Commun. Korean Math. Soc. **39** (2024), No. 2, pp. 407–420
https://doi.org/10.4134/CKMS.c230109
pISSN: 1225-1763 / eISSN: 2234-3024

GEOMETRIC PROPERTIES OF STARLIKENESS INVOLVING HYPERBOLIC COSINE FUNCTION

Om P. Ahuja, Asena Çetinkaya, and Sushil Kumar

Abstract. In this paper, we investigate some geometric properties of starlikeness connected with the hyperbolic cosine functions defined in the open unit disk. In particular, for the class of such starlike hyperbolic cosine functions, we determine the lower bounds of partial sums, Briot-Bouquet differential subordination associated with Bernardi integral operator, and bounds on some third Hankel determinants containing initial coefficients.

1. Preliminaries and the class $\mathcal{S}^*_{\cosh}$

Let $\mathcal{A}$ be the class of all analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

that are defined in the disk $\mathbb{D} = \{z : |z| < 1\}$. Denote by Ω the class of all Schwarz functions w, which are analytic in $\mathbb{D}$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$. If f and g are analytic functions, then f is subordinate to g, written as $f \prec g$ if there exists a Schwarz function w such that $f = g \circ w$ for all $z \in \mathbb{D}$. Also, let $\mathcal{P}$ be the class of all analytic functions $p : \mathbb{D} \to \mathbb{C}$ such that $p(0) = 1$ and $\mathrm{Re}(p(z)) > 0$ for every $z \in \mathbb{D}$. For more details, we refer [7].

In [2], researchers studied the family $\mathcal{S}^*_{\cosh}$ of starlike functions connected with hyperbolic cosine function defined by

$$\mathcal{S}^*_{\cosh} = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \cosh z,\ z \in \mathbb{D} \right\}.$$

Geometrically, the function $zf'(z)/f(z)$ maps $\mathbb{D}$ onto the open disk symmetric with respect to the real axis with center $(\cosh 1 + \cos 1)/2$ and radius $(\cosh 1 - \cos 1)/2$; (see [2]). In [4], the authors discovered that the cosine and

Received May 6, 2023; Accepted January 25, 2024.
2020 *Mathematics Subject Classification.* 30C45, 30C50.
Key words and phrases. Briot-Bouquet differential subordination, hyperbolic cosine function, partial sums, starlike functions, third Hankel determinants.

hyperbolic cosine functions have the same image domain in $\mathbb{D}$. In [2], the researchers found the following sufficient condition.

Lemma 1.1. *If $f \in \mathcal{A}$, $\theta \in [0, 2\pi]$ and*

$$\sum_{n=2}^{\infty} \left| \frac{n - \cosh(e^{i\theta})}{\cosh(e^{i\theta}) - 1} \right| |a_n| \le 1,$$

then $f \in \mathcal{S}^*_{\cosh}$.

For $\theta = 0$, the condition in Lemma 1.1 reduces to

$$\sum_{n=2}^{\infty} (n - \cosh 1)|a_n| \le \cosh 1 - 1. \tag{1.2}$$

Further, if $f \in \mathcal{S}^*_{\cosh}$, then

$$\frac{zf'(z)}{f(z)} = \cosh w(z) \quad \text{for all } z \in \mathbb{D}, \tag{1.3}$$

where $w \in \Omega$. Therefore, we observe that

$$2\pi \sum_{n=1}^{\infty} |a_n|^2 r^{2n} = \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \ge \frac{2\pi}{(\cosh 1)^2} \sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n},$$

where $0 < r < 1$ and $a_1 = 1$. On taking $r \to 1^-$, we get the inequality given by

$$\sum_{n=2}^{\infty} (n^2 - (\cosh 1)^2)|a_n|^2 \le (\cosh 1)^2 - 1.$$

In particular,

$$|a_n| \le \sqrt{\frac{(\cosh 1)^2 - 1}{n^2 - (\cosh 1)^2}} \quad \text{for all } n \ge 2.$$

Let f and g be analytic functions. Then f is majorized to g, written as $f(z) \ll g(z)$ if there exists an analytic function ψ in $\mathbb{D}$ satisfying $|\psi(z)| \le 1$ and $f(z) = \psi(z)g(z)$ for all $z \in \mathbb{D}$ (see [17]). By using the reasoning given in [28, Theorem 2.1, p. 4], we get the following new result:

Theorem 1.2. *Let $f \in \mathcal{A}$ and suppose that $g \in \mathcal{S}^*_{\cosh}$ with $f(z) \ll g(z)$ for all $z \in \mathbb{D}$. Then for $|z| \le r_1$, we have $|f'(z)| \le |g'(z)|$, where r_1 is the smallest positive root of the equation $(1 - r^2)\cos r - 2r = 0$.*

