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HORADAM POLYNOMIALS FOR A NEW SUBCLASS OF SAKAGUCHI-TYPE BI-UNIVALENT FUNCTIONS DEFINED BY (p, q)-DERIVATIVE OPERATOR

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Abstract. In this paper, a new subclass, $\mathcal{SC}_{\sigma}^{\mu,p,q}(r,s;x)$, of Sakaguchi-type analytic bi-univalent functions defined by (p,q)-derivative operator using Horadam polynomials is constructed and investigated. The initial coefficient bounds for $|a_2|$ and $|a_3|$ are obtained. Fekete-Szegö inequalities for the class are found. Finally, we give some corollaries.

1. Introduction

We denote the complex plane by C, the open unit disk by U and the real line by R. Let $f(z)$ be a *normalized analytic function* of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

in U. Let $\mathcal{A}$ be the class of all normalized analytic functions. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of *univalent functions.*

Let f be a member of $\mathcal{S}$. The function $f(z)$ is said to be *bi-univalent* if, in the w-plane, the inverse function, $f^{-1}(w)$, of $f(z)$ has an analytic continuation to $|w| < 1$. Let σ be the class of all bi-univalent functions in U [13]. In 1967, Lewin [10] introduced the class of bi-univalent functions and gave an estimate for the second coefficient for functions belonging to this class as $|a_2| < 1.51$. His result was improved by Brannan and Clunie [3] to $|a_2| \leq \sqrt{2}$. There is an extensive literature on the estimates of the initial coefficients of bi-univalent functions (see [4, 14, 17–21]).

For any compact family of functions, finding sharp bounds for $|a_3 - \kappa a_2^2|$ is called the *Fekete-Szegö* problem. In particular, when $\kappa = 1$, the functional

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represents Schwarzian derivative. In the theory of Geometric functions the role of Schwarzian derivative is remarkable.

Let f_1 and f_2 be members of $\mathcal{A}$. The function f_1 is said to be *subordinate* to f_2, if there exists an analytic function $c(z)$ in U with $c(0) = 0$ and $|c(z)| < 1$, and such that $f_1(z) = f_2(c(z))$. It is written as $f_1(z) \prec f_2(z)$. Sakaguchi [12] introduced a subclass consisting of functions satisfying

$$\Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > \alpha.$$

These functions were named after him as *Sakaguchi type functions* (see [1, 2]). These functions are starlike with respect to symmetric points. Frasin [5] generalized this class which had functions of the form $\Re\left(\frac{(r-s)zf'(z)}{f(rz)-f(sz)}\right) > \alpha$, $0 \le \alpha < 1$, $r, s \in C$ with $r \neq s$, $|s| \le 1$, $z \in U$.

Horadam polynomials are generalized Horadam numbers and second order polynomial sequence. Recently, Horzum and Kocer [8], studied the Horadam polynomials $h_k(x)$, which is defined by the recurrence relation [7]

$$h_k(x) = \varrho x h_{k-1}(x) + \rho h_{k-2}(x), \quad (x \in R,\ k = 3, 4, \ldots)$$

with initial conditions

$$h_1(x) = b, \quad h_2(x) = ax, \tag{2}$$

where $b, a, \varrho, \rho \in R$.

For $k = 3$ we obtain

$$h_3(x) = a\varrho x^2 + b\rho.$$

For more details one can refer to (see [6,7,9,11,15,16]). These polynomials and their generalizations play a vital role in Mathematics, Statistics and Physics. Table 1 gives us some of the special cases of Horadam polynomials.

Table 1. Special cases of the Horadam polynomials.

S. No.	Parameters	Special Cases
1	$b = a = \varrho = \rho = 1$	Fibonacci polynomials $F_k(x)$
2	$b = 2, a = \varrho = \rho = 1$	Lucas polynomials $L_k(x)$
3	$b = \rho = 1, a = \varrho = 2$	Pell polynomials $P_k(x)$
4	$b = a = \varrho = 2, \rho = 1$	Pell-Lucas polynomials $Q_k(x)$
5	$b = a = 1, \varrho = 2, \rho = -1$	Chebyshev polynomials of the first kind $T_k(x)$
6	$b = 1, a = \varrho = 2, \rho = -1$	Chebyshev polynomials of the second kind $U_k(x)$

Lemma 1.1. *The generating function $G(x, z)$ of the Horadam polynomials $h_k(x)$ is given by*

$$G(x, z) = \sum_{k=1}^{\infty} h_k(x) z^{k-1} = \frac{b + (a - b\varrho)xz}{1 - \varrho xz - \rho z^2}.$$

Definition. *For* $0 < q < p \leq 1$, the (p, q)-derivative operator, $D_{p,q}(f(z))$, is defined as

$$D_{p,q}(f(z)) \begin{cases} \dfrac{f(pz) - f(qz)}{(p-q)z}, & z \neq 0, \\ f'(0), & z = 0, \end{cases} \tag{3}$$

provided $f'(0)$ exists.

