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COHEN-MACAULAY DIMENSION FOR COMPLEXES

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Abstract. In this paper, our focus lies in exploring the concept of Cohen-Macaulay dimension within the category of homologically finite complexes. We prove that over a local ring $(R, \mathfrak{m})$, any homologically finite complex X with a finite Cohen-Macaulay dimension possesses a finite *CM-resolution*. This means that there exists a bounded complex G of finitely generated R-modules, such that G is isomorphic to X and each nonzero G_i within the complex G has zero Cohen-Macaulay dimension.

1. Introduction

Throughout this paper, $(R, \mathfrak{m})$ is a local ring and all rings are commutative and Noetherian with identity. The projective dimension of a finitely generated module is a well-known and widely studied numerical invariant in classical homological algebra. However, there exist several refinements and extensions of this dimension that provide valuable information about the algebraic and geometric aspects of modules. One such refinement is the concept of Gorenstein dimension, introduced by Auslander and Bridger [1]. Another refinement is the notion of complete intersection dimension, which was introduced by Avramov, Gasharov, and Peeva [2]. Furthermore, Gerko [9] introduced yet another refinement known as Cohen-Macaulay dimension. Assume that M is a finitely generated R-module. These homological dimensions satisfy in the following inequalities

$$\text{CM-dim}_R M \leq \text{G–dim}_R M \leq \text{CI–dim}_R M \leq \text{pd}_R M,$$

with equality to the left of any finite quantity, see [2, Theorem 1.4] and [10, Theorem 5.6].

The concept of projective dimension, originally defined for individual modules, has been extended to complexes of R-modules by Foxby [6–8]. Similarly, the notion of Gorenstein dimension was developed for complexes of R-modules

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by Yassemi [14] and Christensen [3]. Additionally, Sather-Wagstaff [13] extended the concept of complete intersection dimension to the category of homologically finite complexes. Recently, Sahandi, Sharif, and Yassemi [12] further extended the concept of Cohen-Macaulay dimension to the category of homologically bounded complexes. These extensions and developments allow for a deeper and more comprehensive understanding of the homological properties of complexes of modules, enriching the field of homological algebra.

Since many mathematicians are interested in studying homological dimensions through resolutions, we present a significant theorem that allows the computation of the Cohen-Macaulay dimension for a homologically finite complex of R-modules using its syzygies, see Theorem 3.10. And as an interesting application of this theorem, we demonstrate that any homologically finite complex X with a finite Cohen-Macaulay dimension possesses a finite *CM-resolution*. This means that there exists a bounded complex G of finitely generated R-modules, isomorphic to X, such that each nonzero module G_i in the complex has zero Cohen-Macaulay dimension, see Corollary 3.12.

2. Prerequisites

Throughout this paper, $(R, \mathfrak{m})$ is a commutative Noetherian local ring and we will work within $\mathcal{D}(R)$, the derived category of R-modules. We recall that the objects in $\mathcal{D}(R)$ are complexes of R-modules and symbol $\simeq$ denotes isomorphisms in this category. For a complex

$$X = \cdots \longrightarrow X_{n+1} \xrightarrow{\partial^X_{n+1}} X_n \xrightarrow{\partial^X_n} X_{n-1} \longrightarrow \cdots$$

in $\mathcal{D}(R)$, its *supremum* and *infimum* are defined respectively by $\sup X := \sup\{i \in \mathbb{Z} \,|\, H_i(X) \neq 0\}$ and $\inf X := \inf\{i \in \mathbb{Z} \,|\, H_i(X) \neq 0\}$, with the usual convention that $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. Let n be an integer. The *nth suspension* of X is denoted by $\Sigma^n X$, and it represents the complex X shifted n degrees to the left. The kernel and cokernel of ∂^X_n are denoted Z^X_n and $\Omega^{n-1}(X)$, respectively. For any R-module M, one has $\Omega^n(X \otimes_R M) \cong \Omega^n(X) \otimes_R M$, by the right-exactness of - $\otimes_R M$. The *nth soft left- and right-truncations* of X are the complexes

