Commun. Korean Math. Soc. **39** (2024), No. 2, pp. 399–406
https://doi.org/10.4134/CKMS.c220346
pISSN: 1225-1763 / eISSN: 2234-3024

ON THE CONVERGENCE OF ISHIKAWA ITERATION WITH ERRORS FOR REAL CONTINUOUS FUNCTIONS

Kittithat Boonpot and Satit Saejung

Abstract. We point out an error appeared in the paper of Yuan et al. [3] and present a correction of their result under a more general assumption. Moreover, we discuss the validity of the conditions imposed on the sequences of error terms.

1. Introduction

Iterative sequences are one of powerful tools in solving nonlinear equations. Mann and Ishikawa iterations are two interesting iterative sequences which have been studied and investigated by many authors. In this paper, we focus on the problem of finding a fixed point of a continuous function defined on the real line. More precisely, in this paper, we assume that $\mathbb{R}$ is the set of all real numbers and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with the (probably empty) fixed-point set $\mathrm{Fix}(f) := \{p \in \mathbb{R} : p = f(p)\}$. Suppose that $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are two sequences in $[0, 1]$. The following two iterative sequences are known as Mann iteration and Ishikawa iteration.

Mann iteration: $x_1 \in \mathbb{R}$ is arbitrarily chosen and

$$x_{n+1} := (1 - \alpha_n)x_n + \alpha_n f(x_n), \quad \text{where } n \geq 1.$$

Ishikawa iteration: $x_1 \in \mathbb{R}$ is arbitrarily chosen and

$$\begin{aligned} y_n &:= (1 - \beta_n)x_n + \beta_n f(x_n), \\ x_{n+1} &:= (1 - \alpha_n)x_n + \alpha_n f(y_n), \quad \text{where } n \geq 1. \end{aligned}$$

It is clear that Mann iteration can be deduced from Ishikawa iteration by letting $\beta_n = 0$ for all $n \geq 1$. Borwein and Borwein [1] proved that if $f : [a, b] \to [a, b]$ is continuous, then every iterative sequence generated by Mann iteration with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$ converges to a fixed point of f. Yuan and Liu [4] extended the result to Ishikawa iteration for a continuous function $f : E \to E$, where E is a closed (not necessarily bounded) interval.

Received November 28, 2022; Revised March 17, 2023; Accepted April 6, 2023.
2020 *Mathematics Subject Classification.* 47H10.
Key words and phrases. Fixed point, real continuous function, Mann iteration with error, Ishikawa iteration with error.

Theorem QQ. *Suppose that E is a closed (not necessarily bounded) interval and $f : E \to E$ is continuous. Suppose that $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are two sequences in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then every iterative sequence $\{x_n\}_{n=1}^{\infty}$ generated by Ishikawa iteration converges to a fixed point of f if and only if $\{x_n\}_{n=1}^{\infty}$ is bounded.*

Yuan et al. [3] extended Theorem QQ, where the error in the computation of the iterative sequence is allowed.

Theorem YCQ. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Suppose that $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are two sequences in $(0,1]$ such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose that $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are two real sequences satisfying the following two conditions:*

(C1) $\sum_{n=1}^{\infty} |u_n| < \infty$ *and* $\sum_{n=1}^{\infty} |v_n| < \infty$;
(C2) $\lim_{n\to\infty} |u_n|/\alpha_n = \lim_{n\to\infty} |v_n|/\beta_n = 0$.

Suppose that $\{x_n\}_{n=1}^{\infty}$ is generated by Ishikawa iteration with errors:

Ishikawa iteration with errors: *$x_1 \in \mathbb{R}$ is arbitrarily chosen and*

$$\begin{aligned} y_n &:= (1-\beta_n)x_n + \beta_n f(x_n) + v_n, \\ x_{n+1} &:= (1-\alpha_n)x_n + \alpha_n f(y_n) + u_n, \quad \text{where } n \geq 1. \end{aligned}$$

Then $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f if and only if it is bounded.

