}(e_1, e_1) = \overline{g}(e_2, e_2) = -\overline{g}(e_3, e_3) = -1$, $\overline{g}(e_i, e_j) = 0$, $i \neq j = 1, 2, 3$, that is, the form of the metric becomes

$$\overline{g} = e^{-2x}\left\{-(1 + 4y^2e^{2x})dx^2 - dy^2 + dz^2\right\},$$

which is a semi-Riemannian metric. Let η be the 1-form defined by $\eta(X) = \overline{g}(X, \xi)$ for any $X \in \Gamma(TM)$ with $\xi = e_3$. Let $\overline{\phi}$ be the (1,1)-tensor field defined by $\overline{\phi}e_1 = e_2$, $\overline{\phi}e_2 = -e_1$, $\overline{\phi}\xi = 0$. Using the linearity of $\overline{\phi}$ and $\overline{g}$, we have

$$\overline{\phi}^2X = -X + \eta(X)\xi,\ \ \overline{g}(\overline{\phi}X, \overline{\phi}Y) = \overline{g}(X, Y) - \eta(X)\eta(Y)$$

for any $X, Y \in \Gamma(TM)$. Then we have $[e_1, e_2] = -e^xe_1 + 2e^{2x}\xi$, $[e_1, \xi] = [e_2, \xi] = 0$. The connection $\overline{\nabla}$ of the metric tensor $\overline{g}$ is given by Koszul's formula which is given by

$$\begin{aligned} 2\overline{g}(\overline{\nabla}_XY, W) = {} & X(\overline{g}(Y, W)) + Y(\overline{g}(X, W)) - W(\overline{g}(X, Y)) \\ & - \overline{g}(X, [Y, W]) - \overline{g}(Y, [X, W]) + \overline{g}(W, [X, Y]). \end{aligned}$$

Using Koszul's formula, we get the following

$$\begin{aligned} &\overline{\nabla}_{e_1}e_1 = e^xe_2,\ \overline{\nabla}_{e_1}e_2 = -e^xe_1 + e^{2x}\xi,\ \overline{\nabla}_{e_1}\xi = e^{2x}e_2,\ \overline{\nabla}_{e_2}e_1 = -e^{2x}\xi, \\ &\overline{\nabla}_{e_2}e_2 = 0,\ \overline{\nabla}_{e_2}\xi = -e^{2x}e_1,\ \overline{\nabla}_{\xi}e_1 = e^{2x}e_2,\ \overline{\nabla}_{\xi}e_2 = -e^{2x}e_1,\ \ \overline{\nabla}_{\xi}\xi = 0, \end{aligned}$$

and also

$$\begin{aligned} &(\overline{\nabla}_{e_1}\overline{\phi})e_1 = e^{2x}\xi,\ (\overline{\nabla}_{e_1}\overline{\phi})\xi = e^{2x}e_1,\ (\overline{\nabla}_{e_1}\overline{\phi})e_2 = 0,\ (\overline{\nabla}_{e_2}\overline{\phi})e_1 = 0, \\ &(\overline{\nabla}_{e_2}\overline{\phi})e_2 = e^{2x}\xi,\ (\overline{\nabla}_{e_2}\overline{\phi})\xi = e^{2x}e_2,\ (\overline{\nabla}_{\xi}\overline{\phi})e_1 = 0,\ (\overline{\nabla}_{\xi}\overline{\phi})e_2 = 0. \end{aligned}$$

Therefore, the manifold $\overline{M}$ is a nearly α-Sasakian manifold with $\alpha = -e^{2x}$.

The concepts of nearly cosymplectic and nearly Sasakian manifolds were defined in [1] for Riemannian metric. We adapt the same definitions in the case of semi-Riemannian settings.

Let Ω be the fundamental 2-form of $\overline{M}$ defined by

$$\Omega(\overline{X}, \overline{Y}) = \overline{g}(\overline{X}, \overline{\phi}\,\overline{Y})$$

for any vector fields $\overline{X}$, $\overline{Y}$ on $\overline{M}$. Replacing $\overline{Y}$ by ξ in (3.3) we obtain

$$\overline{\nabla}_{\overline{X}}\xi + \overline{\phi}(\overline{\nabla}_{\xi}\overline{\phi})\overline{X} = -\alpha\overline{\phi}\,\overline{X}, \quad \forall\, \overline{X} \in \Gamma(T\overline{M}). \tag{3.4}$$

Let $\overline{H}$ be the (1,1)-tensor on $\overline{M}$ given by

$$\overline{H}\,\overline{X} = \overline{\phi}(\overline{\nabla}_{\xi}\overline{\phi})\overline{X} \tag{3.5}$$

for any $\overline{X} \in \Gamma(T\overline{M})$ such that (3.4) reduces to

$$\overline{\nabla}_{\overline{X}}\xi = -\overline{H}\,\overline{X} - \alpha\overline{\phi}\,\overline{X}. \tag{3.6}$$

A straightforward calculation shows that the linear operator $\overline{H}$ satisfies the following properties:

$$\overline{H}\,\overline{\phi} + \overline{\phi}\,\overline{H} = 0, \quad \overline{H}\xi = 0, \quad \eta \circ \overline{H} = 0 \quad \text{and} \tag{3.7}$$

$$\overline{g}(\overline{H}\,\overline{X}, \overline{Y}) = -\overline{g}(\overline{X}, \overline{H}\,\overline{Y}) \tag{3.8}$$

for any $X, Y \in \Gamma(TM)$. It is easy to see that $\overline{\nabla}_{\xi}\xi = 0$ and the relation (3.8) means that $\overline{H}$ is skew-symmetric.

Note that, for any $X \in \Gamma(TM)$,

$$\overline{H}\,X = \overline{H}^{T}X + \overline{H}^{N}X, \tag{3.9}$$

where $\overline{H}^{T}X$ and $\overline{H}^{N}X$ are the tangential and normal components of $\overline{H}X$, respectively.

Moreover, $\overline{M}$ is indefinite α-Sasakian if and only if $\overline{H}$ vanishes identically on $\overline{M}$ (see [1]). As an example, we have the following.