$$\begin{aligned} \subset_n X &= \cdots \to 0 \to \Omega^n(X) \xrightarrow{\overline{\partial^X_n}} X_{n-1} \xrightarrow{\partial^X_{n-1}} \cdots \quad \textit{and} \\ X_n \supset &= \cdots \xrightarrow{\partial^X_{n+2}} X_{n+1} \xrightarrow{\partial^X_{n+1}} Z^X_n \to 0 \to \cdots , \end{aligned}$$

respectively, where $\overline{\partial^X_n}$ is the map induced by ∂^X_n. The *nth hard left- and right-truncations* of X are the complexes

$$\begin{aligned} \sqsubset_n X &= \cdots \to 0 \to X_n \xrightarrow{\partial^X_n} X_{n-1} \xrightarrow{\partial^X_{n-1}} \cdots \quad \textit{and} \\ X_n \sqsupset &= \cdots \xrightarrow{\partial^X_{n+2}} X_{n+1} \xrightarrow{\partial^X_{n+1}} X_n \to 0 \to \cdots , \end{aligned}$$

respectively. The full subcategory of complexes homologically bounded to the right (resp. left) is denoted by $\mathcal{D}_{\sqsupset}(R)$ (resp. $\mathcal{D}_{\sqsubset}(R)$). Also, the full subcategories of homologically bounded complexes and of complexes with finitely generated homology modules will be denoted by $\mathcal{D}_{\square}(R)$ and $\mathcal{D}^f(R)$, respectively. The full subcategory of complexes whose homology modules concentrated in degree zero will be denoted by $\mathcal{D}_0(R)$. Since modules can be considered as complexes concentrated in degree zero, we may and do identify the category of R-modules and R-homomorphisms with the subcategory $\mathcal{D}_0(R)$. Throughout, for any two properties $\sharp$ and $\natural$ of complexes, we set $\mathcal{D}_{\sharp}^{\natural}(R) := \mathcal{D}_{\sharp}(R) \cap \mathcal{D}^{\natural}(R)$. So, for instance, $\mathcal{D}_{\square}^{f}(R)$ stands for the full subcategory of homologically bounded complexes with finitely generated homology modules which we call them *homologically finite* complexes. For any complex X in $\mathcal{D}_{\sqsupset}(R)$ (resp. $\mathcal{D}_{\sqsubset}(R)$), there is a bounded to the right (resp. left) complex P (resp. I) of projective (resp. injective) R-modules which is isomorphic to X in $\mathcal{D}(R)$. Such a complex P (resp. I) is called a *projective* (*resp. injective*) *resolution* of X.

The *right derived homomorphism functor* $\mathbf{R}\operatorname{Hom}_R(-,\sim)$ is computed by taking a projective resolution of the first argument or by taking an injective resolution of the second one. Also, we recall that for any ideal $\mathfrak{a}$ of R and any complex $X \in \mathcal{D}_{\square}^{f}(R)$, $\operatorname{depth}(\mathfrak{a}, X)$ is defined by $\operatorname{depth}(\mathfrak{a}, X) := -\sup \mathbf{R}\operatorname{Hom}_R(R/\mathfrak{a}, X)$, and we set $\operatorname{depth}_R X := \operatorname{depth}(\mathfrak{m}, X)$.

An R-module M is said to be *Gorenstein projective* if there exists an exact complex P of projective R-modules such that $M \cong \operatorname{coker}(P_1 \to P_0)$ and the complex $\operatorname{Hom}_R(P, N)$ is exact for all projective R-modules N. Also, for a complex $Y \in \mathcal{D}_{\sqsupset}(R)$, the *Gorenstein projective dimension* of Y is defined by

$$\begin{aligned}\operatorname{Gpd}_R Y := \inf\{\sup\{l \in \mathbb{Z} \mid Q_l \neq 0\} \mid Q &\text{ is a bounded to the right complex of}\\ &\text{Gorenstein projective } R\text{-modules such that } Q \simeq Y\}.\end{aligned}$$

Any finitely generated Gorenstein projective R-module is called *totally reflexive* module. For a complex $X \in \mathcal{D}_{\sqsupset}^{f}(R)$, the *G-dimension* of X is defined by

$$\begin{aligned}\operatorname{G-dim}_R X := \inf\{\sup\{l \in \mathbb{Z} \mid Q_l \neq 0\} \mid Q &\text{ is a bounded to the right complex of}\\ &\text{totally reflexive } R\text{-modules such that } Q \simeq X\}.\end{aligned}$$

