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WHEN EVERY FINITELY GENERATED REGULAR IDEAL IS FINITELY PRESENTED

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Abstract. In this paper, we introduce a weak version of coherent that we call regular coherent property. A ring is called regular coherent, if every finitely generated regular ideal is finitely presented. We investigate the stability of this property under localization and homomorphic image, and its transfer to various contexts of constructions such as trivial ring extensions, pullbacks and amalgamated. Our results generate examples which enrich the current literature with new and original families of rings that satisfy this property.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are nonzero unital. Let R denote such a ring, we denote by $Reg(R)$ and $Z(R)$ the set of all regular elements of R and the set of all zero-divisors of R, respectively. By a "local" ring we mean a (not necessarily Noetherian) ring with a unique maximal ideal. For a nonnegative integer n, an R-module E is called n-presented if there is an exact sequence of R-modules:

$$F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to E \to 0,$$

where each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-modules are, respectively, finitely generated and finitely presented R-modules.

A ring R is coherent if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R. Examples of coherent rings are Noetherian rings, Boolean algebras, von Neumann regular rings, valuation rings, and Prüfer/semihereditary rings. The concept of coherence first sprang up from the study of coherent sheaves in algebraic geometry, and then developed, under the influence of Noetherian ring theory and homology, towards a

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full-fledged topic in algebra. During the past 30 years, several (commutative) coherent-like notions grew out of coherence such as finite conductor, quasi-coherent, v-coherent, and n-coherent. See for instance [2, 12, 14, 15, 20].

Some of our results use the $R \propto M$ construction. Let R be a ring and M be an R-module. Then $R \propto M$, the *trivial (ring) extension of* R *by* M, is the ring whose additive structure is that of the external direct sum $R \oplus M$ and whose multiplication is defined by $(r_1, m_1)(r_2, m_2) := (r_1r_2, r_1m_2 + r_2m_1)$ for all $r_1, r_2 \in R$ and all $m_1, m_2 \in M$. The basic properties of trivial ring extensions are summarized in the books [12, 13]. Mainly, trivial ring extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [1, 2, 8, 9, 12, 13, 16, 20, 26].

Let T be a ring and let M be an ideal of T. Denote by π the natural surjection $\pi : T \longrightarrow T/M$. Let D be a subring of T/M . Then, $R := \pi^{-1}(D)$ is a subring of T and M is a common ideal of R and T such that $D = R/M$. The ring R is known as the pullback associated to the following pullback diagram:

$$\begin{tikzcd} R := \pi^{-1}(D) \arrow[r, "\pi|_R"] \arrow[d, "i"] & D = R/M \arrow[d, "j"] \\ T \arrow[r, "\pi"] & T/M \end{tikzcd}$$

where i and j are the natural injections.

A particular case of this pullback is the $D+M$-construction, when the ring T is of the form $K + M$, where K is a field and M is a maximal ideal of T, and R takes the form $D + M$. See for instance [12].

Let A and B be two rings, let J be an ideal of B and let $f : A \to B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) : a \in A, j \in J\}$$

called the amalgamation of A with B along J with respect to f (introduced and studied by D'Anna, Finocchiaro, and Fontana in [4,5]). This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [3, 6, 7] and denoted by $A \bowtie I$). Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation [4, Examples 2.5 & 2.6] and other classical constructions, such as the Nagata's idealization and the CPI extensions (in the sense of Boisen and Sheldon) are strictly related to it (see [4, Example 2.7 & Remark 2.8]).

One of the key tools for studying $A \bowtie^f J$ is based on the fact that the amalgamation can be studied in the frame of pullback constructions [4, Section 4]. This point of view allows the authors in [4,5] to provide an ample description of various properties of $A \bowtie^f J$, in connection with the properties of A, J and f. Namely, in [4], the authors studied the basic properties of this construction (e.g., characterizations for $A \bowtie^f J$ to be a Noetherian ring, an integral domain, a reduced ring) and they characterized those distinguished pullbacks that can be expressed as an amalgamation. See for instance [3–7, 9–11, 15, 17, 18, 24–26].

In this paper, we investigate the possible transfer of regular coherent property to the direct product of rings and various trivial extension constructions. Also, we examine the transfer of regular coherent property to a particular pullbacks and to the amalgamated duplication of ring along an ideal. Using these results, we construct several classes of examples of non-coherent regular coherent rings.

2. Main results

A ring is called regular coherent and noted reg-coherent, if every finitely generated regular ideal is finitely presented (that is every finitely generated proper regular ideal is finitely presented). Now we give the following natural results.

