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ON THE NUMBER OF EQUIVALENCE CLASSES OF BI-PARTITIONS ARISING FROM THE COLOR CHANGE

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Abstract. We introduce a new class of bi-partition function $c_k(n)$, which counts the number of bi-color partitions of n in which the second color only appears at the parts that are multiples of k. We consider two partitions to be the same if they can be obtained by switching the color of parts that are congruent to zero modulo k. We show that the generating function for $c_k(n)$ involves the partial theta function and obtain the following congruences:

$$c_2(27n+26) \equiv 0 \pmod 3$$

and

$$c_3(4n+2) \equiv 0 \pmod 2.$$

1. Introduction

In the theory of integer partitions [2], involutions such as partition conjugation and Franklin's involution play crucial roles in discovering combinatorial proofs or generating functions. In a joint work with J. Huh [9], the author initiated the study on the number of equivalence classes for integer partitions arising from partition involutions. Specifically, given an involution φ acting on a set of partitions X, we define two partitions λ and μ belonging to X as equivalent if $\varphi(\lambda) = \mu$. For a positive integer n, we denote by $X^{\varphi}(n)$ the number of equivalence classes of weight n in the set X induced by φ, i.e.

$$X^{\varphi}(n) = \frac{1}{2}\left(\sum_{\substack{\lambda \in X \\ |\lambda| = n}} 1 + \sum_{\substack{\lambda \in X, |\lambda|=n \\ \varphi(\lambda)=\lambda}} 1 \right),$$

where $|\lambda|$ denotes the number being partitioned. We may think of $X^{\varphi}(n)$ as a number of partitions of n that belong to the set X, where two partitions are

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considered the same if they are equivalent. For example, let $\mathcal{D}$ be the set of partitions into distinct parts and φ be the Franklin's involution on $\mathcal{D}$, then Huh and the author [9, Theorem 3] showed that for all positive integers n,

$$\mathcal{D}^{\varphi}(n) = dd(n),$$

where $dd(n)$ denotes the number of partitions of n into distinct parts with an even number of parts that are congruent to zero modulo 3.

On the other hand, H.-C. Chan [7] introduced a type of bi-partitions known as cubic partitions [10]. The cubic partition function $c(n)$ is the number of ways to partition n while considering two colors for even parts. Generally, let $\mathcal{C}_k$ denote the set of bi-color partitions in which there are two colored parts that are congruent to zero modulo k. For example, there are 16 partitions of 6 in the set $\mathcal{C}_3$ as

$$\begin{gathered} 6_r, 6_b, 5+1, 4+2, 4+1+1, 3_r+3_r, 3_r+3_b, 3_b+3_b \\ 3_r+2+1, 3_b+2+1, 3_r+1+1+1, 3_b+1+1+1, \\ 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1. \end{gathered}$$

Ahmed et. al. [1] provided congruences for the number of partitions of n in the set $\mathcal{C}_k$. It is clear that switching the color of the parts that are multiples of k in the partition $\lambda \in \mathcal{C}_k$ is an involution. Consequently, we may examine the number of equivalence classes arising from the color change. For this purpose, we define

$$c_k(n) = \mathcal{C}_k^{\varphi}(n) = \frac{1}{2}\left(\sum_{\substack{\lambda\in\mathcal{C}_k\\|\lambda|=n}} 1 + \sum_{\substack{\lambda\in\mathcal{C}_k,|\lambda|=n\\ \varphi(\lambda)=\lambda}} 1\right),$$

where φ is the involution defined by switching the color of the parts that are multiples of k. Thus, we can interpret $c_k(n)$ as the number of partitions of n in the set $\mathcal{C}_k$, while we consider two partitions to be the same if they can be obtained by switching the color of the parts that are congruent to 0 modulo k. For example, $c_3(6) = 12$ since there exist four pairs of partitions of weight 6 in $\mathcal{C}_3$ that are equivalent under the color change:

$$\begin{gathered} 6_r = 6_b, 5+1, 4+2, 4+1+1, 3_r+3_r = 3_b+3_b, 3_r+3_b, \\ 3_r+2+1 = 3_b+2+1, 3_r+1+1+1 = 3_b+1+1+1, \\ 2+2+2, 2+2+1+1, 2+1+1+1+1, 1+1+1+1+1+1. \end{gathered}$$

It is straightforward to see that

$$\sum_{n\geq 0} c_k(n)q^n = \frac{1}{2}\frac{(q^k;q^k)_\infty}{(q)_\infty}\left(\frac{1}{(q^k;q^k)_\infty^2} + \frac{1}{(q^{2k};q^{2k})_\infty}\right),$$

where here and in the sequel we use the standard q-product notation

$$(a)_n = (a;q)_n = \prod_{k=1}^{n}(1-aq^k)$$

for $n \in \mathbb{N}_0 \cup \{\infty\}$. We also use

$$(a_1, a_2, \dots, a_j; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_j; q)_n.$$

Our first result reveals a relationship between the generating function of $c_k(n)$ and a partial theta function.