If the function $f \in \mathcal{A}$, then its sequence of k^{th} partial sums is the polynomial which is given by $f_k(z) = z + \sum_{n=2}^{k} a_n z^n$. In [27], it is noted that the partial sums of univalent functions is univalent in the disk $\mathbb{D}_{1/4}$. Starlikeness and convexity of the partial sums of univalent functions was discussed in [9, 11, 23, 25, 26]. The Briot-Bouquet differential subordination is given by

$$\phi(z) + \frac{z\phi'(z)}{\eta\phi(z) + \mu} \prec \varphi(z) \quad (\eta, \mu \in \mathbb{C}, \eta \ne 0) \tag{1.4}$$

with $\phi(0) = \varphi(0) = 1$. If the univalent function $q(z) = 1 + q_1 z + q_2 z^2 + \cdots$ has the property that $\phi \prec q$ for all analytic functions ϕ satisfying (1.4), then it is called a dominant of (1.4). If $\tilde{q} \prec q$ for all dominants q of (1.4), then a dominant $\tilde{q}$ is called the best dominant of the differential subordination. In [8], Eenigenburg *et al.* investigated the Briot-Bouquet differential subordination. For more details, see [1, 5, 6, 15]. The Bernardi integral operator $\mathcal{L}_\sigma : \mathcal{A} \to \mathcal{A}$ with $\sigma > 0$ is given by

$$\mathcal{L}_\sigma f(z) = \frac{1+\sigma}{z^\sigma} \int_0^z t^{\sigma-1} f(t) dt.$$

From this operator, we easily get

$$z(\mathcal{L}_\sigma f(z))' = (1+\sigma) f(z) - \sigma \mathcal{L}_\sigma f(z). \tag{1.5}$$

For natural numbers q and n, the q^{th} Hankel determinant, which is associated with the coefficients of functions $f \in \mathcal{A}$, is defined by $H_q(n) := \det\{a_{n+i+j-2}\}_{i,j}^q$, $1 \le i, j \le q$, $a_1 = 1$. The third Hankel determinants $H_3^{(1)}$, $H_3^{(2)}$ and $H_3^{(3)}$ are given as

$$H_3(1) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2), \tag{1.6}$$

$$H_3(2) = a_2(a_4 a_6 - a_5^2) - a_3(a_3 a_6 - a_4 a_5) + a_4(a_3 a_5 - a_4^2), \tag{1.7}$$

$$H_3(3) = a_3(a_5 a_7 - a_6^2) - a_4(a_4 a_7 - a_5 a_6) + a_5(a_4 a_6 - a_5^2). \tag{1.8}$$

In [21], Pommerenke computed initially an upper bounds on the Hankel determinants for starlike functions and univalent functions. The best possible upper bound on second Hankel determinant $H_2(2) = a_2 a_4 - a_3^2$ for the Ma-Minda starlike and convex functions were estimated in [16].

In view of the above discussed literature, we determine first lower bounds on $\operatorname{Re}\big(f(z)/f_k(z)\big)$, $\operatorname{Re}\big(f_k(z)/f(z)\big)$, $\operatorname{Re}\big(f'(z)/f'_k(z)\big)$ and $\operatorname{Re}\big(f'_k(z)/f'(z)\big)$ for the functions $f \in \mathcal{S}^*_{\cosh}$. The Briot-Bouquet differential subordination implications related to the Bernardi integral operator are established. In addition, we determine bounds on the third Hankel determinants $H_3(1)$, $H_3(2)$ and $H_3(3)$ for such functions.

2. Partial sums

In this section, we examine the ratios of a function of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ to its sequence of partial sums $f_k(z) = z + \sum_{n=2}^{k} a_n z^n$, when the coefficients of the function f to be in the class $\mathcal{S}^*_{\cosh}$ are sufficiently small.

Theorem 2.1. *If a function $f \in \mathcal{A}$ satisfies the condition* (1.2)*, then*

(a) $\operatorname{Re}\left(\frac{f(z)}{f_k(z)}\right) \ge \frac{k+2-2\cosh 1}{k+1-\cosh 1}$,

(b) $\operatorname{Re}\left(\frac{f_k(z)}{f(z)}\right) \ge \frac{k+1-\cosh 1}{k}$

for all $z \in \mathbb{D}$. Sharpness follows by the function

$$f(z) = z + \frac{\cosh 1 - 1}{k+1-\cosh 1} z^{k+1}$$

for every k.

Proof. (a) Since

$$\frac{n+1-\cosh 1}{\cosh 1 - 1} > \frac{n-\cosh 1}{\cosh 1 - 1}, \ (n \geq 2)$$

it follows from (1.2) that

$$\sum_{n=2}^{k} |a_n| + \frac{k+1-\cosh 1}{\cosh 1 - 1} \sum_{n=k+1}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \frac{n-\cosh 1}{\cosh 1 - 1} |a_n| \leq 1.$$

Thus we write

$$\begin{aligned} \psi_1(z) &= 1 + \frac{k+1-\cosh 1}{\cosh 1 - 1}\left(\frac{f(z)}{f_k(z)} - 1\right) \\ &= \frac{1 + \sum_{n=2}^{k} a_n z^{n-1} + \frac{k+1-\cosh 1}{\cosh 1 - 1}\sum_{n=k+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{k} a_n z^{n-1}}, \end{aligned} \tag{2.1}$$

which is analytic in $\mathbb{D}$ with $\psi_1(0) = 1$. To obtain the lower bound, we need to show that

$$\psi_1(z) = \frac{1 + w_1(z)}{1 - w_1(z)}, \tag{2.2}$$

where w_1 is a Schwarz function with $w_1(0) = 0$ and $|w_1(z)| < 1$ in $\mathbb{D}$. Clearly, from (2.1) and (2.2), we get

$$w_1(z) = \frac{\frac{k+1-\cosh 1}{\cosh 1 - 1}\sum_{n=k+1}^{\infty} a_n z^{n-1}}{2 + 2\sum_{n=2}^{k} a_n z^{n-1} + \frac{k+1-\cosh 1}{\cosh 1 - 1}\sum_{n=k+1}^{\infty} a_n z^{n-1}}$$

and therefore

$$|w_1(z)| \leq \frac{\frac{k+1-\cosh 1}{\cosh 1 - 1}\sum_{n=k+1}^{\infty} |a_n|}{2 - 2\sum_{n=2}^{k} |a_n| - \frac{k+1-\cosh 1}{\cosh 1 - 1}\sum_{n=k+1}^{\infty} |a_n|}.$$