Remark 1.1. It is worth noting that Theorem YCQ deals with only continuous functions $f : \mathbb{R} \to \mathbb{R}$. However, it can be applied for all continuous functions $g : E \to E$, where E is a closed interval, such that the iterative sequences are defined in E as well. In fact, if $g : E \to E$ is continuous, then there exists a continuous extension $\widehat{g} : \mathbb{R} \to \mathbb{R}$ such that $\widehat{g}|_E = g$ and $\mathrm{Fix}(\widehat{g}) = \mathrm{Fix}(g)$. A constructive extension is given in Remark 2.3 of [3].

Unfortunately, there is a mistake in the proof of Theorem YCQ above concerning the induction step (line 15 on page 233). In fact, the expression $|x_{m+1} - x_M| = |u_m| + |v_m|$ is not true. In this paper, we present a correction of Theorem YCQ with a more general assumption. Moreover, we also discuss the validity of the conditions imposed on the sequence of error terms.

2. Main results

The following result improves and strengthens Theorem YCQ.

Theorem 2.1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Suppose that $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are two sequences in $[0,1]$ such that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Suppose that $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are two real sequences satisfying the two following conditions:*

(C1*) $\sum_{n=1}^{\infty} (u_n + \alpha_n v_n)$ *is convergent*;
(C2*) $\lim_{n\to\infty} v_n = 0$.

Suppose that $\{x_n\}_{n=1}^{\infty}$ *is generated by Ishikawa iteration with errors. Then* $\{x_n\}_{n=1}^{\infty}$ *converges to a fixed point of* f *if and only if it is bounded.*

Proof. We assume that $\{x_n\}_{n=1}^{\infty}$ is bounded and hence $\{f(x_n)\}_{n=1}^{\infty}$ is bounded. Since

$$y_n = x_n + \beta_n(f(x_n) - x_n) + v_n$$

and $\{v_n\}_{n=0}^{\infty}$ is bounded, it follows that $\{y_n\}_{n=1}^{\infty}$ is bounded and hence $\{f(y_n)\}_{n=1}^{\infty}$ is bounded. In particular, it follows from $\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} v_n = 0$ that $\lim_{n\to\infty}(y_n - x_n) = 0$. Note that $\lim_{n\to\infty} u_n = \lim_{n\to\infty}(u_n + \alpha_n v_n) = 0$. Similarly, it follows from $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} u_n = 0$ that

$$\lim_{n\to\infty}(x_{n+1} - x_n) = \lim_{n\to\infty}\big(\alpha_n\,(f(y_n) - x_n) + u_n\big) = 0.$$

We now divide the proof into two steps.

Step 1: $\{x_n\}$ *is a convergent sequence.* Suppose that the assertion is not true, that is, $\alpha := \liminf_{n\to\infty} x_n < \limsup_{n\to\infty} x_n =: \beta$. We prove that if $\gamma \in (\alpha, \beta)$, then $\gamma = f(\gamma)$. To see this, let $\alpha < \gamma < \beta$. Without loss of generality, we assume that $f(\gamma) > \gamma$. By the continuity of f at γ, there exists $\delta > 0$ such that $f(x) - x > 0$ for all $x \in \mathbb{R}$ with $|x - \gamma| < \delta$. For convenience, we write

$$\widehat{x}_n := x_n + \sum_{k=n}^{\infty}(u_k + \alpha_k v_k).$$

Note that $\liminf_{n\to\infty} \widehat{x}_n = \alpha$ and $\limsup_{n\to\infty} \widehat{x}_n = \beta$. Moreover,

$$\lim_{n\to\infty}(\widehat{x}_{n+1} - \widehat{x}_n) = \lim_{n\to\infty}(x_{n+1} - x_n - (u_n + \alpha_n v_n)) = 0.$$

Since $\lim_{n\to\infty}(y_n - x_n) = \lim_{n\to\infty}(u_n + \alpha_n v_n) = 0$, there exists an integer N such that

$$\widehat{x}_N > \gamma,\ |y_n - x_n| < \frac{\delta}{4},\ |\widehat{x}_{n+1} - \widehat{x}_n| < \frac{\delta}{2},\ \left|\sum_{k=n}^{\infty}(u_k + \alpha_k v_k)\right| < \frac{\delta}{4} \text{ for all } n \geq N.$$

We consider the following two cases.