Theorem 1.1. *For a positive integer $k > 1$,*

$$\sum_{n\geq 0} c_k(n) q^n = \frac{1}{(q)_\infty (q^k; q^k)_\infty} \sum_{n\geq 0} (-1)^n q^{kn^2}.$$

The function of the form $\sum_{n\geq 0}(-1)^n q^{n^2}$ is known as a partial theta function, which holds a prominent role in the theory of q-series and modular forms [11]. While a number of congruences are known for generating functions that are single q-products or quotients of theta functions, only a few congruences are known when the generating function involves a partial theta function or a linear combination of quotients of theta functions. In this context, the following two congruences are particularly intriguing.

Theorem 1.2. *For all non-negative integers n,*

$$c_2(27n + 26) \equiv 0 \pmod 3, \tag{1}$$

$$c_3(4n + 2) \equiv 0 \pmod 2. \tag{2}$$

Remark 1.3. Several congruences inherited from those of the ordinary partition function exist. For example, from the famous Ramanujan congruences

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7, \quad p(11n+6) \equiv 0 \pmod{11},$$

one can immediately see that

$$c_5(5n + 4) \equiv 0 \pmod 5, \quad c_7(7n + 5) \equiv 0 \pmod 7, \quad c_{11}(11n + 6) \equiv 0 \pmod{11}.$$

The rest of paper is organized as follows. Section 2 presents the basic definitions and transformation formulas for q-series. Section 3 contains the proofs of Theorems 1.1–1.2. We conclude the paper with a remark in Section 4.

2. Preliminaries

For x, y with $|x| < 1$ and $|y| < 1$, we define Ramanujan's theta function [4, Chapter 1]

$$f(x, y) = \sum_{n\in\mathbb{Z}} x^{n(n+1)/2} y^{n(n-1)/2}.$$

Then, in terms of Ramanujan's theta function, the Jacobi triple product identity states

$$f(x, y) = (-x, -y, xy; xy)_\infty.$$

The following theta functions appear frequently in the literature.

$$\varphi(q) = f(q, q) = \sum_{n\in\mathbb{Z}} q^{n^2} = (-q; q^2)_\infty^2 (q^2; q^2)_\infty,$$

$$\psi(q) = f(q, q^3) = \sum_{n\geq 0} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

We also need Jacobi's identity

$$(q)_\infty^3 = \sum_{n\geq 0} (-1)^n (2n+1) q^{n(n+1)/2}.$$

Now we give a q-series transformation formula and decomposition formulas of a product of theta functions. The first transformation identity is a special case of Heine's transformations [8, Appendix III.1 and 3] and the second and the third identities come from Z. Cao's general results on theta functions [5, Corollary 1.1] and [6, Corollary 2.5].

Lemma 2.1. (i) *For* $|z|, |c| < 1$,

$$(z;q)_\infty \sum_{n\geq 0} \frac{z^n}{(q;q)_n (cq;q)_n} = \frac{1}{(cq;q)_\infty} \sum_{n\geq 0} \frac{(z;q)_n}{(q;q)_n} (-c)^n q^{n(n+1)/2}. \tag{3}$$

(ii) *For* $ab = q$ *and* $cd = q^2$,

$$\begin{aligned} f(a,b)f(c,d) =& f(ac,bd)f(a^2dq, b^2cq) + af(acq, bd/q)f(a^2dq^3, b^2c/q) \\ &+ bf(ac/q, bdq)f(a^2d/q, b^2cq^3). \end{aligned} \tag{4}$$

(iii) *We have*

$$\psi(q)\psi(q^3) = \varphi(q^6)\psi(q^4) + q\varphi(q^2)\psi(q^{12}). \tag{5}$$

3. Proofs

We begin with the proof of Theorem 1.1.