It can be written as

$$D_{p,q}(f(z)) = 1 + \sum_{k=2}^{\infty} [k]_{p,q} a_k z^{k-1},$$

where $[k]_{p,q} = \frac{p^k - q^k}{p-q}$, the (p, q)-bracket of k and is also called a twin-basic number. For instance, $D_{p,q}(z^k) = [k]_{p,q} z^{k-1}$. When $p = 1$, the (p, q)-derivative operator $D_{p,q}$ reduces to the q-derivative operator D_q. The inverse Taylor series of (3) is given by

$$\begin{aligned} D_{p,q}(g(w)) &= \frac{g(pw) - g(qw)}{(p-q)w} \\ &= 1 - [2]_{p,q} a_2 w + [3]_{p,q}(2a_2^2 - a_3)w^2 - [4]_{p,q}(5a_2^3 - 5a_2 a_3 + a_4)w^3 + \cdots, \end{aligned}$$

where $g = f^{-1}$.

2. Coefficient bounds for the function class $\mathcal{SC}_{\sigma}^{\mu,p,q}(r, s; x)$

In this section, we define our new class $\mathcal{SC}_{\sigma}^{\mu,p,q}(r, s; x)$ and evaluate the bound for the initial coefficients $|a_2|$ and $|a_3|$ for the functions in $\mathcal{SC}_{\sigma}^{\mu,p,q}(r, s; x)$.

Definition. A function $f \in \sigma$, given by (1), is said to be in the class $\mathcal{SC}_{\sigma}^{\mu,p,q}(r, s; x)$ if

$$(D_{p,q} f)^{\mu}(z) \left(\frac{(r-s)z}{f(rz) - f(sz)} \right) \prec 1 - b + G(x, z) \tag{4}$$

and

$$(D_{p,q} g)^{\mu}(w) \left(\frac{(r-s)w}{g(rw) - g(sw)} \right) \prec 1 - b + G(x, w), \tag{5}$$

where $g = f^{-1}$, $\mu \geq 1$ and $r, s \in C$ with $r \neq s$, $|s| \leq 1$.

Theorem 2.1. *If* $f(z)$*, given by* (1)*, is in* $\mathcal{SC}_{\sigma}^{\mu,p,q}(r, s; x)$*, then*

$$|a_2| \leq \frac{|ax|\sqrt{|ax|}}{\sqrt{|La^2x^2 - M^2(a\varrho x^2 + b\rho)|}} \tag{6}$$

and

$$|a_3| \leq \left| \frac{ax}{N} \right| + \frac{a^2x^2}{|M|^2}, \tag{7}$$

where

$$L = \frac{\mu(\mu-1)}{2}[2]_{p,q}^2 - \mu[2]_{p,q}(r+s) + rs + \mu[3]_{p,q},$$
$$M = \mu[2]_{p,q} - r - s,$$

and

$$N = \mu[3]_{p,q} - r^2 - rs - s^2.$$

Proof. Since $f \in \mathcal{SC}_\sigma^{\mu,p,q}(r,s;x)$, there exist two analytic functions $u, v : U \to U$ given by

$$u(z) = \sum_{k=1}^{\infty} u_k z^k \tag{8}$$

and

$$v(w) = \sum_{k=1}^{\infty} v_k w^k \tag{9}$$

with $u(0) = 0 = v(0)$, $|u(z)| < 1$, $|v(w)| < 1$ for all $z, w \in U$ such that

$$(D_{p,q}f)^{\mu}(z)\left(\frac{(r-s)z}{f(rz)-f(sz)}\right) = 1 - b + G(x, u(z))$$

and

$$(D_{p,q}g)^{\mu}(w)\left(\frac{(r-s)w}{g(rw)-g(sw)}\right) = 1 - b + G(x, v(w)).$$