Example 3.2. Let $\mathbb{R}^7$ be the 7-dimensional real number space. Let us consider $\{x_i\}_{1\leq i\leq 7}$ as Cartesian coordinates on $\mathbb{R}^7$ and define with respect to the natural field of frames $\left\{\frac{\partial}{\partial x_i}\right\}$ a tensor field $\overline{\phi}$ of type $(1,1)$ by:

$$\overline{\phi}(\frac{\partial}{\partial x_1}) = -\frac{\partial}{\partial x_2}, \;\; \overline{\phi}(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_7}, \;\; \overline{\phi}(\frac{\partial}{\partial x_3}) = -\frac{\partial}{\partial x_4},$$
$$\overline{\phi}(\frac{\partial}{\partial x_4}) = \frac{\partial}{\partial x_3} + x_6\frac{\partial}{\partial x_7}, \;\; \overline{\phi}(\frac{\partial}{\partial x_5}) = -\frac{\partial}{\partial x_6}, \;\; \overline{\phi}(\frac{\partial}{\partial x_6}) = \frac{\partial}{\partial x_5}, \;\; \overline{\phi}(\frac{\partial}{\partial x_7}) = 0.$$

The 1-form η is defined by $\eta = dx_7 - x_4dx_1 - x_6dx_3$. The vector field ξ is defined by $\xi = \frac{\partial}{\partial x_7}$. It is easy to check (3.1) and thus $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on $\mathbb{R}^7$. Finally, we define a metric $\overline{g}$ on $\mathbb{R}^7$ by

$$\overline{g} = (x_4^2 - 1)dx_1^2 - dx_2^2 + (x_6^2 + 1)dx_3^2 + dx_4^2 - dx_5^2 - dx_6^2 + dx_7^2$$

$$- x_4 dx_1 \otimes dx_7 - x_4 dx_7 \otimes dx_1 + x_4 x_6 dx_1 \otimes dx_3 + x_4 x_6 dx_3 \otimes dx_1$$
$$- x_6 dx_3 \otimes dx_7 - x_6 dx_7 \otimes dx_3,$$

with respect to the natural field of frames. It is easy to check that $\overline{g}$ is a semi-Riemannian metric and $(\overline{\phi}, \xi, \eta, \overline{g})$ given above is a nearly α-Sasakian structure on $\mathbb{R}^7$ with $\alpha = 1$ and the (1,1)-tensor field $\overline{H}$ vanishes identically.

Also, for any $\overline{X}, \overline{Y} \in \Gamma(T\overline{M})$ and using (3.2) and (3.8), the Lie derivative $\mathcal{L}_\xi$ with respect to ξ is given by

$$\begin{aligned}(\mathcal{L}_\xi \overline{g})(\overline{X}, \overline{Y}) &= -\overline{g}(\overline{H}\,\overline{X}, \overline{Y}) - \alpha \overline{g}(\overline{\phi}\,\overline{X}, \overline{Y}) - \overline{g}(\overline{X}, \overline{H}\,\overline{Y}) - \alpha \overline{g}(\overline{X}, \overline{\phi}\,\overline{Y}) \\ &= 0.\end{aligned}$$

This means that in a nearly α-Sasakian manifold, the characteristic vector field ξ is $\overline{g}$-Killing. Note that, for any $\overline{X}, \overline{Y}, \overline{Z} \in \Gamma(T\overline{M})$,

$$\overline{g}((\overline{\nabla}_{\overline{Z}} \overline{\phi})\overline{X}, \overline{Y}) = -\overline{g}(\overline{X}, (\overline{\nabla}_{\overline{Z}} \overline{\phi})\overline{Y}),$$

which means that the tensor $\overline{\nabla}\,\overline{\phi}$ is skew-symmetric.

Let $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ be an indefinite nearly α-Sasakian manifold and (M, g) be a null hypersurface of $(\overline{M}, \overline{g})$, tangent to the vector field ξ, i.e., $\xi \in TM$.

If E is a local section of $TM^\perp$, then $\overline{g}(\overline{\phi}E, E) = 0$, and $\overline{\phi}E$ is tangent to M. Thus $\overline{\phi}(TM^\perp)$ is a distribution on M of rank 1.

Choose a screen distribution $S(TM)$ of M such that it contains $\overline{\phi}E$ and ξ. Using (3.2), we have

$$\overline{g}(\overline{\phi}N, E) = -\overline{g}(N, \overline{\phi}E) = 0, \;\; \overline{g}(N, \overline{\phi}N) = 0,$$

where $N \in \Gamma(\mathrm{tr}(TM))$. This implies that $\overline{\phi}N$ is also tangent to M and belongs to $S(TM)$. From (3.1), we have $\overline{g}(\overline{\phi}\,N, \overline{\phi}E) = 1$. Therefore,

$$\overline{\phi}(TM^\perp) \oplus \overline{\phi}(\mathrm{tr}(TM)),$$

is a non-degenerate vector subbundle of $S(TM)$ of rank 2.

Therefore, we have the following.

Lemma 3.3. *Let $(M, g, S(TM))$ be a null hypersurface of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ with $\xi \in TM$. If the screen distribution $S(TM)$ of M contains $\overline{\phi}E$ and ξ, then $\overline{g}$ cannot be a Lorentzian metric.*

Proof. Assume that $\overline{g}$ is Lorentzian metric. Then in M, there is a unique null direction, say E. Since $\xi \in TM$, we have

$$\overline{g}(\overline{\phi}E, \overline{\phi}E) = \overline{g}(E, E) - \eta(E)\eta(E) = 0.$$

This means that $\overline{\phi}E$ is also a null vector in M. Then necessarily $\overline{\phi}E = fE$ for a certain function f. Using (2.2), we have $f = 0$. Likewise, $\overline{\phi}N = 0$. This contradicts the fact that E and N are non-zero null vectors and completes the proof. $\square$

Since $S(TM)$ was chosen so that ξ belongs to $S(TM)$ and $\overline{g}(\overline{\phi}\,\cdot,\xi)=0$, there exists a non-degenerate distribution D_0 of rank $(2n-4)$ on M such that

$$S(TM)=\{\overline{\phi}(TM^{\perp})\oplus\overline{\phi}(\mathrm{tr}(TM))\}\perp D_0\perp\mathbb{R}\xi, \tag{3.10}$$

where $\mathbb{R}\xi$ is the distribution spanned by ξ. Then the distribution D_0 is invariant respect $\overline{\phi}$, i.e., $\overline{\phi}(D_0)=D_0$. From (2.1) and (3.10), we obtain the decomposition

$$\begin{aligned} TM&=\{\overline{\phi}(TM^{\perp})\oplus\overline{\phi}(\mathrm{tr}(TM))\}\perp D_0\perp\mathbb{R}\xi\perp TM^{\perp},\\ T\overline{M}&=\{\overline{\phi}(TM^{\perp})\oplus\overline{\phi}(\mathrm{tr}(TM))\}\perp D_0\perp\mathbb{R}\xi\perp(TM^{\perp}\oplus\mathrm{tr}(TM)). \end{aligned}$$