Proposition 2.1. (1) *A reg-coherent ring provided it is a coherent ring.*

(2) *Assume that R is an integral domain. Then R is reg-coherent if and only if it is coherent.*

(3) *A reg-coherent ring provided it is a total ring.*

Proof. Straightforward. □

First, we construct a non-coherent reg-coherent ring.

Example 2.2. Let $R = \mathbb{Z} \propto (\mathbb{Z}/2\mathbb{Z})^\infty$. Then:

(1) R is a non-local reg-coherent ring.

(2) R is non-coherent.

Proof. (1) Let $R = \mathbb{Z} \propto (\mathbb{Z}/2\mathbb{Z})^\infty$. It is clear that R is non-local since so is $\mathbb{Z}$. Now, we claim that $R - Z(R) = \{(n, e) \in R : n \notin 2\mathbb{Z} \text{ and } e \in (\mathbb{Z}/2\mathbb{Z})^\infty\}$.

Indeed, we have $(0, e)(0, e) = 0$ and $(2m, e)(0, e) = 0$ for every $e \in (\mathbb{Z}/2\mathbb{Z})^\infty$ and $m \in \mathbb{Z}$. Hence, $R - Z(R) \subseteq \{(n, e) \in R : n \in \mathbb{Z} - 2\mathbb{Z} \text{ and } e \in (\mathbb{Z}/2\mathbb{Z})^\infty\}$. Conversely, let $(n, e) \in R$ such that $n \in \mathbb{Z} - 2\mathbb{Z}$ and $e \in (\mathbb{Z}/2\mathbb{Z})^\infty$ and let $(m, f) \in R$ such that $(n, e)(m, f) = (0, 0)$. Hence, $(0, 0) = (n, e)(m, f) = (nm, nf + me)$ and so $nm = 0$ and $nf + me = 0$. Since $n \in \mathbb{Z} - 2\mathbb{Z}$ and $nm = 0$, then $m = 0$ and so $nf = 0$. On the other hand, let $n = 2p+1$ since $n \in \mathbb{Z} - 2\mathbb{Z}$. Therefore, $0 = nf = (2p+1)f = 2pf + f = f$ and so $(m, f) = (0, 0)$, as desired.

Now, we show that R is a reg-coherent ring and let I be a finitely generated regular ideal of R. Then, there exists $(p, e) \in I$ such that $p \in \mathbb{Z} - 2\mathbb{Z}$ and $e \in (\mathbb{Z}/2\mathbb{Z})^\infty$. Hence, for every $f \in (\mathbb{Z}/2\mathbb{Z})^\infty$, we have $(p, e)(0, f) = (0, pf) = (0, f)$ since $p \in \mathbb{Z} - 2\mathbb{Z}$. Therefore, $0 \propto (\mathbb{Z}/2\mathbb{Z})^\infty \subseteq I$ and so $I = J \propto (\mathbb{Z}/2\mathbb{Z})^\infty$, where J is an ideal of $\mathbb{Z}$, that is $J = n\mathbb{Z}$, where $n \in \mathbb{Z} - 2\mathbb{Z}$ since I is a regular ideal of R. Hence, $I = n\mathbb{Z} \propto (\mathbb{Z}/2\mathbb{Z})^\infty = R(n, 0) \cong R$ (since $(n, 0)$ is a regular element of R) and so I is finitely presented, as desired.

(2) We have $Ann((2, 0)) = 0 \propto (\mathbb{Z}/2\mathbb{Z})^\infty$ which is a non-finitely generated ideal of R. Hence, R is non-coherent which completes the proof. □

A ring R is called 2-coherent if each 1-presented R-module is 2-presented. A coherent ring is clearly 2-coherent. A ring is called 2-von Neumann regular if it is a $(2,0)$-ring, that is every 2-presented module is projective. A 2-von Neumann regular is 2-coherent. See for instance [19–22] (see Figure 1 below). Hence, we have:

$$\begin{tikzcd} & \text{coherent} \arrow[dl] \arrow[dr] & \\ \text{reg-coherent} & & \text{2-coherent} \end{tikzcd}$$

Figure 1.

Now, we give a new example of a non-coherent reg-coherent ring which is 2-coherent. Also, we show that the notions reg-coherent and 2-coherent are not comparable.

Example 2.3. Let (A, M) be a local ring, $E := (A/M)^{\infty}$ be an infinite (A/M)-vector space, and $R := A \propto E$ be a trivial ring extension of A by E. Then:

(1) R is a local total ring. In particular, R is reg-coherent.
(2) R is 2-coherent since R is 2-von Neumann regular ring by [22, Theorem 2.1(1)].
(3) R is non-coherent by [20, Theorem 2.6(2)] since E is an (A/M)-vector space with infinite rank.