Case 1. $\gamma < \widehat{x}_N < \gamma + \frac{\delta}{2}$. This implies that

$$|x_N - \gamma| \leq |\widehat{x}_N - \gamma| + \left|\sum_{k=N}^{\infty}(u_k + \alpha_k v_k)\right| < \frac{\delta}{2} + \frac{\delta}{4} < \delta.$$

It follows that $f(x_N) - x_N > 0$. Since $|y_N - x_N| < \frac{\delta}{4}$, we have

$$|y_N - \gamma| \leq |y_N - x_N| + |x_N - \gamma| < \frac{\delta}{4} + \frac{3\delta}{4} = \delta.$$

This implies that $f(y_N) - y_N > 0$ and hence

$$\begin{aligned}
x_{N+1} &= x_N + \alpha_N\,(f(y_N) - x_N) + u_N \\
&= x_N + \alpha_N(f(y_N) - y_N) + \alpha_N\,(y_N - x_N) + u_N \\
&= x_N + \alpha_N(f(y_N) - y_N) + \alpha_N\,(\beta_N\,(f(x_N) - x_N)) + u_N + \alpha_N v_N
\end{aligned}$$

$$> x_N + u_N + \alpha_N v_N.$$

In particular, $\widehat{x}_{N+1} > \widehat{x}_N > \gamma$.

Case 2. $\widehat{x}_N \geq \gamma + \frac{\delta}{2}$. In this case, we have

$$\widehat{x}_{N+1} \geq \widehat{x}_N - |\widehat{x}_{N+1} - \widehat{x}_N| > \gamma + \frac{\delta}{2} - \frac{\delta}{2} = \gamma.$$

It follows from the two cases above that $\widehat{x}_{N+1} > \gamma$. We can prove by induction that $\widehat{x}_n > \gamma$ for all $n \geq N$ and hence $\alpha = \liminf_{n\to\infty} \widehat{x}_n \geq \gamma$ which is a contradiction. This implies that $\gamma = f(\gamma)$.

Let $\eta := \frac{1}{4}(\beta - \alpha) > 0$. Since the series $\sum_{n=1}^{\infty}(u_n + \alpha_n v_n)$ converges and $\lim_{n\to\infty}(y_n - x_n) = 0$, there exists an integer M such that

$$\left|\sum_{n=m}^{\infty}(u_n + \alpha_n v_n)\right| < \frac{\eta}{4}, \quad \text{and} \quad |y_m - x_m| < \eta \quad \text{for all } m \geq M.$$

Since $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$, there exists $K \geq M$ such that $|x_K - \frac{1}{2}(\alpha+\beta)| < \frac{\eta}{2}$. We prove by induction that

$$x_{m+1} = x_m + u_m + \alpha_m v_m \in \left(\frac{1}{2}(\alpha+\beta) - \eta, \frac{1}{2}(\alpha+\beta) + \eta\right) \quad \text{for all } m \geq K.$$

The starting point of the induction is true because $x_K \in (\alpha, \beta)$ and hence $f(x_K) = x_K$. Since $|y_K - x_K| < \eta$, we have

$$\left|y_K - \frac{1}{2}(\alpha+\beta)\right| \leq |y_K - x_K| + \left|x_K - \frac{1}{2}(\alpha+\beta)\right| < \eta + \frac{\eta}{2} = \frac{3\eta}{2}.$$

In particular, $y_K \in (\alpha, \beta)$ and hence $f(y_K) = y_K$. This implies that

$$y_K = (1 - \beta_K)x_K + \beta_K f(x_K) + v_K = x_K + v_K$$

and hence

$$x_{K+1} = (1 - \alpha_K)x_K + \alpha_K f(y_K) + u_K = x_K + u_K + \alpha_K v_K.$$

Moreover, we have

$$\begin{aligned}
&\left|x_{K+1} - \frac{1}{2}(\alpha+\beta)\right| \\
\leq\ &\left|x_K - \frac{1}{2}(\alpha+\beta)\right| + |u_K + \alpha_K v_K| \\
=\ &\left|x_K - \frac{1}{2}(\alpha+\beta)\right| + \left|\sum_{n=K}^{\infty}(u_n + \alpha_n v_n) - \sum_{n=K+1}^{\infty}(u_n + \alpha_n v_n)\right| \\
\leq\ &\left|x_K - \frac{1}{2}(\alpha+\beta)\right| + \left|\sum_{n=K}^{\infty}(u_n + \alpha_n v_n)\right| + \left|\sum_{n=K+1}^{\infty}(u_n + \alpha_n v_n)\right| \\
<\ &\frac{\eta}{2} + \frac{\eta}{4} + \frac{\eta}{4} = \eta.
\end{aligned}$$

This implies that $x_{K+1} \in \big(\frac{1}{2}(\alpha+\beta)-\eta, \frac{1}{2}(\alpha+\beta)+\eta\big)$ and the proof of the initial step of the induction is done.