Thus, the last expression is equivalent to

$$\sum_{n=2}^{k} |a_n| + \frac{k+1-\cosh 1}{\cosh 1 - 1} \sum_{n=k+1}^{\infty} |a_n| \leq 1. \tag{2.3}$$

It is sufficient to show that the left hand side of (2.3) is bounded above by $\sum_{n=2}^{\infty}((n-\cosh 1)/(\cosh 1 - 1))|a_n|$. Thus we get

$$\sum_{n=2}^{k}\left(\frac{n+1-2\cosh 1}{\cosh 1 - 1}\right)|a_n| + \sum_{n=k+1}^{\infty}\left(\frac{n-k-1}{\cosh 1 - 1}\right)|a_n| \geq 0.$$

The function $f(z) = z + \frac{\cosh 1 - 1}{k+1-\cosh 1} z^{k+1}$ gives the sharpness as we observe that for $z = re^{i\pi/k}$ and $r \to 1^-$ that

$$\frac{f(z)}{f_k(z)} = 1 + \frac{\cosh 1 - 1}{k+1-\cosh 1} z^k \to 1 - \frac{\cosh 1 - 1}{k+1-\cosh 1} = \frac{k+2-2\cosh 1}{k+1-\cosh 1}.$$

(b) To prove the second part of the theorem, similarly

$$\psi_2(z) = 1 + \frac{k}{\cosh 1 - 1}\left(\frac{f_k(z)}{f(z)} - 1\right)$$
$$= \frac{1 + \sum_{n=2}^{\infty} a_n z^{n-1} - \frac{k}{\cosh 1 - 1}\sum_{n=k+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}} := \frac{1 + w_2(z)}{1 - w_2(z)},$$

so that we obtain

$$|w_2(z)| \leq \frac{\frac{k}{\cosh 1 - 1}\sum_{n=k+1}^{\infty} |a_n|}{2 - 2\sum_{n=2}^{k} |a_n| - \frac{k+2-2\cosh 1}{\cosh 1 - 1}\sum_{n=k+1}^{\infty} |a_n|}.$$

This inequality is bounded above by 1 and gives

$$\sum_{n=2}^{k} |a_n| + \frac{k + 1 - \cosh 1}{\cosh 1 - 1}\sum_{n=k+1}^{\infty} |a_n| \leq 1. \tag{2.4}$$

The left hand side of (2.4) is bounded above by

$$\sum_{n=2}^{\infty}((n - \cosh 1)/(\cosh 1 - 1))|a_n|,$$

and we get the proof. □

Theorem 2.2. *If a function $f \in \mathcal{A}$ satisfies the condition* (1.2)*, then*

(a) $\operatorname{Re}\left(\frac{f'(z)}{f_k'(z)}\right) \geq \frac{2k+2-(k+2)\cosh 1}{k+1-\cosh 1}$,

(b) $\operatorname{Re}\left(\frac{f_k'(z)}{f'(z)}\right) \geq \frac{k+1-\cosh 1}{k+1-\cosh 1+(k+1)(\cosh 1-1)}$

for all $z \in \mathbb{D}$. For every k, these results are sharp for the function

$$f(z) = z + \frac{\cosh 1 - 1}{k + 1 - \cosh 1} z^{k+1}.$$

Proof. (a) If the function $f \in \mathcal{S}^*_{\cosh}$, then from (1.2) we get

$$\sum_{n=2}^{k} n|a_n| + \frac{k + 1 - \cosh 1}{(k + 1)(\cosh 1 - 1)}\sum_{n=k+1}^{\infty} n|a_n| \leq 1.$$

Let

$$\psi_3(z) = 1 + \frac{k + 1 - \cosh 1}{(k + 1)(\cosh 1 - 1)}\left(\frac{f'(z)}{f_k'(z)} - 1\right)$$
$$= \frac{1 + \sum_{n=2}^{k} n a_n z^{n-1} + \frac{k+1-\cosh 1}{(k+1)(\cosh 1-1)}\sum_{n=k+1}^{\infty} n a_n z^{n-1}}{1 + \sum_{n=2}^{k} n a_n z^{n-1}}$$
$$:= \frac{1 + w_3(z)}{1 - w_3(z)},$$

where ψ_3 is analytic in $\mathbb{D}$ with $\psi_3(0) = 1$ and w_3 is a Schwarz function with $w_3(0) = 0$ and $|w_3(z)| < 1$ in $\mathbb{D}$ so that

$$w_3(z) = \frac{\frac{k+1-\cosh 1}{(k+1)(\cosh 1-1)} \sum_{n=k+1}^{\infty} n a_n z^{n-1}}{2 + 2\sum_{n=2}^{k} n a_n z^{n-1} + \frac{k+1-\cosh 1}{(k+1)(\cosh 1-1)} \sum_{n=k+1}^{\infty} n a_n z^{n-1}}$$

and

$$|w_3(z)| \leq \frac{\frac{k+1-\cosh 1}{(k+1)(\cosh 1-1)} \sum_{n=k+1}^{\infty} n|a_n|}{2 - 2\sum_{n=2}^{k} n|a_n| - \frac{k+1-\cosh 1}{(k+1)(\cosh 1-1)} \sum_{n=k+1}^{\infty} n|a_n|}.$$

