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SOME PROPERTIES OF CRITICAL POINT EQUATIONS METRICS ON THE STATISTICAL MANIFOLDS

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Abstract. The aim of this paper is to investigate some properties of the critical points equations on the statistical manifolds. We obtain some geometric equations on the statistical manifolds which admit critical point equations. We give a relation only between potential function and difference tensor for a CPE metric on the statistical manifolds to be Einstein.

1. Introduction

Let M be an n-dimensional compact (without boundary) oriented Riemannian manifold with dimension at least three. As we know the total scalar curvature functional $\mathcal{R} : \mathcal{M} \to \mathbb{R}$ is as follows:

$$\mathcal{R}(g) = \int_M s_g dvol_g,$$

where s_g is the scalar curvature and $\mathcal{M}$ is the space of Riemannian metrics on the manifold M. The Euler-Lagrangian equation of the total scalar curvature functional restricted to the space of metrics with constant scalar curvature of unitary volume is given by

$$\text{Ric} - \frac{s_g}{n} g = Hess(f) - (\text{Ric} - \frac{s_g}{n-1} g) f, \tag{1.1}$$

where Ric and $Hess$ stand, respectively, for the Ricci tensor, and the Hessian form on M^n [4, 6]. We recall the definition of critical point equations (CPE metrics).

Definition 1.1 ([2]). A CPE metric is a 3-tuple (M^n, g, f), where (M^n, g) is a compact oriented Riemannian manifold of dimension at least three with constant scalar curvature and $f : M^n \to \mathbb{R}$ is a non-constant smooth function satisfying equation (1.1). Such a function f is called a potential.

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Considering $\mathring{\mathrm{Ric}} = \mathrm{Ric} - \frac{s_g}{n} g$, the equation (1.1) yields

$$(1+f)\mathring{\mathrm{Ric}} = \nabla^2 f + \frac{s_g f}{n(n-1)} g. \tag{1.2}$$

Also, in local coordinates, we have

$$(1+f)\mathring{\mathrm{R}}_{ij} = \nabla_i \nabla_j f + \frac{s_g f}{n(n-1)} g_{ij}. \tag{1.3}$$

Computing the trace in (1.3) gives

$$-\Delta f = \frac{s_g f}{(n-1)}. \tag{1.4}$$

In 1987, Besse proposed a conjecture in [6] that the critical point metrics of the total scalar curvature functional $\mathcal{R}$ restricted to the space of constat scalar curvatures metrics, i.e., $\mathcal{C} = \{g \,|\, s_g \text{ is constant}\}$ are Einstein. The conjecture with the notations of some papers have been presented in the following way [4].

Conjecture 1.2 ([4,6])**.** *A* CPE *metric is always Einstein.*

There are many researches around critical point equations. For example, you can see [3, 5, 8, 11] and references therein. In [8] provided a necessary and sufficient condition on the norm of the gradient of the potential function for a CPE metric to be Einstein as follows:

Theorem 1.3 ([8])**.** *Let (M, g, f) be an n-dimensional* CPE *metric. Then M is Einstein if and only if*

$$|\nabla f|^2 + \frac{s_g f^2}{n(n-1)} = \Lambda, \tag{1.5}$$

where Λ is a constant.

On the other hand the statistical manifolds have been the subject of many investigations in the recent years. We first recall some notions and definitions of them. For more details, see [1,7,9]. Let (M, g) be an n-dimensional Riemannian manifold with Levi-Civita connection $\hat{\nabla}$ and ∇ as affine connection.

Definition 1.4. A pair (∇, g) is called a statistical structure on M, when ∇ is a torsion-free affine connection and ∇ satisfies the following Codazzi condition

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z) \tag{1.6}$$

for all vector fields $X, Y, Z \in TM$.

A Riemannian manifold (M, g) with statistical structure (∇, g) is called a Riemannian statistical manifold.