Assume that there exists $m \geq K$ and the induction hypothesis is true for all integers j such that $K \leq j \leq m$. We show that

$$x_{m+2} = x_{m+1} + u_{m+1} + \alpha_{m+1} v_{m+1} \in \left(\frac{1}{2}(\alpha+\beta)-\eta, \frac{1}{2}(\alpha+\beta)+\eta\right).$$

Note that $x_{m+1} = x_m + u_m + \alpha_m v_m \in \big(\frac{1}{2}(\alpha+\beta)-\eta, \frac{1}{2}(\alpha+\beta)+\eta\big)$ and hence $f(x_{m+1}) = x_{m+1}$. Since $|y_{m+1} - x_{m+1}| < \eta$, we have $y_{m+1} \in (\alpha, \beta)$ and hence $f(y_{m+1}) = y_{m+1}$. This implies $y_{m+1} = x_{m+1} + v_{m+1}$ and hence

$$\begin{aligned} x_{m+2} &= (1-\alpha_{m+1})x_{m+1} + \alpha_{m+1} f(y_{m+1}) + u_{m+1} \\ &= x_{m+1} + u_{m+1} + \alpha_{m+1} v_{m+1}. \end{aligned}$$

In particular, we have

$$x_{m+2} = x_K + \sum_{n=K}^{m+1} (u_n + \alpha_n v_n).$$

Moreover, we have

$$\begin{aligned} &\left| x_{m+2} - \frac{1}{2}(\alpha+\beta) \right| \\ &\leq \left| x_K - \frac{1}{2}(\alpha+\beta) \right| + \left| \sum_{n=K}^{m+1} (u_n + \alpha_n v_n) \right| \\ &= \left| x_K - \frac{1}{2}(\alpha+\beta) \right| + \left| \sum_{n=K}^{\infty} (u_n + \alpha_n v_n) - \sum_{n=m+2}^{\infty} (u_n + \alpha_n v_n) \right| \\ &\leq \left| x_K - \frac{1}{2}(\alpha+\beta) \right| + \left| \sum_{n=K}^{\infty} (u_n + \alpha_n v_n) \right| + \left| \sum_{n=m+2}^{\infty} (u_n + \alpha_n v_n) \right| \\ &< \frac{\eta}{2} + \frac{\eta}{4} + \frac{\eta}{4} = \eta. \end{aligned}$$

This implies that $x_{m+2} \in \big(\frac{1}{2}(\alpha+\beta)-\eta, \frac{1}{2}(\alpha+\beta)+\eta\big)$. We now complete the proof of the induction. In particular,

$$\limsup_{n\to\infty} x_n = x_K + \sum_{n=K}^{\infty} (u_n + \alpha_n v_n) \leq \frac{1}{2}(\alpha+\beta) + \frac{\eta}{2} + \frac{\eta}{4} = \frac{1}{2}(\alpha+\beta) + \frac{3\eta}{4} < \beta$$

which is a contradiction.

Step 2: $\lim_{n\to\infty} x_n = p$ *for some* $p \in \mathrm{Fix}(f)$. We assume from **Step 1** that $\lim_{n\to\infty} x_n = p$ for some $p \in \mathbb{R}$. We prove that $f(p) = p$. Without loss of generality, we suppose that $f(p) > p$. Note that $\lim_{n\to\infty} y_n = p$ because $\lim_{n\to\infty}(x_n - y_n) = 0$. This implies that $\lim_{n\to\infty}(f(x_n) - x_n) =$

$\lim_{n\to\infty}(f(y_n) - y_n) = f(p) - p$. Then, there exists N such that $f(x_n) - x_n > \frac{1}{2}(f(p) - p) > 0$ and $f(y_n) - y_n > \frac{1}{2}(f(p) - p)$ for all $n \geq N$. Note that

$$x_{n+1} - x_n = \alpha_n \left(f(y_n) - y_n\right) + \alpha_n\beta_n(f(x_n) - x_n) + u_n + \alpha_n v_n.$$