Let us consider the distributions on M, $D=TM^{\perp}\perp\overline{\phi}(TM^{\perp})\perp D_0$ and $D^{'}=\overline{\phi}(\mathrm{tr}(TM))$. Then D is invariant respect $\overline{\phi}$, i.e., $\overline{\phi}(D)=D$ and

$$TM=(D\oplus D^{'})\perp\mathbb{R}\xi. \tag{3.11}$$

Now, we consider the local null vector fields $U=-\overline{\phi}N$ and $V=-\overline{\phi}E$. Then, from (3.11), a vector field X on M is decomposed as

$$X=RX+QX+\eta(X)\xi,\quad QX=\omega(X)U, \tag{3.12}$$

where R and Q are the projection morphisms of TM into D and $D^{'}$, respectively, and ω is a differential 1-form on M defined by

$$\omega(\cdot)=\overline{g}(\cdot,V). \tag{3.13}$$

Now applying $\overline{\phi}$ to (3.12) and we note that $\overline{\phi}^{2}N=-N$, and using (3.1) we obtain

$$\overline{\phi}X=\phi X+\omega(X)N,\ \ \forall\, X\in\Gamma(TM), \tag{3.14}$$

where ϕ is a tensor field of type $(1,1)$ defined on M given by $\phi X:=\overline{\phi}RX$, and we also have

$$\phi^2X=-X+\eta(X)\xi+\omega(X)U. \tag{3.15}$$

Now applying ϕ to ϕ^2X and since $\phi U=0$, we obtain $\phi^3+\phi=0$, which shows that ϕ is an f-structure of constant rank [18]. By using (3.1), we derive, for any X, $Y\in\Gamma(TM)$,

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y)-v(X)\omega(Y)-v(Y)\omega(X), \tag{3.16}$$

where v is a 1-form locally defined on M by

$$v(\cdot)=g(U,\cdot). \tag{3.17}$$

For any X, $Y\in\Gamma(TM)$,

$$g(X,\phi Y)+g(\phi X,Y)=-\{\theta(X)\omega(Y)+\theta(Y)\omega(X)\}. \tag{3.18}$$

Also,

$$\begin{aligned} \overline{g}(h(\phi X,Y),E)=&-\overline{g}(h(X,\phi Y),E)-(\nabla_X\omega)Y-(\nabla_Y\omega)X+2\overline{g}(\overline{\phi}h(X,Y),E)\\ &-\tau(X)\omega(Y)-\tau(Y)\omega(X). \end{aligned} \tag{3.19}$$

Since $h = B \otimes N$, we have $\overline{g}(\overline{\phi}h(\cdot,\cdot), E) = B(\cdot,\cdot)\overline{g}(\overline{\phi}N, E) = 0$, and therefore

$$\begin{aligned} B(\phi X, Y) = & -B(X, \phi Y) - \{(\nabla_X \omega)Y + (\nabla_Y \omega)X \\ & + \tau(X)\omega(Y) + \tau(Y)\omega(X)\}. \end{aligned} \tag{3.20}$$

Therefore, we have the following useful identities.

Proposition 3.4. *Let $(M, g, S(TM))$ be a null hypersurface of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ with $\xi \in TM$. Then, we have*

$$\nabla_X \xi = -\overline{H}^T X - \alpha\phi X, \tag{3.21}$$

$$B(X, \xi) = -\overline{g}(\overline{H}X, E) - \alpha\,\omega(X), \tag{3.22}$$

$$\begin{aligned} (\nabla_X \phi)Y + (\nabla_Y \phi)X = & \alpha\{2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\} + \omega(X)A_N Y \\ & + \omega(Y)A_N X + 2B(X,Y)\overline{\phi}N \end{aligned} \tag{3.23}$$

for any vector fields X and Y on M.

Proof. For any $X \in \Gamma(TM)$,

$$\nabla_X \xi + B(X, \xi)N = -\overline{H}^T X - \overline{H}^N X - \alpha\phi X - \alpha\omega(X)N.$$

Comparing the tangential and normal components, we obtain (3.21) and

$$B(X, \xi)N = -\overline{H}^N X - \alpha\omega(X)N.$$

By $\overline{g}$-doting this relation by E and using the fact that $\overline{g}(\overline{H}^N \cdot, E) = \overline{g}(\overline{H}\cdot, E)$, we get (3.22). Now, for any X, $Y \in \Gamma(TM)$,

$$\begin{aligned} (\nabla_X \phi)Y = & (\overline{\nabla}_X \overline{\phi})Y - X(\omega(Y))N + \omega(Y)A_N X - \tau(X)\omega(Y)N - B(X, \phi Y)N \\ & + B(X,Y)\overline{\phi}N + \omega(\nabla_X Y)N. \end{aligned} \tag{3.24}$$

Therefore, using (3.20) and (3.24), one obtains

$$\begin{aligned} & (\nabla_X \phi)Y + (\nabla_Y \phi)X \\ = & (\overline{\nabla}_X \overline{\phi})Y + (\overline{\nabla}_Y \overline{\phi})X - \{X(\omega(Y)) + Y(\omega(X))\}N \\ & + \omega(X)A_N Y + \omega(Y)A_N X - \{\tau(X)\omega(Y) + \tau(Y)\omega(X)\}N \\ & - \{B(X, \phi Y) + B(Y, \phi X)\}N + \{\omega(\nabla_X Y) + \omega(\nabla_Y X)\}N \\ & + 2B(X,Y)\overline{\phi}N \\ = & \alpha\{2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\} + \omega(X)A_N Y + \omega(Y)A_N X \\ & + 2B(X,Y)\overline{\phi}N, \end{aligned}$$

which completes the proof. □

Note that, using (3.12), the decomposition in (3.9) is explicitly defined on the null hypersurface M as follows: for any $X \in \Gamma(TM)$,

$$\overline{H}X = \overline{H}^T X + \overline{H}^N X,$$

where $\overline{H}^T X := \overline{H}RX$ and $\overline{H}^N X := \overline{H}QX = \omega(X)\overline{H}U$.