The conjugate (dual) connection $\bar{\nabla}$ of any connection ∇ relative to metric g has been defined by the following formula

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \bar{\nabla}_X Z). \tag{1.7}$$

It is easy to see that if (g, ∇) is a statistical structure, then $(g, \bar{\nabla})$ is also statistical structure. In this paper we assume that ∇ is the statistical connection. The difference tensor K between statistical connection ∇ and Levi-Civita connection $\hat{\nabla}$ is that

$$\nabla_X Y = \hat{\nabla}_X Y + K_X Y, \tag{1.8}$$

therefore we have

$$\bar{\nabla}_X Y = \hat{\nabla}_X Y - K_X Y. \tag{1.9}$$

The notation $K(X, Y) := K_X Y$ is used for the difference tensor. It is known that since $\nabla, \hat{\nabla}$ are torsion free K is a $(1, 2)$ symmetric tensor. A statistical structure (g, ∇) is trace-free if $\mathrm{tr}_g K(\dot{;}) = 0$ (equivalently $\mathrm{tr}_g K_X = 0$ for every vector filed X on M). If we let $R, \bar{R}, \hat{R}$ as curvature tensors of statistical connection ∇, its dual connection $\bar{\nabla}$ and Levi-Civita connection $\hat{\nabla}$, respectively, then relations between curvature tensors have been expressed as the following equations [10]:

$$\mathrm{R}(X, Y) = \hat{\mathrm{R}}(X, Y) + (\hat{\nabla}_X K)_Y - (\hat{\nabla}_Y K)_X + [K_X, K_Y]. \tag{1.10}$$

Writing the same equality for $\bar{\nabla}$ and adding both equalities gives

$$\mathrm{R}(X, Y) + \bar{\mathrm{R}}(X, Y) = 2\hat{\mathrm{R}}(X, Y) + 2[K_X, K_Y]. \tag{1.11}$$

Now, if $\mathrm{R} = \bar{\mathrm{R}}$, then

$$\mathrm{R}(X, Y) = \hat{\mathrm{R}}(X, Y) + [K_X, K_Y]. \tag{1.12}$$

Also the following equation is satisfied [9]

$$g(\mathrm{R}(X, Y)Z, W) = -g(\bar{\mathrm{R}}(X, Y)W, Z). \tag{1.13}$$

Then, we have

$$\bar{\mathrm{Ric}}(Y, W) = -\mathrm{tr}_g g(\mathrm{R}(\cdot, Y)\cdot, W), \tag{1.14}$$

where $\hat{\mathrm{Ric}}$ is the Ricci tensor of $\hat{\nabla}$. If the statistical structure (g, ∇) is trace-free and using (1.12), the following equation will get [9]:

$$\mathrm{Ric}(Y, Z) + \bar{\mathrm{Ric}}(Y, Z) = 2\hat{\mathrm{Ric}}(Y, Z) - 2g(K_Y, K_Z). \tag{1.15}$$

It is important that the condition $\mathrm{R} = \bar{\mathrm{R}}$ gives the symmetry of Ric. If we denote the scalar curvatures of $(g, \bar{\nabla})$ and $(g, \hat{\nabla})$, by $\bar{s_g}$ and $\hat{s_g}$, respectively, then according to $s_g = \mathrm{tr}_g \mathrm{Ric}(\cdot, \cdot)$ and equation (1.14) we have

$$s_g = \bar{s_g}. \tag{1.16}$$

Taking the trace relative to g on both sides of (1.15), one gets

$$\hat{s_g} = s_g + \|K\|^2 \tag{1.17}$$

for a trace-free statistical structure, [10].

The authors studied the critical point equation metrics on three-dimensional cosymplectic manifolds [12]. The aim of this paper is to investigate some properties of the critical points equations on the statistical manifolds. In the next section, we obtain some geometric equations on the statistical manifolds which admit critical point equations. We give a relation only between potential function f and difference tensor K for a CPE metric on the statistical manifolds to be Einstein.

2. Theorems and results

In this section we investigate some properties of metrics on the statistical manifolds. Let (g, ∇) be a statistical structure and $(g, \bar{\nabla})$ be its dual structure on the Riemannian manifold (M, g) with Levi-Civita Connection $\hat{\nabla}$. From now we assume that the statistical structure is trace-free and $\mathrm{R} = \bar{\mathrm{R}}$. If g is a critical point equation metrics on the M with potesioal function f, then the equation (1.1) in terms of the above notation is as follows:

$$\hat{\mathrm{Ric}} - \frac{\hat{s_g}}{n} g = \hat{\mathrm{Hess}} f - \left(\hat{\mathrm{Ric}} - \frac{\hat{s_g}}{n-1} g \right) f. \tag{2.18}$$

Proposition 2.1. *If (g, ∇) is a trace-free statistical structure on the Riemannian manifold (M, g), then critical point equation on the M is as follows*:

$$\begin{aligned} &(1+f)\left(\mathrm{Ric}(X,Y) + g(K_X, K_Y)\right) - \frac{s_g + \|K\|^2}{n} g \\ &= \mathrm{Hess} f + (K_X Y) f - \left(\frac{s_g + \|K\|^2}{n-1} g \right) f. \end{aligned} \tag{2.19}$$