In particular,

$$\sum_{n=N}^{\infty}(x_{n+1}-x_n) = \sum_{n=N}^{\infty}\Big(\alpha_n \left(f(y_n) - y_n\right)+\alpha_n\beta_n(f(x_n)-x_n)\Big)+\sum_{n=N}^{\infty}(u_n+\alpha_n v_n).$$

Since the series $\sum_{n=N}^{\infty}(x_{n+1} - x_n)$ and $\sum_{n=N}^{\infty}(u_n + \alpha_n v_n)$ are convergent, we have

$$\sum_{n=N}^{\infty}\Big(\alpha_n \left(f(y_n) - y_n\right) + \alpha_n\beta_n(f(x_n) - x_n)\Big) \text{ is convergent.}$$

Since $f(x_n) - x_n > \frac{1}{2}(f(p) - p) > 0$ and $f(y_n) - y_n > \frac{1}{2}(f(p) - p)$ for all $n \geq N$, we have

$$\frac{1}{2}(f(p) - p)\sum_{n=N}^{\infty}(\alpha_n + \alpha_n\beta_n) \text{ is convergent.}$$

This is impossible because $\sum_{n=N}^{\infty}\alpha_n = \infty$. The proof is finished. $\square$

Remark 2.2. It is clear that (C1) $\Longrightarrow$ (C1*); and (C2) $\Longrightarrow$ (C2*). Moreover, our result is a *strict* generalization of Theorem YCQ. In fact, the following choices $\alpha_n = \beta_n := 1/\sqrt{n}$ and $u_n = v_n := (-1)^n/n$ for all $n \geq 1$ are applicable in our result but not in Theorem YCQ.

Consequently, we obtain Mann type iteration with errors which is deduced from Theorem 2.1 by letting $\beta_n = v_n := 0$ for all $n \geq 1$.

Corollary 2.3. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous. Suppose that $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $[0, 1]$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$ and suppose that $\{u_n\}_{n=1}^{\infty}$ is a real sequence such that $\sum_{n=1}^{\infty}u_n$ converges. Suppose that $\{x_n\}_{n=1}^{\infty}$ is generated by*

Mann iteration with errors: *$x_1 \in \mathbb{R}$ is arbitrarily chosen and*

$$x_{n+1} := (1 - \alpha_n)x_n + \alpha_n f(x_n) + u_n, \quad \text{where } n \geq 1.$$

Then $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f if and only if it is bounded.

Finally we discuss the following result of Cholamjiak [2, Theorem 2.3]. Let us recall his result.

Theorem Ch. *Let E be a closed interval on $\mathbb{R}$ such that $E + E \subset E$ and let $f : E \to E$ be a continuous function. Let $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ be two sequences in E and let $\{x_n\}_{n=1}^{\infty}$ be generated by the following scheme: $x_1 \in E$ and*

$$\begin{aligned} y_n &:= (1 - \beta_n)x_n + \beta_n f(x_n) + v_n, \\ x_{n+1} &:= (1 - \alpha_n)y_n + \alpha_n f(y_n) + u_n \quad \text{for all } n \geq 1, \end{aligned}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0, 1]$ satisfying the conditions:

(A1) $\lim_{n\to\infty}\alpha_n = 0$, *$\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\sum_{n=1}^{\infty}\beta_n < \infty$;*
(A2) *$\sum_{n=1}^{\infty}|u_n| < \infty$ and $\sum_{n=1}^{\infty}|v_n| < \infty$;*
(A3) $\lim_{n\to\infty}|u_n|/\alpha_n = \lim_{n\to\infty}|v_n|/\beta_n = 0$.

Then $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f if and only if it is bounded.

Remark 2.4. We remark the following issues on Theorem Ch.