Hence, the last expression is equivalent to

$$\sum_{n=2}^{k} n|a_n| + \frac{k+1-\cosh 1}{(k+1)(\cosh 1 - 1)} \sum_{n=k+1}^{\infty} n|a_n| \leq 1. \tag{2.5}$$

It is sufficient to show that the left hand side of (2.5) is bounded above by $\sum_{n=2}^{\infty}((n - \cosh 1)/(\cosh 1 - 1))|a_n|$, and we get

$$\sum_{n=2}^{k} \left(\frac{2n - (n+1)\cosh 1}{\cosh 1 - 1} \right) |a_n| + \sum_{n=k+1}^{\infty} \left(\frac{(n-k-1)\cosh 1}{(k+1)(\cosh 1 - 1)} \right) |a_n| \geq 0.$$

For $z = re^{i\pi/k}$ and $r \to 1^-$, we observe that

$$\begin{aligned} \frac{f'(z)}{f_k'(z)} &= 1 + \frac{(k+1)(\cosh 1 - 1)}{k+1-\cosh 1} z^k \\ &\to 1 - \frac{(k+1)(\cosh 1 - 1)}{k+1-\cosh 1} = \frac{2k+2-(k+2)\cosh 1}{k+1-\cosh 1} \end{aligned}$$

which shows the desired sharpness.

(b) It is similar to the proof of part (a), therefore it is omitted. □

3. Differential subordination

By using a technique based upon Briot-Bouquet differential subordination [18], we establish some subordination properties. In order to prove further results, we need the following lemmas.

Lemma 3.1 ([10]). *Let φ be convex univalent in $\mathbb{D}$ with $\varphi(0) = 1$, and let ϕ of the form $\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be analytic in $\mathbb{D}$ with $\phi(0) = 1$. If*

$$\phi(z) + \frac{1}{\mu} z\phi'(z) \prec \varphi(z), \quad (\mu \neq 0,\ \operatorname{Re}\mu \geq 0)$$

then

$$\phi(z) \prec \tilde{h}(z) = \frac{\mu}{z^\mu} \int_0^z t^{\mu-1} \varphi(t)dt \prec \varphi(z)$$

and $\tilde{h}$ is the best dominant of (1.4).

Lemma 3.2 ([18])**.** *Let* η $(\eta \neq 0)$ *and* μ *be complex constants, and let* φ $(\varphi(0) = 1)$ *be a convex univalent function in* $\mathbb{D}$ *with* $\mathrm{Re}(\eta\varphi(z) + \mu) > 0$*. Let* ϕ *be analytic in* $\mathbb{D}$ *and satisfy* (1.4)*. If the Briot-Bouquet differential equation*

$$q(z) + \frac{zq'(z)}{\eta q(z) + \mu} = \varphi(z), \quad (q(0) = 1) \tag{3.1}$$

has a univalent solution q*, then*

$$\phi(z) \prec q(z) \prec \varphi(z)$$

and q *is the best dominant of* (1.4)*.*

The differential equation (3.1) *has a formal solution given by*

$$q(z) = z^{\mu}[H(z)]^{\eta}\left(\eta \int_0^z [H(t)]^{\eta} t^{\mu-1} dt\right)^{-1} - \mu/\eta, \tag{3.2}$$

where

$$H(z) = z \exp \int_0^z \frac{\varphi(t) - 1}{t} dt.$$

Theorem 3.3. *Let* $0 < \lambda < 1$ *and* $\zeta \geq 1$*. If a function* $f \in \mathcal{A}$ *satisfies the condition*

$$(1 - \lambda)\frac{f(z)}{z} + \lambda\frac{\mathcal{L}_\sigma f(z)}{z} \prec \cosh z, \tag{3.3}$$

then

$$\mathrm{Re}\left\{\left(\frac{\mathcal{L}_\sigma(z)}{z}\right)^{1/\zeta}\right\} > \left(\frac{1+\sigma}{1-\lambda}\int_0^1 u^{\frac{1+\sigma}{1-\lambda}-1} \cos u du\right)^{1/\zeta}. \tag{3.4}$$

The result is sharp.

Proof. Consider the analytic function

$$\phi(z) = \frac{\mathcal{L}_\sigma f(z)}{z}, \quad (z \in \mathbb{D})$$

with $\phi(0) = 1$ in $\mathbb{D}$. Differentiating both sides of this identity, and using (1.5), we get

$$\frac{f(z)}{z} = \phi(z) + \frac{1}{1+\sigma} z\phi'(z).$$

Applying (3.3), we conclude that

$$(1 - \lambda)\frac{f(z)}{z} + \lambda\frac{\mathcal{L}_\sigma f(z)}{z} = \phi(z) + \frac{1-\lambda}{1+\sigma} z\phi'(z) \prec \cosh z.$$