Or equivalently

$$\begin{aligned} &(D_{p,q}f)^{\mu}(z)\left(\frac{(r-s)z}{f(rz)-f(sz)}\right) \\ &= 1 + h_2(x)u_1 z + \left[h_2(x)u_2 + h_3(x)u_1^2\right] z^2 + \cdots \end{aligned} \tag{10}$$

and

$$\begin{aligned} &(D_{p,q}g)^{\mu}(w)\left(\frac{(r-s)w}{g(rw)-g(sw)}\right) \\ &= 1 + h_2(x)v_1 w + \left[h_2(x)v_2 + h_3(x)v_1^2\right] w^2 + \cdots . \end{aligned} \tag{11}$$

Since $|u(z)| < 1$ and $|v(w)| < 1$, it is clear that

$$|u_k| \le 1, \tag{12}$$
$$|v_k| \le 1 \tag{13}$$

for $k = 1, 2, \ldots$. From (10) and (11), we have

$$\left(\mu[2]_{p,q} - r - s\right) a_2 = h_2(x)u_1, \tag{14}$$

$$\begin{aligned}(15)\qquad &\left(\frac{\mu(\mu-1)}{2}[2]_{p,q}^2+(r+s)^2-\mu[2]_{p,q}(r+s)\right)a_2^2\\ &+\left(\mu[3]_{p,q}-r^2-rs-s^2\right)a_3\\ &=h_2(x)u_2+h_3(x)u_1^2,\end{aligned}$$

$$(16)\qquad -\left(\mu[2]_{p,q}-r-s\right)a_2=h_2(x)v_1$$

and

$$\begin{aligned}(17)\qquad &\left(\frac{\mu(\mu-1)}{2}[2]_{p,q}^2+(r+s)^2-\mu[2]_{p,q}(r+s)\right)a_2^2\\ &+\left(\mu[3]_{p,q}-r^2-rs-s^2\right)\left(2a_2^2-a_3\right)\\ &=h_2(x)v_2+h_3(x)v_1^2.\end{aligned}$$

From (14) and (16), we get

$$(18)\qquad u_1=-v_1$$

and

$$(19)\qquad 2\left(\mu[2]_{p,q}-r-s\right)^2a_2^2=h_2^2(x)(u_1^2+v_1^2).$$

Upon adding (15) and (17), we get

$$\begin{aligned}(20)\qquad &2\left(\tfrac{\mu(\mu-1)}{2}[2]_{p,q}^2-\mu[2]_{p,q}(r+s)+rs+\mu[3]_{p,q}\right)a_2^2\\ &=h_2(x)(u_2+v_2)+h_3(x)(u_1^2+v_1^2).\end{aligned}$$

By using (19) in (20), we have

$$\begin{aligned}(21)\qquad &2\Big[\left(\tfrac{\mu(\mu-1)}{2}[2]_{p,q}^2-\mu[2]_{p,q}(r+s)+rs+\mu[3]_{p,q}\right)h_2^2(x)\\ &-\left(\mu[2]_{p,q}-r-s\right)^2h_3(x)\Big]a_2^2\\ &=h_2^3(x)(u_2+v_2)\end{aligned}$$

which implies

$$|a_2|\le\frac{|ax|\sqrt{|ax|}}{\sqrt{|La^2x^2-M^2\left(a\varrho x^2+b\rho\right)|}}.$$

Now subtracting (17) from (15) and using (18), we get

$$(22)\qquad a_3-a_2^2=\frac{h_2(x)(u_2-v_2)}{2\left(\mu[3]_{p,q}-r^2-rs-s^2\right)}.$$

Then, in aid of (19), we get

$$(23)\qquad a_3=\frac{h_2(x)(u_2-v_2)}{2N}+\frac{h_2^2(x)\left(u_1^2+v_1^2\right)}{2M^2}.$$

Thus

$$|a_3|\le\left|\frac{ax}{N}\right|+\frac{a^2x^2}{|M|^2}.$$

□

Corollary 2.2. *If $f(z)$, given by (1), is in $\mathcal{SC}_{\sigma}^{1,p,q}(r,s;x)$, then*

$$|a_2| \leq \frac{|ax|\sqrt{|ax|}}{\sqrt{\left|\left([3]_{p,q}+rs-[2]_{p,q}(r+s)\right)a^2x^2-\left([2]_{p,q}-r-s\right)^2\left(a\varrho x^2+b\rho\right)\right|}} \tag{24}$$

and

$$|a_3| \leq \left|\frac{ax}{[3]_{p,q}-r^2-rs-s^2}\right| + \frac{a^2x^2}{\left|[2]_{p,q}-r-s\right|^2}. \tag{25}$$