Now using (3.21) and (3.22), the Lie derivative of g with respect to the characteristic vector field ξ is given by, for any X, $Y \in \Gamma(TM)$,

$$(\mathcal{L}_\xi g)(X,Y) = -\{\overline{g}(\overline{H}X,E) + \alpha\omega(X)\}\theta(Y) - \{\overline{g}(\overline{H}Y,E) + \alpha\omega(Y)\}\theta(X) \\ - \overline{g}(\overline{H}^T X,Y) - \overline{g}(X,\overline{H}^T Y) - \alpha\{g(\phi X,Y) + g(X,\phi Y)\}. \tag{3.25}$$

Using (3.18), the relation (3.25) becomes,

$$(\mathcal{L}_\xi g)(X,Y) = -\overline{g}(\overline{H}X,E)\theta(Y) - \overline{g}(\overline{H}Y,E)\theta(X) \\ - \overline{g}(\overline{H}^T X,Y) - \overline{g}(X,\overline{H}^T Y). \tag{3.26}$$

Therefore we have the following.

Lemma 3.5. *Let $(M,g,S(TM))$ be a null hypersurface of an indefinite nearly α-Sasakian manifold $(\overline{M},\overline{\phi},\xi,\eta,\overline{g})$ with $\xi \in TM$. If $\overline{H}U = 0$, then*:

(i) *The characteristic vector field ξ is Killing on M if and only if for any $X \in \Gamma(TM)$, $\overline{H}X$ does not have a component in the direction of* $\mathrm{tr}(TM)$.

(ii) *If the screen distribution $S(TM)$ is integrable, the line bundle $\mathbb{R}\xi$ defined in* (3.11) *is a g-Killing distribution as a subbundle of $S(TM)$.*

4. Invariant null rigging hypersurfaces

In this section, we study invariant null rigging hypersurface of indefinite nearly α-Sasakian manifolds.

Let $(M,g,S(TM))$ be a null hypersurface of an indefinite nearly α-Sasakian manifold $(\overline{M},\overline{\phi},\xi,\eta,\overline{g})$ of index $q \in \{1,2,\ldots,2n\}$ with $\xi \in TM$.

The null hypersurface M is said to be *invariant* in $\overline{M}$ [11] if M is *tangent to the structure vector field* ξ and, for any $X \in \Gamma(TM)$,

$$\overline{\phi}X \in \Gamma(TM),$$

that is,

$$\overline{\phi}X = \phi X. \tag{4.1}$$

As an example of invariant hypersurfaces, we have the following.

Example 4.1. Let M be a hypersurface of $(\mathbb{R}^7,\overline{\phi},\xi,\eta,\overline{g})$ in Example 3.2 defined as $M = \{(x_1,\ldots,x_7) \in \mathbb{R}^7 : x_5 = x_4\}$. Thus, the tangent space TM is spanned by $\{e_i\}_{1\le i\le 6}$, where

$$e_1 = \frac{\partial}{\partial x_1},\ e_2 = \frac{\partial}{\partial x_2},\ e_3 = \frac{\partial}{\partial x_3},\ e_4 = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5},\ e_5 = \frac{\partial}{\partial x_6},\ e_6 = \xi$$

and the 1-dimensional distribution $TM^\perp$ of rank 1 is spanned by E, where

$$E = \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}.$$

It follows that $TM^{\perp} \subset TM$. Then M is a 6-dimensional lightlike hypersurface of $\mathbb{R}^7$. Also, the transversal bundle $\mathrm{tr}(TM)$ is spanned by

$$N = \frac{1}{2}(\frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5}).$$

On the other hand, by using the almost contact structure of $\mathbb{R}^7$ and also by taking into account of the decomposition (3.10), the distribution D_0 is spanned by $\{F, \overline{\phi}F\}$, where $F = e_2$, $\overline{\phi}F = e_1 + x_4\xi$ and the distributions $\mathbb{R}\xi$, $\overline{\phi}(TM^{\perp})$ and $\overline{\phi}(\mathrm{tr}(TM))$ are spanned, respectively, by ξ, $\overline{\phi}E = e_3 - e_5 + x_6\xi$ and $\overline{\phi}N = \frac{1}{2}(e_3 + e_5 + x_6\xi)$. Hence M is a null hypersurface of $\mathbb{R}^7$. Denote by $\overline{\nabla}$ the Levi-Civita connection on $\mathbb{R}^7$. Then, by straightforward calculations, the non-vanishing covariant derivative components of N and E along M are given by,

$$\begin{aligned}
\nabla_{e_1}E &= 2\overline{\nabla}_{e_1}N = -\frac{1}{2}x_4e_1 - \frac{1}{2}(x_4^2+1)\xi,\\
\nabla_{e_3}E &= 2\overline{\nabla}_{e_3}N = -\frac{1}{2}x_6e_1 - \frac{1}{2}x_4x_6\xi,\\
\nabla_{\xi}E &= 2\overline{\nabla}_{\xi}N = \frac{1}{2}e_1 + \frac{1}{2}x_4\xi.
\end{aligned}$$

Using these equations above, the differential 1-form τ vanishes, i.e., $\tau(X) = 0$ for any $X \in \Gamma(TM)$. So, from the Gauss and Weingarten formulae, the non-vanishing components of the shape operators A_N and A^*_E are given by $A^*_E e_1 = 2A_N e_1 = \frac{1}{2}x_4e_1 + \frac{1}{2}(x_4^2+1)\xi$, $A^*_E e_3 = 2A_N e_3 = \frac{1}{2}x_6e_1 + \frac{1}{2}x_4x_6\xi$ and $A^*_E\xi = 2A_N\xi = -\frac{1}{2}e_1 - \frac{1}{2}x_4\xi$. From these relations $trA_N = 0$, i.e., the shape operator A_N is trace-free. The non-vanishing components of the local second fundamental form B are given by $B(e_1, e_1) = -x_4$, $B(e_1, e_3) = -\frac{1}{2}x_6$ and $B(e_1, e_6) = \frac{1}{2}$. It is easy to check that $B(X, \xi) = 0$ for any $X \in \Gamma(TM)$. Hence, M is an invariant null hypersurface.