It follows from Lemma 3.1 that

$$\phi(z) \prec \frac{1+\sigma}{1-\lambda} z^{-\frac{1+\sigma}{1-\lambda}} \int_0^z t^{\frac{1+\sigma}{1-\lambda}-1} \cosh t dt$$

or

$$\frac{\mathcal{L}_\sigma f(z)}{z} = \frac{1+\sigma}{1-\lambda}\int_0^1 u^{\frac{1+\sigma}{1-\lambda}-1} \cosh uw(z) du, \tag{3.5}$$

where w is a Schwarz function. Let $w(z) = Re^{it}$ with $R \leq |z| = r$, $-\pi \leq t \leq \pi$. Now, we find the minimum value of $\mathrm{Re}(\cosh w(z))$ with $w(z) = Re^{it}$. A computation shows that

$$\mathrm{Re}(\cosh Re^{it}) = \cosh(Rx)\cos(Ry),$$

where $x = \cos t$, $y = \sin t$ and $x, y \in [-1, 1]$. Thus we get

$$\mathrm{Re}(\cosh w(z)) > \cos r.$$

It follows that

$$\mathrm{Re}(\cosh uw(z)) > \cos ur.$$

Hence, for $0 < \lambda < 1$, and letting $r \to 1^-$, from (3.5) we arrive at

$$\mathrm{Re}\left(\frac{\mathcal{L}_\sigma f(z)}{z}\right) > \frac{1+\sigma}{1-\lambda}\int_0^1 u^{\frac{1+\sigma}{1-\lambda}-1}\cos u du > 0, \quad z \in \mathbb{D}. \tag{3.6}$$

Since $\mathrm{Re}(w^{1/\zeta}) \geq \mathrm{Re}(w)^{1/\zeta}$ for $\mathrm{Re}(w) > 0$ and $\zeta \geq 1$, from (3.6) we prove the inequality (3.4).

To prove sharpness, we take $f \in \mathcal{A}$ defined by

$$\frac{\mathcal{L}_\sigma f(z)}{z} = \frac{1+\sigma}{1-\lambda}\int_0^1 u^{\frac{1+\sigma}{1-\lambda}-1}\cosh uz du.$$

For this function we find that

$$(1-\lambda)\frac{f(z)}{z} + \lambda\frac{\mathcal{L}_\sigma f(z)}{z} \prec \cosh z,$$

and

$$\frac{\mathcal{L}_\sigma f(z)}{z} \to \frac{1+\sigma}{1-\lambda}\int_0^1 u^{\frac{1+\sigma}{1-\lambda}-1}\cosh u du$$

as $z \to 1^-$. This completes the proof. □

Theorem 3.4. *Let $\mathcal{L}_\sigma f(z) \neq 0$ and $\sigma > 0$. If f belongs to the class $\mathcal{S}^*_{\cosh}$ and*

$$\mathrm{Re}\big(\cosh z + \sigma\big) > 0,$$

then $\mathcal{L}_\sigma f \in \mathcal{S}^*_{\cosh}$. Furthermore, if $f \in \mathcal{S}^*_{\cosh}$ then

$$\frac{z\big(\mathcal{L}_\sigma f(z)\big)'}{\mathcal{L}_\sigma f(z)} \prec q(z) \prec \cosh z, \tag{3.7}$$

where

$$q(z) = z^\sigma e^{Chi(z)-\gamma}\left(\int_0^z t^{\sigma-1}e^{Chi(t)-\gamma}dt\right)^{-1} - \sigma, \tag{3.8}$$

and q is the best dominant of (3.7).

Proof. Consider the analytic function

$$\phi(z)=\frac{z\big(\mathcal{L}_\sigma f(z)\big)'}{\mathcal{L}_\sigma f(z)},\quad (z\in\mathbb{D})$$

with $\phi(0)=1$. By using (1.5), we get

$$(1+\sigma)\frac{f(z)}{\mathcal{L}_\sigma f(z)}=\phi(z)+\sigma.$$

Differentiating logarithmically with respect to z, from the last equation, we obtain

$$\frac{zf'(z)}{f(z)}=\phi(z)+\frac{z\phi'(z)}{\phi(z)+\sigma}\prec\cosh z. \tag{3.9}$$

Now, let us consider the differential equation

$$q(z)+\frac{zq'(z)}{q(z)+\sigma}=\varphi(z):=\cosh z, \tag{3.10}$$

where the function q is analytic with $q(0)=1$ and the function $\varphi(z)=\cosh z$ is convex univalent with $\varphi(0)=1$ in $\mathbb{D}$, and let $P(z)=\eta\varphi(z)+\mu$. According to (3.10) and Lemma 3.2, we observe that since $\eta=1$ and $\mu=\sigma$, then $P(z)=\cosh z+\sigma$. For proving $\mathrm{Re}(\cosh z+\sigma)>0$, it is enough to set $z=e^{it},\ t\in[0,\pi]$, and we get

$$\mathrm{Re}(\cosh z+\sigma)=\cosh(\cos t)\cos(\sin t)+\sigma>0$$

under the condition $\sigma>0$. Furthermore, $P(z)$ and $1/P(z)$ are convex. Hence, there is a univalent solution of the equation (3.10). To get this solution, we apply to Lemma 3.2. Since $\varphi(z)=\cosh z$, we find

$$\begin{aligned}H(z)&=z\exp\int_0^z\frac{\varphi(t)-1}{t}dt\\&=z\exp\int_0^z\frac{\cosh t-1}{t}dt=e^{\mathrm{Chi}(z)-\gamma},\end{aligned}$$

where Chi is the cosh integral and γ is the Euler's constant. Setting this result together with $\eta=1$ and $\mu=\sigma$ into the formula (3.2), we obtain (3.8) which is the solution of the differential equation given by (3.10). Since ϕ is analytic function satisfying the relation (3.9), then we have

$$\phi(z)\prec q(z)\prec\varphi(z):=\cosh z$$

and q is the best dominant of (3.7). □

4. Hankel determinants

In 2010, Babalola [3] determined the upper bound on $H_3(1)$ for the classes of starlike and convex functions. Kowalczyk *et al.* [12] computed the best possible bound 4/135 on $H_3(1)$ for convex functions. Recently, Kumar and Çetinkaya [13] computed an estimate on third order Hankel determinants for starlike functions associated with nephroid shaped domain. For more recent work on third Hakel determinant, see [19,20,22]. In this section, we determine an upper bound on the Hankel determinants $H_3(1)$, $H_3(2)$ and $H_3(3)$ of the third order for the function $f \in \mathcal{S}^*_{\cosh}$.