Proof. Since the statistical structure (g, ∇) is trace-free and $\mathrm{R} = \bar{\mathrm{R}}$ applying in the equation (1.15), we have

$$\mathrm{Ric}(X,Y) = \hat{\mathrm{Ric}}(X,Y) - g(K_X, K_Y). \tag{2.20}$$

Now substituting the relations of (1.17), (2.20) in the equation (2.18), we have

$$\begin{aligned} &\mathrm{Ric}(X,Y) + g(K_X, K_Y) - \frac{s_g + \|K\|^2}{n} g \\ &= \mathrm{Hess} f(X,Y) + \mathrm{d} f\,(K(X,Y)) - \left(\mathrm{Ric}(X,Y) + g(K_X, K_Y) - \frac{s_g + \|K\|^2}{n-1} g \right) f. \end{aligned}$$

Consequently

$$\begin{aligned} &(1+f)\left(\mathrm{Ric}(X,Y) + g(K_X, K_Y)\right) - \frac{s_g + \|K\|^2}{n} g \\ &= \mathrm{Hess} f + (K_X Y) f - \left(\frac{s_g + \|K\|^2}{n-1} g \right) f. \end{aligned}$$

□

Now in the following theorem we show that the scalar curvature of a statistical structure on the CPE metric is related to tensor K.

Theorem 2.2. *If (g, ∇) is a trace-free statistical structure on the Riemannian manifold (M, g) with CPE metric g, then the scalar curvature s_g satisfies in the following equation*:

$$s_g = \frac{n\|K\|^2(f-1)}{f}. \tag{2.21}$$

Proof. Taking the trace with respect to g on both sides of (2.19) and taking into account that $tr_g K = 0$, we get

$$\|K\|^2 = \Delta f - s_g f - \frac{n s_g f + n\|K\|^2 f}{n-1}$$

$$\Rightarrow \Delta f = \frac{2n-1}{n(n-1)} s_g f + \|K\|^2 - \frac{n}{n-1}\|K\|^2 f. \tag{2.22}$$

On the other hand applying (1.17) in the equation (1.4) we have

$$-\Delta f = \frac{(s_g + \|K\|^2) f}{n-1}, \tag{2.23}$$

using now formulas (2.22) and (2.23) we see that

$$\frac{\mathrm{Rc}}{n-1} f + \frac{\|K\|^2}{n-1} f = -\frac{2n-1}{n(n-1)} \mathrm{Rc} f - \|K\|^2 + \frac{n}{n-1}\|K\|^2 f$$

$$\Rightarrow \quad \frac{1}{n}\mathrm{Rc} f = \|K\|^2 (f-1). \tag{2.24}$$

Hence,

$$s_g = \frac{n\|K\|^2(f-1)}{f}.$$

□

Remark 2.3. Note that K_X is a $(1,1)$-tensor for any $X \in TM$, and it can be considered as an endomorphism of the vector space $T_x M$. That is,

$$K_X \in \mathcal{T}_1^1(TM \otimes T^*M)$$

also

$$K_X = (K_X)_n^m \mathrm{d}x^n \otimes \partial_m.$$

Therefore, in coordinate we have

$$\begin{aligned} K_X(\partial_n) &= \nabla_X(\partial_n) - \hat{\nabla}_X(\partial_n) \\ &= \nabla_{X^i \partial_i}(\partial_n) - \hat{\nabla}_{X^i \partial_i}(\partial_n) \\ &= X^i \nabla_{\partial_i}(\partial_n) - X^i \hat{\nabla}_{\partial_i}(\partial_n) \\ &= X^i \Gamma_{in}^m(\partial_m) - X^i \hat{\Gamma}_{in}^m(\partial_m). \end{aligned}$$

Hence

$$K_X = (X^i \Gamma_{in}^m - X^i \hat{\Gamma}_{in}^m) \mathrm{d}x^n \otimes \partial_m. \tag{2.25}$$

Theorem 2.4. *Let (M, g, f) be a* CPE *metric and (∇, g) a statistical (trace-free) structure. If M is Einstein, then statistical connection ∇ is the Levi-Civita connection.*