(1) The condition $E + E \subset E$ is not needed as discussed in Remark 1.1. Moreover, this condition is very restricted. In fact, one can see that any closed interval E such that $E + E \subset E$ is of the form $[a,\infty)$ or $(-\infty,-a]$ where $a \geq 0$.
(2) The iterative scheme studied in Theorem Ch is nothing but the following Mann iteration with errors (of Corollary 2.3): $\widehat{x}_1$ is arbitrarily chosen and

$$\begin{aligned}\widehat{x}_{n+0.5} &:= (1-\beta_n)\widehat{x}_n + \beta_n f(\widehat{x}_n) + v_n,\\ \widehat{x}_{n+1} &:= (1-\alpha_n)\widehat{x}_{n+0.5} + \alpha_n f(\widehat{x}_{n+0.5}) + u_n \quad \text{for all } n \geq 1.\end{aligned}$$

In particular, the conclusion of Theorem Ch follows from our Corollary 2.3 with the following weaker assumptions:
(A1*) $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=1}^{\infty}(\alpha_n+\beta_n) = \infty$;
(A2*) $\sum_{n=1}^{\infty}(u_n+v_n)$ converges.

3. A further discussion on Condition (C1*)

In this section, we show that Condition (C1*) on the real sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ is not too strong.

Definition. Suppose that $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are two sequences in $[0,1]$ such that $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. We say that two real sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are *admissible* if for every continuous function $f : \mathbb{R} \to \mathbb{R}$ and for every bounded iterative sequence $\{x_n\}_{n=1}^{\infty}$ generated by Ishikawa iteration with errors it follows that $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of f.

Theorem 3.1. *Suppose that $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ are two sequences in $[0,1]$ such that $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. If two real sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are admissible and $\{\sum_{k=1}^{n}(u_k+\alpha_k v_k)\}_{n=1}^{\infty}$ is bounded, then Condition* (C1*) *holds.*

Proof. We assume that two real sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are admissible and $\{\sum_{k=1}^{n}(u_k+\alpha_k v_k)\}_{n=1}^{\infty}$ is bounded. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := x$ for all $x \in \mathbb{R}$ and $x_1 := 0$ and let

$$\begin{aligned}y_n &:= (1-\beta_n)x_n + \beta_n f(x_n) + v_n = x_n + v_n,\\ x_{n+1} &:= (1-\alpha_n)x_n + \alpha_n f(y_n) + u_n = x_n + u_n + \alpha_n v_n\end{aligned}$$

for all $n \geq 1$. Note that $x_{n+1} = \sum_{k=1}^{n}(u_k+\alpha_k v_k)$ and the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. Since $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ are admissible, it follows that $\lim_{n\to\infty}x_n = p$ for some $p \in \mathbb{R}$. Hence $\sum_{n=1}^{\infty}(u_n+\alpha_n v_n)$ is convergent. $\square$

Acknowledgement. The authors would like to thank the referee for his/her comments and suggestions. S. Saejung was supported by National Research Council of Thailand and Khon Kaen University under grant N42A650290; and the Research and Graduate Studies at Khon Kaen University.

References

[1] D. Borwein and J. M. Borwein, *Fixed point iterations for real functions*, J. Math. Anal. Appl. **157** (1991), no. 1, 112–126. https://doi.org/10.1016/0022-247X(91)90139-Q

[2] P. Cholamjiak, *On Ishikawa-type iteration with errors for a continuous real function on an arbitrary interval*, Appl. Math. Sci. (Ruse) **7** (2013), no. 37-40, 1901–1907. https://doi.org/10.12988/ams.2013.13172

[3] Q. Yuan, S. Y. Cho, and X. L. Qin, *Convergence of Ishikawa iteration with error terms on an arbitrary interval*, Commun. Korean Math. Soc. **26** (2011), no. 2, 229–235. https://doi.org/10.4134/CKMS.2011.26.2.229

[4] Q. Yuan and Q. H. Liu, *The necessary and sufficient condition for the convergence of Ishikawa iteration on an arbitrary interval*, J. Math. Anal. Appl. **323** (2006), no. 2, 1383–1386. https://doi.org/10.1016/j.jmaa.2005.11.058

Kittithat Boonpot
Department of Mathematics
Faculty of Science
Khon Kaen University
Khon Kaen, 40002, Thailand
Email address: kitithat.b@kkumail.com

Satit Saejung
Department of Mathematics
Faculty of Science
Khon Kaen University
Khon Kaen, 40002, Thailand
and
Research Center for Environmental and
Hazardous Substance Management (EHSM)
Khon Kaen University
Khon Kaen 40002, Thailand
Email address: saejung@kku.ac.th