Theorem 4.1. *For the function $f \in \mathcal{S}^*_{\cosh}$, the following bounds on third order Hankel determinants hold*:

$$|H_3(1)| \leq \frac{23}{288} + \frac{3}{4\sqrt{131}} \approx 0.145389, \quad |H_3(2)| \leq \frac{509}{8640} \approx 0.058912,$$

and

$$|H_3(3)| \leq \frac{97\sqrt{\frac{97}{13647}}}{14400} + \frac{1256939}{13271040} + \frac{1}{16\sqrt{35}} \approx 0.105845.$$

In order to determine estimates on third order Hankel determinants, we use the following lemmas.

Lemma 4.2 ([24, Lemma 2.3, p. 507])**.** *Let $p \in \mathcal{P}$. Then for all $n, m \in \mathbb{N}$,*

$$|\mu p_n p_m - p_{m+n}| \leq \begin{cases} 2, & 0 \leq \mu \leq 1; \\ 2|2\mu - 1|, & \textit{elsewhere.} \end{cases}$$

If $0 < \mu < 1$, then the inequality is sharp for the function $p(z) = (1 + z^{m+n})/(1 - z^{m+n})$. In the other cases, the inequality is sharp for the function $p_0(z) = (1 + z)/(1 - z)$.

Lemma 4.3 ([14, Lemma 2.1, p. 1055])**.** *Let $p \in \mathcal{P}$. Then,*

$$|\mu p_3 - p_1^3| \leq \begin{cases} 2|\mu - 4|, & \mu \leq \frac{4}{3}; \\ 2\mu\sqrt{\frac{\mu}{\mu-1}}, & \mu > \frac{4}{3} \end{cases}$$

for any real number μ. The result is sharp. If $\mu \leq \frac{4}{3}$, then equality holds for the function $p_0(z) := (1 + z)/(1 - z)$ and if $\mu > \frac{4}{3}$, then equality holds for the function

$$p_1(z) := \frac{1 - z^2}{z^2 - 2\sqrt{\frac{\mu}{\mu-1}}\, z + 1}.$$

Proof of Theorem 4.1. Since $w \in \Omega$ and $w(z) = (p(z) - 1)/(p(z) + 1)$ where $p \in \mathcal{P}$ has the Taylor series expansion $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$, then on comparing the coefficients of like power terms in equation (1.3) by using the function given by (1.1) and $p \in \mathcal{P}$, we get

(4.1) $a_2 = 0,$

(4.2) $a_3 = \dfrac{1}{16} p_1^2,$

(4.3) $a_4 = -\dfrac{1}{24} p_1 \left(p_1^2 - 2p_2\right),$

(4.4) $a_5 = \dfrac{1}{192} \left(5p_1^4 - 18p_1^2 p_2 + 12 p_1 p_3 + 6p_2^2\right),$

$$a_6 = \frac{1}{1920}(-31p_1^5 + 158p_1^3p_2 - 144p_1^2p_3 - 144p_1p_2^2 + 96p_1p_4 + 96p_2p_3), \tag{4.5}$$

$$a_7 = \frac{1}{552960}(5537p_1^6 - 36360p_1^4p_2 + 37680p_1^3p_3 + 120p_1^2(469p_2^2 - 288p_4) - 23040p_1(3p_2p_3 - p_5) - 11520(p_2^3 - 2p_2p_4 - p_3^2)). \tag{4.6}$$

On substituting the values of a_2, a_3, a_4, a_5, a_6 and a_7 from (4.1), (4.2), (4.3), (4.4), (4.5), and (4.6), respectively, in the expressions (1.6), (1.7) and (1.8) of Hankel determinants of third order, we get

$$H_3(1) = \frac{1}{36864}(p_1^2\left(-13p_1^4 + 40p_1^2p_2 + 144p_1p_3 - 184p_2^2\right)), \tag{4.7}$$

$$H_3(2) = \frac{1}{4423680}(-p_1^3(p_1^6 - 18p_1^4p_2 + 144p_1^3p_3 - 96p_1^2(p_2^2 - 9p_4) - 2016p_1p_2p_3 + 1120p_2^3)), \tag{4.8}$$

and

(4.9)

$$\begin{aligned} H_3(3) = \frac{1}{25480396800}&(-37p_1^{12} + 442p_1^{10}p2 - 777600p_2^6 + 1284p_1^9p_3 \\ &+ 1969920p_1p_2^4p_3 - 2p_1^8(647p_2^2 + 10368p_4) \\ &- 96p_1^6(407p_2^3 + 582p_3^2 - 5568p_2p_4) \\ &- 1152p_1^2p_2^2(265p_2^3 + 2336p_3^2 - 1160p_2p_4) + 192p_1^7(241p_2p_3 - 600p_5) \\ &- 192p_1^5(3463p_2^2p_3 + 4752p_3p_4 - 6000p_2p_5) \\ &+ 4608p_1^3(147p_2^3p_3 - 900p_3^3 + 2052p_2p_3p_4 - 1150p_2^2p_5) \\ &+ 48p_1^4(7081p_2^4 + 31872p_2p_3^2 - 45408p_2^2p_4 + 3456(-24p_4^2 + 25p_3p_5))). \end{aligned}$$