Proof. According to (1.15) and $\mathrm{Ric} = \bar{\mathrm{Ric}}$, since M is Einstein, we have

$$\mathrm{Ric} = \lambda g - g(K_X, K_Y). \tag{2.26}$$

Equation (2.19) can be rewritten as follows:

$$\begin{aligned} &\mathrm{Ric} - \frac{s_g + \|K\|^2}{n} g + \mathrm{Ric} f - \left(\frac{s_g + \|K\|^2}{n} g \right) f \\ = &- g(K_X, K_Y) - g(K_X, K_Y) f + \left(\frac{s_g + \|K\|^2}{n(n-1)} g \right) f \\ &+ \mathrm{Hess} f + (K_X Y) f. \end{aligned} \tag{2.27}$$

Therefore, substituting (2.26) into (2.27) we infer

$$(1+f)\left(\lambda g - \frac{s_g + \|K\|^2}{n} g \right) = \left(\lambda g - \frac{s_g + \|K\|^2}{n(n-1)} g \right) f + \mathrm{Hess} f + (K_X Y) f. \tag{2.28}$$

Since M is assumed to be Einstein from (1.17), we have:

$$s_g = n\lambda - \|K\|^2. \tag{2.29}$$

Using formula (2.28) in (2.29) we get

$$0 = \frac{\lambda f}{n-1} g + \mathrm{Hess} f + (K_X Y) f. \tag{2.30}$$

Proceeding, in local coordinates and applying (2.25), we have

$$0 = \frac{\lambda f}{n-1} g_{ij} + \nabla_i \nabla_j f + (\Gamma^l_{ij} - \hat{\Gamma}^l_{ij}) \partial_l f. \tag{2.31}$$

Taking the trace of (2.31), one yields

$$\begin{aligned} 0 &= g^{ij} \frac{\lambda f}{n-1} g_{ij} + g^{ij} \nabla_i \nabla_j f + g^{ij} (\Gamma^l_{ij} - \hat{\Gamma}^l_{ij}) \partial_l f \\ &= \frac{n\lambda f}{n-1} + \Delta f + g^{ij} (\Gamma^l_{ij} - \hat{\Gamma}^l_{ij}) \partial_l f. \end{aligned}$$

Now, it follows from (2.23) that

$$0 = \frac{n\lambda f}{n-1} - \frac{(\mathrm{Rc} + \|K\|^2) f}{n-1} + g^{ij} (\Gamma^l_{ij} - \hat{\Gamma}^l_{ij}) \partial_l f.$$

Hence, from (1.17) we have

$$0 = g^{ij} (\Gamma^l_{ij} - \hat{\Gamma}^l_{ij}) = K_X Y. \tag{2.32}$$

Finally, from (1.8), we conclude

$$\nabla_X Y = \hat{\nabla}_X Y.$$

This completes the proof of theorem. □

Theorem 2.5. *Let* (M, g, f) *be a* CPE *metric and* (∇, g) *a statistical* (*trace-free*) *structure. Then* M *is Einstein when the following equation is satisfied*:

$$\text{Hess}f = \frac{s_g + \|K\|^2}{n(n-1)}g - fK. \tag{2.33}$$

Proof. Considering assumption of theorem, the equation (2.19) is equivalent to (2.34)

$$\text{Ric} = -g(K_X, K_Y) + \frac{s_g + \|K\|^2}{n}g + \frac{f}{1+f}(\frac{s_g + \|K\|^2}{n(n-1)}g + K) + \frac{1}{1+f}\text{Hess}f.$$

Applying (2.33), we get

$$\begin{aligned}\text{Ric} &= -g(K_X, K_Y) + \frac{s_g + \|K\|^2}{n}g + \frac{s_g + \|K\|^2}{n(n-1)}g \\ &= -g(K_X, K_Y) + \frac{s_g + \|K\|^2}{n-1}g,\end{aligned}$$

now, let $\lambda := \frac{s_g+\|K\|^2}{n-1}$ and λ as a constant. Hence we have

$$\text{Ric} = \lambda g - g(K_X, K_Y). \tag{2.35}$$

According to (2.26), the manifold M is Einstein. □

Corollary 2.6. *Let* (M, g, f) *be a* CPE *metric and* (∇, g) *a statistical* (*trace-free*) *structure. Then* M *is Einstein if and only if*

$$|\nabla f|^2 + \frac{(s_g + \|K\|^2)f^2}{n(n-1)} = \Lambda, \tag{2.36}$$

where Λ *is a constant.*

Proof. Substituting formula (1.17) in the conditions that proved by Neto in (1.5), we get

$$|\nabla f|^2 + \frac{(s_g + \|K\|^2)f^2}{n(n-1)} = \Lambda.$$ □

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