Example 2.4. Let R be a non-coherent 2-coherent integral domain (See [21, Theorem 3.1)]). Then:

(1) R is non-reg-coherent since it is a non-coherent integral domain.
(2) R is 2-coherent.

Example 2.5. Let (A, M) be an integral local domain with a non finitely generated maximal ideal M (for instance, take $A = K[[X_1, \ldots, X_n, \ldots]]$ be a power series ring with infinite indeterminates over a field K) and set $R := A \propto (A/M)$. Then:

(1) R is a local total ring. In particular, R is reg-coherent.
(2) R is a non-2-coherent ring by using [19, Theorem 1.1].

Now, we study the transfer of reg-coherent notion to a direct product.

Proposition 2.6. *Let $R := \prod_{i=1}^{n} R_i$ be the direct product of a rings R_i. Then R is a reg-coherent ring if and only if so is R_i, for every $i = 1, \ldots, n$.*

Proof. By induction, it suffices to show the proof for $n = 2$. Assume that R_1 and R_2 are reg-coherents and let J be a finitely generated regular ideal of R. Then, it is easy to see that $J = I_1 \times I_2$, where I_i is a finitely generated regular ideal of R_i for $i = 1, 2$. Hence, I_i is a finitely presented ideal of R_i and so $J := I_1 \times I_2$ is a finitely presented ideal of R by [21, Lemma 2.5(1)], as desired.

Conversely, assume that R is reg-coherent and let I_1 be a finitely generated regular ideal of R_1. Then, $I_1 \times R_2$ is a finitely generated regular ideal of a reg-coherent ring R, hence $I_1 \times R_2$ is a finitely presented ideal of R. Therefore, I_1 is a finitely presented ideal of R_1 by [21, Lemma 2.5(1)], as desired.

By the same argument, we show that R_2 is also a reg-coherent ring which completes the proof. □

We know that a localization of a coherent ring is coherent. Now, we give an example showing that the localization of a reg-coherent ring is not always a reg-coherent.

Example 2.7. Let $A = \mathbb{Z}_{2\mathbb{Z}} + X\mathbb{R}[[X]]$ be a local non-coherent integral domain with maximal ideal $M := 2\mathbb{Z}_{2\mathbb{Z}} + X\mathbb{R}[[X]]$, $E := (A/M)^{\infty}$ be a K-vector space with infinite rank and set $R = A \propto E$ be the trivial ring extension of A by E. Let $S_0 = \{2^n : n \in \mathbb{N}\}$ be a multiplicative set of A and set $S := S_0 \propto 0$ a multiplicative set of R. Then:

(1) R is reg-coherent.
(2) $S^{-1}R$ is a non-reg-coherent ring.

Proof. (1) R is reg-coherent since R is a total ring.

(2) If we take $S_0 = \{2^n : n \in \mathbb{N}\}$ and $S = S_0 \propto 0$, we have $S^{-1}R \cong S_0^{-1}A = \mathbb{Q} + X\mathbb{R}[[X]]$ which is a non-coherent integral domain by [12, Theorem 5.2.3]. Hence, $S^{-1}R$ is a non-reg-coherent ring, as desired. □

Now, we study the transfer of reg-coherent property in trivial ring extension.

Theorem 2.8. *Let A be a ring, E be an A-module and set $R := A \propto E$ be the trivial ring extension of A by E. Then:*

(1) *Assume that A is an integral domain which is not a field, $K := qf(A)$, E is a K-vector space and $R := A \propto E$ is the trivial ring extension of A by E. Then, $R := A \propto E$ is a reg-coherent ring if and only if A is coherent.*
(2) *Assume that (A, M) is a local ring, E is an (A/M)-vector space, and set $R := A \propto E$ the trivial ring extension of A by E. Then R is reg-coherent.*

Proof. (1) Assume that A is coherent and let J be a finitely generated proper regular ideal of R. Then there exists $(a, e) \in J$ such that $a \neq 0$ (since $(0 \propto E)(0, e) = 0$). Since $(a, e)R = aA \propto E$, hence $J := I \propto E$ for some finitely generated proper ideal I of A. Therefore, I is a finitely presented ideal of A since A is coherent, hence $J := I \bigotimes_A R = IR = I \propto E$ (since R is a flat A-module) is a finitely presented ideal of R, as desired.