After rearranging the terms and by making use of triangle inequality in the expressions (4.7), (4.8), and (4.9), respectively, we have

$$36864|H_3(1)| \leq 13|p_1^3| \left|\frac{144}{13}p_3 - p_1^3\right| + 184|p_1^2p_2| \left|\frac{5}{23}p_1^2 - p_2\right|, \tag{4.10}$$

$$4423680|H_3(2)| \leq 18|p_1^7| \left|\frac{1}{18}p_1^2 - p_2\right| + 1120|p_1^3p_2^2| \left|\frac{3}{35}p_1^2 - p_2\right| + 2016|p_1^4p_3| \left|\frac{1}{14}p_1^2 - p_2\right| + 864|p_1^5p_4|, \tag{4.11}$$

and

$$\begin{aligned} &25480396800|H_3(3)| \\ &\leq 442|p_1^{10}| \left|-\frac{37}{442}p_1^2 + p_2\right| + 339888|p_1^4p_2^3| \left|-\frac{814}{7081}p_1^2 + p_2\right| \\ &\quad + 46272|p_1^7p_2| \left|-\frac{27}{964}p_1p_2 + p_3\right| + 677376|p_1^3p_2^2p_3| \left|-\frac{3463}{3528}p_1^2 + p_2\right| \end{aligned} \tag{4.12}$$

$$+ 2691072|p_1^2p_2p_3^2| \left|\frac{83}{146}p_1^2 - p_2\right| + 534528|p_1^6p_4| \left|-\frac{9}{232}p_1^2 + p_2\right|$$
$$+ 1336320|p_1^2p_2^2p_4| \left|-\frac{473}{290}p_1^2 + p_2\right| + 9455616|p_1^3p_3p_4| \left|-\frac{11}{114}p_1^2 + p_2\right|$$
$$+ 5299200|p_1^3p_2p_5| \left|\frac{5}{23}p_1^2 - p_2\right| + 1969920|p_1p_2^5| \left|-\frac{53}{342}p_1p_2 + p_3\right|$$
$$+ 1284|p_1^6p_3| \left|p_1^3 - \frac{4656}{107}p_3\right| + 115200|p_1^4p_5| \left|-p_1^3 + 36p_3\right|$$
$$+ 777600|p_2^6| + 4147200|p_1^3p_3^3| + 3981312|p_1^4p_4^2|.$$

On repeated use of Lemma 4.2, Lemma 4.3 and the inequality $|p_n| \leq 2$ for all $n \in \mathbb{N}$ in expressions (4.10), (4.11) and (4.12), we get the desired estimates as

$$|H_3(1)| \leq \frac{1}{36864}\left(13(2)^4\sqrt{\frac{144}{131}} + 184(2)^4\right) \approx \frac{23}{288} + \frac{3}{4\sqrt{131}},$$

$$|H_3(2)| \leq \frac{1}{4423680}\left(18(2)^8 + 1120(2)^6 + 2016(2)^6 + 864(2)^6\right) = \frac{509}{8640},$$

$$\begin{aligned}|H_3(3)| \leq \frac{1}{25480396800}&(442(2)^{11} + 339888(2)^8 + 46272(2)^9 + 677376(2)^7\\ &+ 2691072(2)^6 + 534528(2)^8 + 1336320(2)^6(\frac{473}{290}) + 9455616(2)^6\\ &+ 5299200(2)^6 + 1969920(2)^7 + 1284(2)^8\frac{4656}{107}\sqrt{\frac{4656}{4549}}\\ &+ 777600(2)^6 + 4147200(2)^6 + 3981312(2)^6 + 115200(2)^6(216)\frac{1}{\sqrt{35}})\\ = \frac{97}{14400}\sqrt{\frac{97}{13647}} &+ \frac{1256939}{13271040} + \frac{1}{16\sqrt{35}}.\end{aligned}$$

This completes the proof. □

References

[1] O. P. Ahuja, S. Kumar, and V. Ravichandran, *Applications of first order differential subordination for functions with positive real part*, Stud. Univ. Babeş-Bolyai Math. **63** (2018), no. 3, 303–311. https://doi.org/10.24193/subbmath.2018.3.02

[2] A. Alotaibi, M. Arif, M. A. Alghamdi, and S. Hussain, *Starlikness associated with cosine hyperbolic function*, Mathematics **8** (2020), no. 7, 1118. https://doi.org/10.3390/math8071118

[3] K. O. Babalola, *On $H_3(1)$ Hankel determinant for some classes of univalent functions*, Ineq. Theory Appl. **6** (2010), 1–7.

[4] K. Bano and M. Raza, *Starlike functions associated with cosine functions*, Bull. Iranian Math. Soc. **47** (2021), no. 5, 1513–1532.