Proof. Note that

$$\frac{1}{(zq)_\infty} = \sum_{n\geq 0} \sum_{m\geq 0} p(m,n) z^m q^n,$$

where $p(m,n)$ is the number of partitions of n into m parts. Therefore, we see that

$$\begin{aligned} \frac{1}{2}\left(\frac{1}{(q)_\infty} + \frac{1}{(-q)_\infty} \right) &= \sum_{n\geq 0} p_e(n) q^n \\ &= \sum_{j\geq 0} \frac{q^{2j}}{(q)_{2j}}, \end{aligned}$$

where $p_e(n)$ is the number of partitions of n with even number of parts. Therefore, we observe that

$$\begin{aligned}\sum_{n\geq 0} c_k(n)q^n &= \frac{1}{(q)_\infty}\sum_{n\geq 0} p_e(n)q^{kn} \\ &= \frac{1}{(q)_\infty}\sum_{n\geq 0}\frac{q^{kn}}{(q^k;q^k)_{2n}} \\ &= \frac{1}{(q)_\infty(q^k;q^k)_\infty}\sum_{n\geq 0}(-1)^n q^{kn^2},\end{aligned}$$

where we have used (3) for the last equality. □

Now we prove the congruences in Theorem 1.2. We first consider the congruence modulo 2.

Proof of (2). Throughout the proof, the modulus for the congruence equation is 2. We observe that

$$\begin{aligned}\sum_{n\geq 0} c_3(n)q^n &= \frac{(q)_\infty^3(q^3;q^3)_\infty^3}{(q)_\infty^4(q^3;q^3)_\infty^4}\sum_{n\geq 0}(-1)^n q^{3n^2} \\ &\equiv \frac{\psi(q)\psi(q^3)}{(q^4;q^4)_\infty(q^{12};q^{12})_\infty}\sum_{n\geq 0} q^{3n^2} \\ &= \frac{\varphi(q^6)\psi(q^4)+q\varphi(q^2)\psi(q^{12})}{(q^4;q^4)_\infty(q^{12};q^{12})_\infty}\sum_{n\geq 0} q^{3n^2} \\ &\equiv \frac{\psi(q^4)+q\psi(q^{12})}{(q^4;q^4)_\infty(q^{12};q^{12})_\infty}\sum_{n\geq 0} q^{3n^2},\end{aligned}$$

where we have used (5) for the penultimate congruence. Since there is no term for the coefficient of q^{4n+2} in the right-hand side, we can conclude that

$$c_3(4n+2)\equiv 0 \pmod 2$$

as desired. □

Proof of (1). Throughout the proof, the modulus for the congruence relation is 3. From Cao's work [5, Theorem 1.5], we find the following 3-dissection identity

$$\frac{1}{(q)_\infty(q^2;q^2)_\infty} = \frac{(q^9;q^9)_\infty(q^{18};q^{18})_\infty}{(q^3;q^3)_\infty^4(q^6;q^6)_\infty^4}\left((A_0^2-A_1A_2)+(A_2^2-A_0A_1)+3A_1^2\right),$$

where

$$\begin{aligned}A_0 &= f(q^3,q^6)\varphi(-q^9) \\ A_1 &= -qf(q^3,q^6)f(-q^3,-q^{15}) \\ A_2 &= -2q^2\psi(q^9)f(-q^3,-q^{15})\end{aligned}$$

Therefore, we find that

$$\sum_{n\geq 0} c_2(3n{+}2)q^{3n+2} \equiv \frac{(q^9;q^9)_\infty(q^{18};q^{18})_\infty}{(q^3;q^3)_\infty^4(q^6;q^6)_\infty^4}(A_0^2{-}A_1A_2)\sum_{n\in\mathbb{Z}}(-1)^{n-1}q^{18n^2+12n+2},$$

which implies that

$$\begin{aligned}\sum_{n\geq 0} c_2(3n+2)q^n &\equiv \frac{(q^3;q^3)_\infty(q^6;q^6)_\infty}{(q;q)_\infty^4(q^2;q^2)_\infty^4}\big(-f(q,q^2)^2\varphi(-q^3)^2\\ &\quad + 2qf(q,q^2)f(-q,-q^5)^2\psi(q^3)\big)\sum_{n\in\mathbb{Z}}(-1)^nq^{6n^2+4n}\\ &\equiv \frac{f(q,q^2)}{(q)_\infty(q^2;q^2)_\infty}\big(-f(q,q^2)\varphi(-q^3)^2\\ &\quad + 2qf(-q,-q^5)^2\psi(q^3)\big)\sum_{n\in\mathbb{Z}}(-1)^nq^{6n^2+4n}.\end{aligned}$$