Conversely, assume that R is a reg-coherent ring and let I be a finitely generated proper ideal of A. Then, $J := I \bigotimes_A R = IR = I \propto E$ (since R is a flat A-module) is a finitely generated regular ideal of R and so J is a finitely presented ideal of R since R is a reg-coherent ring. Hence, I is a finitely presented ideal of A (since R is a faithfully-flat A-module), as desired.

(2) Straightforward since R is a total ring. $\square$

By Theorem 2.8(1), we obtain the following example:

Example 2.9. Let $R = \mathbb{Z} \propto \mathbb{Q}$. Then:

(1) R is reg-coherent by Theorem 2.8(1).
(2) R is non-coherent by [20, Theorem 2.8(1)].

Now, we study the transfer of reg-coherent property in a particular case of pullbacks.

Theorem 2.10. *Let $T = K + M$ be a local ring, where K is a field and M is a maximal ideal of T such that for each $m \in M$, there exists $n \in M$ such that $mn = 0$ (take for instance $M^n = 0$ for some a positive integer n). Let $D \subseteq K$ be a subring of K and set $R = D + M$. Then R is reg-coherent if and only if D is coherent.*

Proof. Assume that R is a reg-coherent ring and let I be a finitely generated proper ideal of D. Set $J := I \bigotimes_A R = IR = I + M$ (since R is a flat D-module) which is a finitely generated ideal of R. We claim that J is a regular ideal of R.

Indeed, let $d \in I - \{0\} \subseteq J$ and let $a + m \in R$ such that $d(a + m) = 0$, where $a \in D$ and $m \in M$. Then $0 = da + dm$ and so $da = 0$ in D and $dm = 0$ in M. Therefore, $a = 0$ since D is an integral domain and $d \in D - \{0\}$ and $m = 0$ since $0 = dm \in M$ and d is invertible in K, hence d is a regular element in J. Hence, J is a finitely presented ideal of R since R is reg-coherent and so I is a finitely presented ideal of D (since R is a faithfully flat D-module). Hence, D is a coherent domain.

Conversely, assume that D is coherent and let J be a finitely generated proper regular ideal of R. Then $J \subsetneq M$ since J is a regular ideal of R and so there exists $d + m \in J$, where $d \in D - \{0\}$ and $m \in M$. Hence, $J \supseteq (d + m)M = dM + mM = M$ (since $mM \subseteq M = dM$) and so $J = I + M = IR = I \bigotimes_A R$ (since R is a faithfully flat D-module), where I is a finitely generated proper ideal of D. Therefore, I is a finitely presented ideal of D (since D is coherent) and so $J(= I \bigotimes_A R)$ is a finitely presented ideal of R, and so R is a reg-coherent ring which completes the proof of Theorem 2.10. $\square$

Example 2.11. Let $T = \frac{\mathbb{R}[[\mathbb{X}]]}{\langle X^n \rangle} = \mathbb{R} + XT$, where X is an indeterminates over $\mathbb{R}$, $\mathbb{R}[[\mathbb{X}]]$ is the power series ring over $\mathbb{R}$, and $\langle X^n \rangle = X^n\mathbb{R}[[X]]$, where n is a positive integer. Set $R = \mathbb{Z} + XT$. Then:

(1) R is a reg-coherent ring.
(2) R is non-coherent.

Proof. (1) R is a reg-coherent ring by Theorem 2.10.

(2) It is clear that $Ann(X^{n-1}R) = XT$ which is a non finitely generated ideal of R. Hence, R is non-coherent, as desired. □

We end this work by studying the transfer of reg-coherent to the amalgamation.

Theorem 2.12. *Let A and B be two rings, J be an ideal of B, $f : A \to B$ be a ring homomorphism and let $R := A \bowtie^f J$ the amalgamation of A with B along J with respect to f. Then*:

1. *Assume that $f(Reg(A)) \subseteq Reg(B)$. Then A is reg-coherent provided so is $A \bowtie^f J$.*
2. *Assume that A is a total ring and $J \subseteq J(B)$, where $J(B)$ is the Jacobson radical of B. Assume that $J \subseteq f(A)$ or J is a torsion A-module (with the A-module structure inherited by f). Then, $R := A \bowtie^f J$ is reg-coherent.*

Proof. (1) Let I_0 be a finitely generated regular ideal of A generated by $(a_i)_{i=1,\ldots,n}$ and let b be a regular element of I_0. Set J_0 be the ideal of $R := A \bowtie^f J$ generated by $(a_i, f(a_i))_{i=1,\ldots,n}$. Hence, J_0 is a finitely generated regular ideal of R since $(b, f(b))$ is a regular element in J_0 (since b is a regular element in I_0 and $f(Reg(A)) \subseteq Reg(B)$). Therefore, J_0 is a finitely presented ideal of R since R is reg-coherent. We claim that I_0 is a finitely presented ideal of A.