Corollary 2.3. *If $f(z)$, given by (1), is in $\mathcal{SC}_{\sigma}^{\mu,p,q}(1,0;x)$, then*

$$|a_2| \leq \frac{|ax|\sqrt{|ax|}}{\sqrt{\left|\left(\frac{\mu(\mu-1)}{2}[2]_{p,q}^2+\mu[3]_{p,q}-\mu[2]_{p,q}\right)a^2x^2-\left(\mu[2]_{p,q}-1\right)^2\left(a\varrho x^2+b\rho\right)\right|}} \tag{26}$$

and

$$|a_3| \leq \left|\frac{ax}{\mu[3]_{p,q}-1}\right| + \frac{a^2x^2}{\left(\mu[2]_{p,q}-1\right)^2}. \tag{27}$$

Corollary 2.4. *If $f(z)$, given by (1), is in $\mathcal{SC}_{\sigma}^{\mu,p,q}(1,-1;x)$, then*

$$|a_2| \leq \frac{|ax|\sqrt{|ax|}}{\sqrt{\left|\left(\frac{\mu(\mu-1)}{2}[2]_{p,q}^2+\mu[3]_{p,q}-1\right)a^2x^2-\mu^2[2]_{p,q}^2\left(a\varrho x^2+b\rho\right)\right|}} \tag{28}$$

and

$$|a_3| \leq \left|\frac{ax}{\mu[3]_{p,q}-1}\right| + \frac{a^2x^2}{\mu^2[2]_{p,q}^2}. \tag{29}$$

Corollary 2.5. *If $f(z)$, given by (1), is in $\mathcal{SC}_{\sigma}^{\mu,1,q}(r,s;x)$ and $q \to 1^-$, then*

$$|a_2| \leq \frac{|ax|\sqrt{|ax|}}{\sqrt{\left|\left(2\mu^2+(1-2r-2s)\mu+rs\right)a^2x^2-\left(2\mu-r-s\right)^2\left(a\varrho x^2+b\rho\right)\right|}} \tag{30}$$

and

$$|a_3| \leq \left|\frac{ax}{3\mu-r^2-rs-s^2}\right| + \frac{a^2x^2}{|2\mu-r-s|^2}. \tag{31}$$

3. Fekete-Szegö inequalities for the function class $\mathcal{SC}_{\sigma}^{\mu,p,q}(r,s;x)$

In this section, we estimate Fekete-Szegö inequalities $|a_3-\kappa a_2^2|$ for the functions belonging to the class $\mathcal{SC}_{\sigma}^{\mu,p,q}(r,s;x)$.

Theorem 3.1. *If $f(z)$, given by (1), is in $\mathcal{SC}_{\sigma}^{\mu,p,q}(r,s;x)$ and $\kappa \in R$, then*

$$|a_3-\kappa a_2^2| \leq \left(\frac{1}{|N|}+2\left|\Psi\left(\mu,p,q,r,s;x\right)\right|\right)|h_2(x)|,$$

where

$$\Psi(\mu, p, q, r, s; x) = \frac{(1-\kappa)h_2^2(x)}{2\left(Lh_2^2(x) - M^2 h_3(x)\right)},$$

L, M and N are as in Theorem 2.1.

Proof. For $\kappa \in R$ and from (22), we get

$$a_3 - \kappa a_2^2 = \frac{h_2(x)(u_2 - v_2)}{2N} + (1-\kappa)a_2^2. \tag{32}$$

By using (21), we have

$$\begin{aligned} & a_3 - \kappa a_2^2 \\ &= \frac{h_2(x)(u_2 - v_2)}{2N} + (1-\kappa)\left(\frac{h_2^3(x)(u_2+v_2)}{2\left(Lh_2^2(x) - M^2 h_3(x)\right)}\right) \\ &= h_2(x)\left[\left(\frac{1}{2N} + \Psi(\mu, p, q, r, s; x)\right)u_2 + \left(\frac{-1}{2N} + \Psi(\mu, p, q, r, s; x)\right)v_2\right], \end{aligned}$$

where

$$\Psi(\mu, p, q, r, s; x) = \frac{(1-\kappa)h_2^2(x)}{2\left(Lh_2^2(x) - M^2 h_3(x)\right)}.$$