Since the null hypersurface M is invariant, the 1-form ω vanishes identically on M and the relations (3.20) becomes, for any $X, Y \in \Gamma(TM)$,

$$B(\phi X, Y) = -B(X, \phi Y). \tag{4.2}$$

Using (3.15) and (4.2), one obtains

$$B(\phi X, \phi Y) = B(X, Y) + \eta(Y)\overline{g}(\overline{H}X, E). \tag{4.3}$$

Since the local fundamental form B is symmetric, the relation (4.3) leads to, for any $X, Y \in \Gamma(TM)$,

$$\overline{g}(\overline{H}X, E)Y = \overline{g}(\overline{H}Y, E)X.$$

This implies that $\overline{g}(\overline{H}X, E) = 0$. Likewise $\overline{g}(\overline{H}\phi X, V) = 0$. Therefore, we have the following.

Lemma 4.2. *Let $(M, g, S(TM))$ be an invariant null hypersurface of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then, for any*

$X \in \Gamma(TM)$,

$$\overline{H}X \in \Gamma(D). \tag{4.4}$$

Moreover, for any $X, Y \in \Gamma(TM)$,

$$B(\phi X, \phi Y) = B(X, Y). \tag{4.5}$$

Proof. The proof follows from a straightforward calculation. □

Next we introduce the concept of rigging for our null hypersurfaces by the adapting the one introduced by Gutierrez and Olea in [4] for null hypersurfaces of Lorentzian manifolds.

Definition 4.3. Let (M, g) be a null hypersurface of semi-Riemannian manifold $(\overline{M}, \overline{g})$. A rigging for M is a vector field L defined on some open set containing M such that $L_p \notin T_x M$ for any $p \in M$.

Let L be a rigging for M. Then L is a vector field over $\overline{M}$ and one can set α^L to be the 1-form $\overline{g}$-metrically equivalent to L, i.e., $\alpha^L = \overline{g}(L, \cdot)$. One sets

$$\theta^L = i^* \alpha^L \text{ and } \widetilde{g} = g + \theta^L \otimes \theta^L.$$

Here $i : M \hookrightarrow \overline{M}$ is the canonical inclusion.

Lemma 4.4. $\widetilde{g}$ *is a non-degenerate metric on* M.

Proof. Let $u \in T_p M$ such that $\widetilde{g}_p(u, v) = 0$ for every $v \in T_p M$. In particular, for $E \in TM^{\perp}$, one has $0 = \widetilde{g}_p(u, E_p) = \overline{g}(E_p, L_p)\overline{g}(u, L_p)$. Since $E \in \Gamma(TM^{\perp})$ and $\overline{g}$ is non-degenerate, $\overline{g}_p(E_p, L_p) \neq 0$ and then $\overline{g}_p(u, L_p) = 0$. Putting this together with the fact that $T_p\overline{M}|_M = \mathrm{Span}\{L_p\} \oplus T_p M$, one has $\overline{g}(u, v) = 0$ for every $v \in T_p M$, which implies that $u = 0$, since $\overline{g}_p$ is non-degenerate metric. □

The *rigged vector field* of L is the $\widetilde{g}$-metrically equivalent vector to the 1-form θ^L and it is denoted by E.

From now on $L = N$ is a null rigging N and E is the associated rigged vector field. All of them are defined in an open set containing M (thus globally on M) such that (2.1), (2.2) and (2.3) hold. In this case, we denote the 1-form θ^N by θ, that is,

$$\theta(\cdot) = i^* \alpha(\cdot), \tag{4.6}$$

with $\alpha = \alpha^N = \overline{g}(N, \cdot)$. In fact, the rigged vector field E is the unique null vector field in M such that $\overline{g}(N, E) = 1$. Moreover, E is $\widetilde{g}$-unitary.

The triple (M, g, N) is called a rigged null hypersurface and $\widetilde{g}$ the associated metric, it is a semi-Riemannian metric of index $(q - 1)$. One defines the screen distribution and the transversal bundle associated to the chosen rigging N by

$$S(TM) = \ker(\theta) \text{ and } \mathrm{tr}(TM) = \mathrm{Span}\{N\}.$$

If (M, g, N) is a rigged null hypersurface of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$, then

$$\ker(\theta) = \{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(\mathrm{tr}(TM))\} \perp D_0 \perp \mathbb{R}\xi.$$

A rigged null hypersurface (M, g, N) is said to be *closed* if the rigging N is closed, i.e., the 1-form θ is closed.

We need the relationship between the Levi-Civita connections of both g and $\widetilde{g}$ acting on vector fields in TM. Call $\widetilde{\nabla}$ the Levi-Civita connection induced on M by $\widetilde{g}$ and $\sigma = \overline{\nabla} - \widetilde{\nabla}$ which is a symmetric tensor on TM. We can suppose that the involved Lie brackets vanish. The Koszul identity leads us to write, for any X, Y, $Z \in \Gamma(TM)$,

$$\begin{aligned}2\overline{g}(\overline{\nabla}_X Y, Z) = 2\widetilde{g}(\widetilde{\nabla}_X Y, Z) &- \{\widetilde{\nabla}_X(\theta\otimes\theta)(Y, Z) + \widetilde{\nabla}_Y(\theta\otimes\theta)(X, Z)\}\\ &+ \widetilde{\nabla}_Z(\theta\otimes\theta)(X, Y).\end{aligned}$$

Taking into account the following identities

$$2d\theta(X, Y) = (\widetilde{\nabla}_X\theta)Y - (\widetilde{\nabla}_Y\theta)X \quad \text{and} \quad (\mathcal{L}_E\widetilde{g})(X, Y) = (\widetilde{\nabla}_X\theta)Y + (\widetilde{\nabla}_Y\theta)X,$$

one obtains

$$\begin{aligned}2\overline{g}(\overline{\nabla}_X Y, Z) = 2\widetilde{g}(\widetilde{\nabla}_X Y, Z) &- \theta(Z)(\mathcal{L}_E\widetilde{g})(X, Y) + 2\theta(Y)d\theta(Z, X)\\ &+ 2\theta(X)d\theta(Z, Y).\end{aligned}$$

Therefore,

$$\overline{g}(\sigma_X Y, Z) = -\frac{1}{2}\theta(Z)(\mathcal{L}_E\widetilde{g})(X, Y) + \theta(Y)d\theta(Z, X) + \theta(X)d\theta(Z, Y). \tag{4.7}$$

Proposition 4.5. *Given $X, Y, Z \in \Gamma(TM)$,*

$$\sigma_X Y = -\frac{1}{2}(\mathcal{L}_E\widetilde{g})(X, Y)\xi + \theta(Y)(i_X d\theta)^{\sharp\overline{g}} + \theta(X)(i_Y d\theta)^{\sharp\overline{g}},$$

where $\overline{g}((i_X d\theta)^{\sharp\overline{g}}, Y) = d\theta(X, Y)$ and $\mathcal{L}_E$ is the Lie derivative with respect to the vector field E.