Using

$$f(q,q^2) = (-q,-q^2,q^3;q^3)_\infty = \frac{(-q)_\infty(q^3;q^3)_\infty}{(-q^3;q^3)_\infty}$$

and

$$f(-q,-q^5) = (q,q^5,q^6;q^6)_\infty = \frac{(q;q^2)(q^6;q^6)_\infty}{(q^3;q^6)_\infty},$$

we deduce that

$$\begin{aligned}\sum_{n\geq 0} c_2(3n+2)q^n &\equiv -\frac{\varphi(-q^3)^2(q^9;q^9)_\infty}{(q^6;q^6)_\infty^2}(q^2;q^2)_\infty\sum_{n\in\mathbb{Z}}(-1)^nq^{6n^2+4n}\\ &\quad + 2q\psi(q^3)(q^6;q^6)_\infty^2(q^2;q^2)_\infty\sum_{n\in\mathbb{Z}}(-1)^nq^{6n^2+4n}.\end{aligned}$$

By using (4) with replacing q by q^6, we find that

$$\begin{aligned}&(q^2;q^2)_\infty\sum_{n\in\mathbb{Z}}(-1)^nq^{6n^2+4n}\\ &= f(-q^2,-q^4)f(-q^{10},-q^2)\\ &= f(q^{12},q^6)f(-q^{12},-q^{24}) - q^2f(q^{18},1)f(-q^{24},-q^{12})\\ &\quad - q^4f(q^6,q^{12})f(-1,q^{36})\\ &= f(q^6,q^{12})f(-q^{12},-q^{24}) - 2q^2\psi(q^{18})f(-q^{12},-q^{24}).\end{aligned}$$

Therefore, we arrive at

$$\sum_{n\geq 0} c_2(9n+8)q^n \equiv 2\frac{\varphi(-q)^2(q^3;q^3)_\infty}{(q^2;q^2)_\infty^2}\psi(q^6)f(-q^4,-q^8).$$

Finally, we observe that

$$\begin{aligned}\sum_{n\geq 0} c_2(9n+8)q^n &\equiv 2\frac{\varphi(-q)^2(q^4;q^4)_\infty}{(q^2;q^2)_\infty^2}(q^3;q^3)_\infty\psi(q^6)\\ &\equiv 2(-q;q^2)_\infty^4(q^4;q^4)_\infty(q^3;q^3)_\infty\varphi(q^6)\\ &\equiv 2(-q)_\infty(q^2;q^2)_\infty(-q^3;q^6)_\infty(q^3;q^3)_\infty\varphi(q^6)\\ &\equiv 2\psi(q)(-q^3;q^6)_\infty(q^3;q^3)_\infty\varphi(q^6).\end{aligned}$$

Since the triangular numbers are congruent to 0 or 1 modulo 3, we have proven that

$$c_2(27n+26)\equiv 0 \pmod 3.$$

□

4. Concluding Remark

While the generating function for $c_k(n)$ involves a partial theta function, the proof of the congruences in Theorem 1.2 reveals that the congruences arise from the fact that the generating functions for certain arithmetic progressions are congruent to products of theta functions. A natural question is whether there exists a congruence for $c_k(n)$ that does not arise from theta functions.

Andrews and Newman [3] define $\mathrm{mex}_{A,a}(\lambda)$ to be the smallest integer $\equiv a \pmod A$ that is not a part of the partition λ and $p_{A,a}(n)$ to be the number of partitions of n with $\mathrm{mex}_{A,a} \equiv a \pmod{2A}$. They obtain the generating function for $p_{2k,k}(n)$ [3, Lemma 8] as

$$\sum_{n\geq 0} p_{2k,k}(n)q^n = \frac{1}{(q)_\infty}\sum_{n\geq 0}(-1)^n q^{kn^2}.$$

By Theorem 1.1, we see that

$$\sum_{n\geq 0} c_k(n)q^n = \frac{1}{(q^k;q^k)_\infty}\sum_{n\geq 0} p_{2k,k}(n)q^n,$$

which implies that $c_k(n)$ is equal to the number of bi-partitions (λ,π) of n such that $\mathrm{mex}_{2k,k}(\lambda)\equiv k \pmod{4k}$ and all parts of π are multiples of k. It would be interesting if one can find a bijective proof of this observation.

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