Thus

$$|a_3 - \kappa a_2^2| \le \left(\frac{1}{|N|} + 2\left|\Psi(\mu, p, q, r, s; x)\right|\right)|h_2(x)|.$$

□

Corollary 3.2. *If $f(z)$, given by (1), is in $\mathcal{SC}_\sigma^{1,p,q}(r, s; x)$ and $\kappa \in R$, then*

$$|a_3 - \kappa a_2^2| \le \left(\frac{1}{|N_1|} + 2\left|\Psi(1, p, q, r, s; x)\right|\right)|h_2(x)|,$$

where

$$\Psi(1, p, q, r, s; x) = \frac{(1-\kappa)h_2^2(x)}{2\left(L_1 h_2^2(x) - M_1^2 h_3(x)\right)},$$

$$\begin{aligned} L_1 &= -[2]_{p,q}(r+s) + rs + [3]_{p,q}, \\ M_1 &= [2]_{p,q} - r - s, \end{aligned}$$

and

$$N_1 = [3]_{p,q} - r^2 - rs - s^2.$$

Corollary 3.3. *If $f(z)$, given by (1), is in $\mathcal{SC}_\sigma^{\mu,p,q}(1, 0; x)$ and $\kappa \in R$, then*

$$|a_3 - \kappa a_2^2| \le \begin{cases} \left|\frac{ax}{N_2}\right|, & |\kappa - 1| \le \left|\frac{L_2}{N_2} - \frac{M_2^2(a\varrho x^2 + b\rho)}{N_2 a^2 x^2}\right|, \\ \frac{|\kappa-1||a^3x^3|}{|L_2 a^2 x^2 - M_2^2(a\varrho x^2 + b\rho)|}, & |\kappa - 1| \ge \left|\frac{L_2}{N_2} - \frac{M_2^2(a\varrho x^2 + b\rho)}{N_2 a^2 x^2}\right|, \end{cases}$$

where

$$\begin{aligned} L_2 &= \frac{\mu(\mu-1)}{2}[2]_{p,q}^2 + \mu[3]_{p,q} - \mu[2]_{p,q}, \\ M_2 &= \mu[2]_{p,q} - 1, \end{aligned}$$

and

$$N_2 = \mu[3]_{p,q} - 1.$$

Corollary 3.4. *If $f(z)$, given by (1), is in $\mathcal{SC}_{\sigma}^{\mu,p,q}(1,-1;x)$ and $\kappa \in R$, then*

$$|a_3 - \kappa a_2^2| \leq \begin{cases} \left|\frac{ax}{N_2}\right|, & |\kappa - 1| \leq \left|\frac{L_3}{N_2} - \frac{\mu^2[2]_{p,q}^2(a\varrho x^2+b\rho)}{N_2 a^2 x^2}\right|, \\ \frac{|\kappa-1||a^3x^3|}{|L_3 a^2 x^2 - \mu^2[2]_{p,q}^2(a\varrho x^2+b\rho)|}, & |\kappa - 1| \geq \left|\frac{L_3}{N_2} - \frac{\mu^2[2]_{p,q}^2(a\varrho x^2+b\rho)}{N_2 a^2 x^2}\right|, \end{cases}$$

where $L_3 = \frac{\mu(\mu-1)}{2}[2]_{p,q}^2 + \mu[3]_{p,q} - 1$ and N_2 is as in Corollary 3.3.

Corollary 3.5. *If $f(z)$, given by (1), is in $\mathcal{SC}_{\sigma}^{\mu,1,q}(r,s;x)$, $q \to 1^-$ and $\kappa \in R$, then*

$$|a_3 - \kappa a_2^2| \leq \left(\frac{1}{|N_3|} + 2\left|\Psi\left(\mu, r, s; x\right)\right|\right)|h_2(x)|,$$

where

$$\Psi\left(\mu, r, s; x\right) = \frac{(1-\kappa)h_2^2(x)}{2\left(L_4 h_2^2(x) - M_3^2 h_3(x)\right)},$$

$$L_4 = 2\mu^2 + (1 - 2r - 2s)\,\mu + rs,$$
$$M_3 = 2\mu - r - s,$$

and

$$N_3 = 3\mu - r^2 - rs - s^2.$$

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