Now taking $\sigma^M = \nabla - \widetilde{\nabla}$, which is also symmetric and we have

$$\sigma - \sigma^M = B \otimes N.$$

Therefore,

$$\begin{aligned}\sigma^M_X Y = &-\frac{1}{2}(\mathcal{L}_E\widetilde{g})(X, Y)\xi + \theta(Y)(i_X d\theta)^{\sharp\overline{g}}\\ &+ \theta(X)(i_Y d\theta)^{\sharp\overline{g}} - B(X, Y)N.\end{aligned}$$

The fact that both ∇ and $\widetilde{\nabla}$ are connections on M makes the computations easier with σ^M instead of σ. The following basic identities holds. For any $W \in \Gamma(TM)$ and $X, Y, Z \in \Gamma(S(TM))$, we have the following [4]:

$$\widetilde{g}(\sigma^M_X W, X) = g(\sigma^M_X W, X) = 0, \tag{4.8}$$

$$\widetilde{g}(\sigma^M_X Y, Z) = g(\sigma^M_X Y, Z) = 0, \tag{4.9}$$

$$\widetilde{g}(\sigma^M_W E, E) = -\tau(W). \tag{4.10}$$

The null mean curvature of the null hypersurface M is related with $\widetilde{g}$ as follows. From (4.8), (4.9) and (4.10), there exists a smooth function φ on M such that

$$\sigma^M_X Y = \varphi E.$$

Thus, for any $X, Y, Z \in \Gamma(S(TM))$,

$$\widetilde{\nabla}_X Y = \nabla^*_X Y + (C(X,Y) - \varphi)E,$$

where

$$C(X,Y) - \varphi = \widetilde{g}(\widetilde{\nabla}_X Y, E) = -\widetilde{g}(\widetilde{\nabla}_X E, Y).$$

This implies that, for any $X, Y \in \Gamma(S(TM))$,

$$\sigma^M_X Y = \left\{ C(X,Y) + \widetilde{g}(\widetilde{\nabla}_X E, Y) \right\} E.$$

Lemma 4.6. *For any* $X, Y, Z \in \Gamma(S(TM))$,

$$\widetilde{g}(\sigma^M_X Y, Z) = -\widetilde{g}(Y, \sigma^M_X Z) \ \ \textit{and} \ \ \overline{g}(\sigma_X Y, Z) = -\overline{g}(Y, \sigma_X Z). \tag{4.11}$$

Proof. For any $X, Y, Z \in \Gamma(S(TM))$,

$$\begin{aligned}\widetilde{g}(\sigma^M_X Y, Z) &= \widetilde{g}(\nabla_X Y, Z) - \widetilde{g}(\widetilde{\nabla}_X Y, Z) \\ &= -(\nabla_X g)Y + X(g(Y,Z)) - g(Y, \nabla_X Z) - X(\widetilde{g}(Y,Z)) \\ &\quad + \widetilde{g}(Y, \widetilde{\nabla}_X Z) \\ &= -\widetilde{g}(Y, \sigma^M_X Z).\end{aligned}$$

The second equality follows from a similar calculation and completes the proof. □

The following identities are very important.

Lemma 4.7. *For any* $X, Y, Z \in \Gamma(S(TM))$,

$$\widetilde{g}(\sigma^M_X Z, Y) = -\widetilde{g}(X, \sigma^M_Y Z) \ \ \textit{and} \ \ \overline{g}(\sigma_X Z, Y) = -\overline{g}(X, \sigma_Y Z).$$

Proof. For any $X, Y, Z \in \Gamma(S(TM))$, and using the symmetry proper of σ^M and (4.11), we have

$$\widetilde{g}(\sigma^M_X Z, Y) = \widetilde{g}(\sigma^M_Z X, Y) = -\widetilde{g}(X, \sigma^M_Z Y) = -\widetilde{g}(X, \sigma^M_Y Z).$$

Similarly, one obtains the second relation. □

Using Lemmas 4.6 and 4.7, the Lie derivative $\mathcal{L}_E$ with respect to the vector field E is given by

$$\begin{aligned}(\mathcal{L}_E \widetilde{g})(X,Y) &= \widetilde{g}(\widetilde{\nabla}_X E, Y) + \widetilde{g}(X, \widetilde{\nabla}_Y E) \\ &= \overline{g}(\overline{\nabla}_X E, Y) - \widetilde{g}(\sigma^M_X E, Y) + \overline{g}(X, \overline{\nabla}_Y E) - \widetilde{g}(X, \sigma^M_Y E) \\ &= -2B(X,Y) - \widetilde{g}(\sigma^M_X E, Y) + \widetilde{g}(Y, \sigma^M_X E) \\ &= -2B(X,Y). \qquad (4.12)\end{aligned}$$

This means that M is totally geodesic if and only if E is $\widetilde{g}$-orthogonally Killing and it is totally umbilic if and only if E is $\widetilde{g}$-orthogonally conformal.

If we consider the rigging vector field N to be closed, then its rigged vector field E is also closed, so the screen distribution $S(TM)$ is integrable. In this case and using (4.7), we have, for any $X \in \Gamma(S(TM))$,

$$\overline{g}(\widetilde{\nabla}_E E, X) = -\overline{g}(\sigma_E E, X) = 2d\theta(E, X) = 0.$$

This means that $\widetilde{\nabla}_E E = 0$ and using (4.12), for any $X, Y \in \Gamma(S(TM))$,

$$-2B(X, Y) = (\mathcal{L}_E \widetilde{g})(X, Y) = 2\widetilde{g}(\widetilde{\nabla}_X E, Y).$$

Therefore,

$$\widetilde{\nabla}_X Y = \nabla^*_X Y + B(X, Y)E. \tag{4.13}$$

Moreover, since $B(X, Y) = \overline{g}(A^*_E X, Y)$, it follows

$$\widetilde{\nabla}_X E = -A^*_E X.$$

We have the almost contact version of Corollary 3.14 given in [4].