Indeed, $I_0 := \sum_{i=1}^n Aa_i$, $J_0 := \sum_{i=1}^n (A \bowtie^f J)(a_i, f(a_i))$, and consider the exact sequence of $(A \bowtie^f J)$-modules:

$$0 \to Ker(u) \to (A \bowtie^f J)^n = A^n \bowtie^f J^n \xrightarrow{u} J_0 \to 0,$$

where

$$\begin{aligned} u((b_i, f(b_i) + j_i)_{i=1,\ldots,n}) &= \sum_{i=1}^n (b_i, f(b_i) + j_i)(a_i, f(a_i)) \\ &= (\sum_{i=1}^n b_i a_i, \sum_{i=1}^n (f(b_i) + j_i) f(a_i)). \end{aligned}$$

On the other hand, consider the exact sequence of A-modules:

$$0 \to Ker(v) \to A^n \xrightarrow{v} I_0 \to 0,$$

where $v((b_i)_{i=1,\ldots,n}) = \sum_{i=1}^n a_i b_i$. Hence,

$$\begin{aligned} Ker(u) &= \{((b_i, f(b_i) + j_i)_{i=1,\ldots,n}) \in A^n \bowtie^f I^n : \sum_{i=1}^n b_i a_i = \sum_{i=1}^n j_i f(a_i) = 0\} \\ &= ker(v) \bowtie^f G_0, \end{aligned}$$

where $G_0 = \{j_i \in I^n : \sum_{i=1}^n j_i f(a_i) = 0\}$. But J_0 is finitely presented, that is $Ker(u)$ is a finitely generated $(A \bowtie^f J)$-module. Therefore, $Ker(v)$ is a finitely generated A-module and so I_0 is a finitely presented ideal of A, as desired.

(2) By [11, Proposition 1.74, p. 45], $R := A \bowtie^f J$ is a total ring and so it is reg-coherent, as desired. □

Corollary 2.13. *Let I be an ideal of a total ring A, $J(\subseteq J(B))$ be an ideal of $B(:= A/I)$ (where $J(B)$ is the Jacobson radical of B), $f : A \to B$ be a ring homomorphism and let $R := A \bowtie^f J$ the amalgamation of A with B along J with respect to f. Then, $R := A \bowtie^f J$ is reg-coherent.*

Corollary 2.14. *Let A be a ring, I be an ideal of A, and $R := A \bowtie I$ be the duplication of A by I. Then*:

(1) *A is reg-coherent provided so is $A \bowtie I$.*
(2) *Assume that A is a total ring and $I \subseteq J(A)$, where $J(A)$ is the Jacobson radical of A. Then $R := A \bowtie I$ is a reg-coherent ring.*

Now, we construct a non-coherent reg-coherent ring by using Theorem 2.12.

Example 2.15. Let (A_0, M_0) be a non-coherent local integral domain (see for instance [21, Theorem 3.1]). Set $A := A_0 \propto E$, where E be an (A_0/M_0)-vector space with infinite rank, and set $R := A \bowtie^f J$, where $J := M_0$ is a maximal ideal of a local ring $A/(0 \propto E) = A_0$. Then:

(1) R is a reg-coherent ring by Theorem 2.12(2).
(2) R is a non-coherent ring by [15, Theorem 2.2] since A is non-coherent (since A_0 is non-coherent).

Example 2.16. Let $A_0 = K[[X_1, \ldots, X_n, \ldots]] = K + M$ be a local power series ring with infinite indeterminate $\{X_i : i \in \mathbb{N}^*\}$ over a field K, where M is its maximal ideal generated by $\{X_i : i \in \mathbb{N}^*\}$. Let $A := \frac{A_0}{M^n} = K + \frac{M}{M^n}$ be a local ring with $I := \frac{M}{M^n}$ its maximal ideal, where $n \geq 3$ is a positive integers and set $R := A \bowtie I$. Then:

(1) R is a reg-coherent ring.
(2) R is a non-coherent ring.

Proof. (1) R is a reg-coherent ring since A is a local total ring with maximal ideal $I := \frac{M}{M^n}$ and $I^n = 0$.

(2) R is a non-coherent ring since A is a non-coherent ring (since $Ann(X_1^{n-1}A) = \frac{M}{M^n}$ which is not finitely generated ideal of A). □

Remark 2.17. The above example is neither a trivial ring extension (since $I^2 = \frac{M^2}{M^n} \neq 0$) nor a pullback of the type studied in this paper.

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