[5] N. Bohra, S. Kumar, and V. Ravichandran, *Some special differential subordinations*, Hacet. J. Math. Stat. **48** (2019), no. 4, 1017–1034. https://doi.org/10.15672/hjms.2018.570

[6] N. Bohra, S. Kumar, and V. Ravichandran, *Applications of Briot-Bouquet differential subordination*, J. Class. Anal. **18** (2021), no. 1, 17–28. https://doi.org/10.7153/jca-2021-18-02
[7] P. L. Duren, *Univalent functions*, Grundlehren der mathematischen Wissenschaften, 259, Springer, New York, 1983.
[8] P. J. Eenigenburg, S. S. Miller, P. T. Mocanu, and M. O. Reade, *On a Briot-Bouquet differential subordination*, in General inequalities, 3 (Oberwolfach, 1981), 339–348, Internat. Schriftenreihe Numer. Math., 64, Birkhäuser, Basel, 1983.
[9] R. M. Goel, *On the partial sums of a class of univalent functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **19** (1965), 17–23.
[10] D. J. Hallenbeck and S. Ruscheweyh, *Subordination by convex functions*, Proc. Amer. Math. Soc. **52** (1975), 191–195. https://doi.org/10.2307/2040127
[11] K. Hu and Y. F. Pan, *On a theorem of Szegő*, J. Math. Res. Exposition **4** (1984), no. 1, 41–44.
[12] B. Kowalczyk, A. Lecko, and Y. J. Sim, *The sharp bound for the Hankel determinant of the third kind for convex functions*, Bull. Aust. Math. Soc. **97** (2018), no. 3, 435–445. https://doi.org/10.1017/S0004972717001125
[13] S. Kumar and A. Çetinkaya, *Coefficient inequalities for certain starlike and convex functions*, Hacet. J. Math. Stat. **51** (2022), no. 1, 156–171. https://doi.org/10.15672/hujms.778148
[14] V. Kumar, N. E. Cho, V. Ravichandran, and H. M. Srivastava, *Sharp coefficient bounds for starlike functions associated with the Bell numbers*, Math. Slovaca **69** (2019), no. 5, 1053–1064. https://doi.org/10.1515/ms-2017-0289
[15] S. Kumar and V. Ravichandran, *Subordinations for functions with positive real part*, Complex Anal. Oper. Theory **12** (2018), no. 5, 1179–1191. https://doi.org/10.1007/s11785-017-0690-4
[16] S. K. Lee, V. Ravichandran, and S. Supramaniam, *Bounds for the second Hankel determinant of certain univalent functions*, J. Inequal. Appl. **2013** (2013), 281, 17 pp. https://doi.org/10.1186/1029-242X-2013-281
[17] T. H. MacGregor, *Majorization by univalent functions*, Duke Math. J. **34** (1967), 95–102. http://projecteuclid.org/euclid.dmj/1077376845
[18] S. S. Miller and P. T. Mocanu, *Univalent solutions of Briot-Bouquet differential equations*, J. Differential Equations **56** (1985), no. 3, 297–309. https://doi.org/10.1016/0022-0396(85)90082-8
[19] A. Naz, S. Kumar, and V. Ravichandran, *Coefficient functionals and radius problems of certain starlike functions*, Asian-Eur. J. Math. **15** (2022), no. 5, Paper No. 2250089, 19 pp. https://doi.org/10.1142/S1793557122500899
[20] M. Obradović, N. Tuneski, and P. Zaprawa, *Sharp bounds of the third Hankel determinant for classes of univalent functions with bounded turning*, Math. Bohem. **147** (2022), no. 2, 211–220.
[21] C. Pommerenke, *On the coefficients and Hankel determinants of univalent functions*, J. London Math. Soc. **41** (1966), 111–122. https://doi.org/10.1112/jlms/s1-41.1.111
[22] B. Rath, K. Sanjay Kumar, D. V. Krishna, and A. Lecko, *The sharp bound of the third Hankel determinant for starlike functions of order* 1/2, Complex Anal. Oper. Theory **16** (2022), no. 5, Paper No. 65, 8 pp. https://doi.org/10.1007/s11785-022-01241-8
[23] V. Ravichandran, *Geometric properties of partial sums of univalent function*, Mathematics Newsletter **22** (2012), no. 3, 208–221.
[24] V. Ravichandran and S. Verma, *Bound for the fifth coefficient of certain starlike functions*, C. R. Math. Acad. Sci. Paris **353** (2015), no. 6, 505–510. https://doi.org/10.1016/j.crma.2015.03.003
[25] H. Silverman, *Partial sums of starlike and convex functions*, J. Math. Anal. Appl. **209** (1997), no. 1, 221–227. https://doi.org/10.1006/jmaa.1997.5361

[26] E. M. Silvia, *On partial sums of convex functions of order α*, Houston J. Math. **11** (1985), no. 3, 397–404.

[27] G. Szegő, *Zur Theorie der schlichten Abbildungen*, Math. Ann. **100** (1928), no. 1, 188–211. https://doi.org/10.1007/BF01448843

[28] H. Tang, H. M. Srivastava, S. Li, and G. Deng, *Majorization results for subclasses of starlike functions based on the sine and cosine functions*, Bull. Iranian Math. Soc. **46** (2020), no. 2, 381–388. https://doi.org/10.1007/s41980-019-00262-y

Om P. Ahuja
Department of Mathematical Sciences
Kent State University
Ohio, 44021, U.S.A
Email address: oahuja@kent.edu

Asena Çetinkaya
Department of Mathematics and Computer Science
İstanbul Kültür University
İstanbul, Turkey
Email address: asnfigen@hotmail.com

Sushil Kumar
Bharati Vidyapeeth's College of Engineering
Delhi–110063, India
Email address: sushilkumar16n@gmail.com