Proposition 4.8. *Let (M, g, N) be an invariant closed rigged null hypersurface of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then,*

(i) *M is totally geodesic if and only if the rigged vector field E is $\widetilde{g}$-parallel.*
(ii) *M is totally geodesic (resp. umbilical) if and only if each leaf of $S(TM)$ is totally geodesic (resp. umbilical) as a hypersurface of (M, g, N).*

In general, if the leaves of an integrable screen distribution are totally umbilical in $(\overline{M}, \overline{g})$, then M is totally umbilical. The converse does not hold and this lack of symmetric hide the geometric meaning of umbilicity in the null case. However, the converse does hold in (M, g, N), which suggests the convenience of the rigging construction.

Lemma 4.9. *Let (M, g, N) be an invariant closed rigged null hypersurface of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then,*

$$C(X, \xi) = -\theta(\overline{H}X) - \alpha v(X)$$

for any $X \in \Gamma(TM)$. Moreover,

$$\widetilde{\nabla}_X \xi = -\overline{H}X - \alpha\,\phi X + \{\theta(\overline{H}X) + \alpha v(X)\}E. \tag{4.14}$$

Proof. From (2.6), (3.21) and (4.13) and Lemma 4.2, one has, for any $X \in \Gamma(TM)$, $B(X, \xi) = 0$ and

$$\begin{aligned}\widetilde{\nabla}_X \xi = \nabla^*_X \xi &= \nabla_X \xi - C(X, \xi)E \\ &= -\overline{H}^T X - \alpha\phi X - C(X, \xi)E.\end{aligned}$$

By $\widetilde{g}$-doting this relation with E, we obtain

$$\begin{aligned}0 = \widetilde{g}(\widetilde{\nabla}_X \xi, E) &= -\widetilde{g}(\overline{H}X, E) - \alpha\widetilde{g}(\phi X, E) - C(X, \xi) \\ &= -\theta(\overline{H}X) - \alpha v(X) - C(X, \xi),\end{aligned}$$

which completes the proof. $\square$

In the same settings as in Lemma 4.9, and using (3.8) and (3.18), the Lie derivative of $\widetilde{g}$ with respect to the characteristic vector field is given by, for any $X, Y \in \Gamma(S(TM))$,

$$\begin{aligned}(\mathcal{L}_\xi \widetilde{g})(X,Y) &= \widetilde{g}(\widetilde{\nabla}_X \xi, Y) + \widetilde{g}(X, \widetilde{\nabla}_Y \xi)\\ &= -\widetilde{g}(\overline{H}X, Y) - \alpha\, \widetilde{g}(\phi X, Y) - \widetilde{g}(\overline{H}Y, X)\\ &\quad - \alpha\, \widetilde{g}(\phi Y, X) = 0.\end{aligned}$$

Therefore, we have the following.

Lemma 4.10. *Let (M, g, N) be an invariant closed rigged null hypersurface of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then, the line bundle $\mathbb{R}\xi$ is a $\widetilde{g}$-Killing distribution as a subbundle of $S(TM)$.*

Now, let (M, g, N) be an invariant closed rigged null hypersurface of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then, using (3.23) and (4.2), for any $X, Y \in \Gamma(TM)$,

$$\begin{aligned}&(\widetilde{\nabla}_X \phi)Y + (\widetilde{\nabla}_Y \phi)X\\ &= (\nabla_X \phi)Y + (\nabla_Y \phi)X - \sigma^M_X \phi Y + \phi \sigma^M_X Y - \sigma^M_Y \phi X + \phi \sigma^M_Y X\\ &= \alpha\{2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\} + 2B(X,Y)\overline{\phi}N\\ &\quad - \{C(X, \phi Y) + C(Y, \phi X)\}E + 2\{C(X,Y) - B(X,Y)\}\overline{\phi}E.\end{aligned}$$

On the other hand,

$$(\widetilde{\nabla}_X \phi)Y + (\widetilde{\nabla}_Y \phi)X = (\nabla^*_X \phi)Y + (\nabla^*_Y \phi)X - 2B(X,Y)\overline{\phi}E.$$

Therefore, for any $X, Y \in \Gamma(S(TM))$,

$$\begin{aligned}(\nabla^*_X \phi)Y + (\nabla^*_Y \phi)X &= \alpha\{2g(X,Y)\xi - \eta(Y)X - \eta(X)Y\} + 2B(X,Y)\overline{\phi}N\\ &\quad - \{C(X, \phi Y) + C(Y, \phi X)\}E + 2C(X,Y)\overline{\phi}E. \qquad (4.15)\end{aligned}$$

We have the following.

Theorem 4.11. *Let (M, g, N) be an invariant closed rigged null hypersurface of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Assume that the smooth* 1*-form v defined in* (3.17) *vanishes identically on a chosen parallel screen distribution $S(TM)$. Then M is totally geodesic if and only if each leaf of $S(TM)$, immersed in (M, g, N) as a submanifold, has a nearly α-Sasakian structure.*

Proof. The proof follows from the relation (4.15) and the fact that if $v = 0$ on $S(TM)$, then for any $X \in \Gamma(S(TM))$, $\phi X \in \Gamma(S(TM))$. □

The notion of totally geodesic or umbilic hypersurface also has sense in the degenerate case and they do not depend on the choice of the null section neither the screen distribution. Indeed, M is totally geodesic if $B = 0$ and totally umbilical if $B = \rho\, \overline{g}$ for certain $\rho \in C^\infty(M)$. If the function $\rho \neq 0$, M is said to be *proper totally umbilical.*

If the null hypersurface M with $\xi \in TM$ is totally umbilical, then

$$\rho = B(\xi, \xi) = -\overline{g}(\overline{H}\xi, E) - \alpha\, \omega(\xi) = 0.$$

Therefore, M is totally geodesic.

Theorem 4.12. *There exist no proper totally umbilical invariant* (*rigged*) *null hypersurfaces in nearly α-Sasakian manifolds.*

Proof. Let (M, g) be an invariant null hypersurface of a nearly α-Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Assume that M is proper totally umbilical. Then, for any $X, Y \in \Gamma(TM)$, $B(X, Y) = \rho g(X, Y)$. Using (3.16) and (4.5) and the fact that $\omega(X) = \omega(Y) = 0$, we have

$$\rho g(X, Y) = B(X, Y) = B(\phi X, \phi Y) = \rho\{g(X, Y) - \eta(X)\eta(Y)\}.$$

This leads to $\rho\eta(X)\eta(Y) = 0$, that is, $\rho = 0$, a contradiction. $\square$

It follows from Theorem 4.12 that a nearly α-Sasakian manifold $\overline{M}$ does not admit any non-totally geodesic, totally umbilical null hypersurface. In other words, totally umbilical invariant (rigged) null hypersurfaces in nearly α-Sasakian manifolds are always minimal.

Let us consider the following distribution

$$\widehat{D} = \big\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(\mathrm{tr}(TM))\big\} \perp D_0, \tag{4.16}$$

so that the tangent space of M is written

$$TM = \widehat{D} \perp \langle\xi\rangle \perp TM^{\perp}. \tag{4.17}$$

Let $\widehat{P}$ be the morphism of $S(TM)$ on $\widehat{D}$ with respect to the orthogonal decomposition of $S(TM)$ such that

$$\widehat{P}X = PX - \eta(X)\xi, \ \ \forall\, X \in \Gamma(TM).$$

It is easy to check that $\widehat{P}$ is also a projection. We have, for any $X, Y \in \Gamma(TM)$,

$$B(X, PY) = B(X, \widehat{P}Y) - \eta(Y)\{\overline{g}(\overline{H}Y, E) + \alpha\omega(Y)\},$$
$$C(X, PY) = C(X, \widehat{P}Y) + \eta(Y)\{\overline{g}(\overline{H}Y, N) + \alpha v(Y))\}.$$

Now, referring to the decomposition (4.17), for any $X \in \Gamma(TM)$, $Y \in \Gamma(\widehat{D})$, we have

$$\nabla_X Y = \widehat{\nabla}_X Y + \widehat{h}(X, Y),$$

where $\widehat{\nabla}$ is a linear connection on the bundle $\widehat{D}$ and

$$\widehat{h} : \Gamma(TM) \times \Gamma(\widehat{D}) \longrightarrow \Gamma(\langle\xi\rangle \perp TM^{\perp})$$

is $\mathcal{F}(M)$-bilinear. Let $\mathcal{U} \subset M$ be a coordinate neighborhood. Then, using (4.17), the relation (4.18) can be rewritten (locally) in the following way:

$$\begin{aligned}\nabla_X Y &= \widehat{\nabla}_X Y + g(\nabla_X Y, \xi)\xi + g(\nabla_X Y, N)E\\ &= \widehat{\nabla}_X Y + \{\overline{g}(\overline{H}X, Y) + \alpha g(\phi X, Y)\}\xi + C(X, Y)E,\end{aligned} \tag{4.18}$$

and the local expression of $\widehat{h}$ is defined as

$$\widehat{h}(X,Y) = \{\overline{g}(\overline{H}X,Y) + \alpha g(\phi X,Y)\}\xi + C(X,Y)E.$$

The tensor $\widehat{h}$ is not symmetric, in general. Using (4.18), then, the distribution $\widehat{D}$ is integrable if and only if it is symmetric, i.e.,

$$\overline{g}(\overline{H}X,Y) = -\alpha g(\phi X,Y) \text{ and } C(X,Y) = C(Y,X)$$

for any $X, Y \in \Gamma(\widehat{D})$. Therefore, we have the following.

Lemma 4.13. *Let $(M, g, S(TM))$ be an invariant null hypersurface of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then the distribution $\widehat{D}$ defined in* (4.16) *is integrable if and only if*

$$\overline{g}(\overline{H}X,Y) = -\alpha g(\phi X,Y) \text{ and } C(X,Y) = C(Y,X) \tag{4.19}$$

for any $X, Y \in \Gamma(\widehat{D})$.

On the other hand and in the case where the relation (4.13) is defined, (4.18) can be rewritten as, for any X, $Y \in \Gamma(\widehat{D})$,

$$\widetilde{\nabla}_X Y - B(X,Y)E = \nabla^*_X Y = \widehat{\nabla}_X Y + \widehat{h}(X,Y) - C(X,Y)E,$$

that is,

$$\widetilde{\nabla}_X Y = \widehat{\nabla}_X Y + \{\overline{g}(\overline{H}X,Y) + \alpha g(\phi X,Y)\}\xi + B(X,Y)E.$$

From the relation (4.19) and Lemma 4.2,

$$\overline{H}X = -\alpha\phi X + \left\{\theta(\overline{H}X) + \alpha v(X)\right\} E$$

for any $X \in \Gamma(\widehat{D})$ and using (4.14), we have

$$\widetilde{\nabla}_X \xi = 0.$$

Lemma 4.14. *Let (M, g, N) be an invariant closed rigged null hypersurface of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Then, the line bundle $\mathbb{R}\xi$ is a $\widehat{D}$-parallel.*

Also, we have the following.

Theorem 4.15. *Let (M, g, N) be an invariant closed rigged null hypersurface of an indefinite nearly α-Sasakian manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. Let M' be a leaf of an integrable distribution $\widehat{D}$. Then, on M', the $(1,1)$-tensor field $\overline{H}$ defined in* (3.5) *proportionally acts like the $(1,1)$ tensor field $\overline{\phi}_{|_{M'}}$ given in* (3.1)*, i.e.,*

$$\overline{g}(\overline{H}X, \overline{H}Y) = \alpha^2\, \overline{g}(X,Y)$$

for any $X, Y \in \Gamma(\widehat{D})$. Moreover,

$$\overline{g}(\overline{H}X, \overline{H}Y) = \alpha^2\, \{\overline{g}(X,Y) - \eta(X)\eta(Y)\}$$

for any $X, Y \in \Gamma(TM)$.

The results above also hold when the screen distribution is conformal. The latter means that the shape operators A_N and A^*_E are related by [3]

$$A_N = \varphi A^*_E,$$

where φ is a non-vanishing smooth function on $\mathcal{U}$ in M. In case $\mathcal{U} = M$ the screen conformality is said to be global. Such a submanifold has some important and desirable properties, for instance, the integrability of its screen distribution.

As an example, we have Example 4.1 in which M is a screen conformal invariant null hypersurface with $\varphi = \frac{1}{2}$.

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