An R-module M is said to be *Gorenstein flat* if there exists an exact complex F of flat R-modules such that $M \cong \operatorname{coker}(F_1 \to F_0)$ and the complex $J \otimes_R F$ is exact for all injective R-modules J. For a complex $T \in \mathcal{D}_{\sqsupset}(R)$, the *Gorenstein flat dimension* of T is defined by

$$\begin{aligned}\operatorname{Gfd}_R T := \inf\{\sup\{l \in \mathbb{Z} \mid Q_l \neq 0\} \mid Q &\text{ is a bounded to the right complex of}\\ &\text{Gorenstein flat } R\text{-modules such that } Q \simeq T\}.\end{aligned}$$

Clearly, any projective (resp. flat) module is Gorenstein projective (resp. Gorenstein flat). One can see that for $X \in \mathcal{D}_{\sqsupset}^{f}(R)$, we have $\operatorname{Gpd}_R X = \operatorname{G-dim}_R X = \operatorname{Gfd}_R X$ by [5, Proposition 3.8].

3. Cohen-Macaulay dimension for complexes

In this section, we delve into the proofs of various results concerning the Cohen-Macaulay dimension of homologically finite complexes. However, before doing that, let us recall some relevant definitions.

Definition 3.1. Let $(R, \mathfrak{m})$ be a local ring. The ideal $\mathfrak{a}$ of R is called *G-perfect* if $\text{G–dim}_R\, R/\mathfrak{a} = \operatorname{depth}(\mathfrak{a}, R)$.

Definition 3.2. Let $(R, \mathfrak{m})$ be a local ring. A *CM-quasi-deformation* of R is a diagram of local homomorphisms $R \longrightarrow R' \longleftarrow Q$, where $R \longrightarrow R'$ is a flat extension and $R' \longleftarrow Q$ is a CM-deformation, that is, a surjective homomorphism whose kernel is a G-perfect ideal.

We recall that the Cohen-Macaulay dimension for a finitely generated R-module M is defined by Gerko [9, Definition 3.2] as

$$\text{CM-dim}_R\, M := \inf\{\text{G–dim}_Q(M \otimes_R R') - \text{G–dim}_Q(R') \mid R \longrightarrow R' \longleftarrow Q \text{ is a CM-quasi-deformation}\}.$$

This definition has been extended to the category of homologically bounded complexes of R-modules as follows.

Definition 3.3 (cf. [12, Definition 3.1])**.** Let $(R, \mathfrak{m})$ be a local ring and Y be a homologically bounded complex of R-modules. The *Cohen-Macaulay projective dimension* of Y is defined as

$$\text{CM}_*\text{-pd}\,_R Y := \inf\{\text{Gpd}_Q(Y \otimes_R R') - \text{Gfd}_Q(R') \mid R \longrightarrow R' \longleftarrow Q \text{ is a CM-quasi-deformation}\}.$$

Remark 3.4. We notice that:

(i) If X is a homologically finite complex of R-modules, then by employing [5, Proposition 3.8], we obtain the following equality:

$$\text{CM}_*\text{-pd}\,_R X = \inf\{\text{G–dim}_Q(X \otimes_R R') - \text{G–dim}_Q(R') \mid R \longrightarrow R' \longleftarrow Q \text{ is a CM-quasi-deformation}\}. \quad (\ddagger)$$

(ii) For any finitely generated R-module M, $\text{CM}_*\text{-pd}\,_R M = \text{CM-dim}_R\, M$.

Notation 3.5. Remark 3.4(ii) demonstrates that the Cohen-Macaulay projective dimension serves as an extension of the Cohen-Macaulay dimension within the category of finitely generated R-modules. Given our particular focus on studying the category of homologically finite complexes, we find it more appropriate to utilize the notation $\text{CM-dim}_R(-)$, which represents the Cohen-Macaulay dimension introduced by Gerko [9], instead of $\text{CM}_*\text{-pd}\,_R(-)$. Therefore, for any homologically finite complex X, we use the notation of $\text{CM-dim}_R\, X$ instead of $\text{CM}_*\text{-pd}\,_R X$, and we call it *Cohen-Macaulay dimension* of X. As a result, we can rewrite (‡) accordingly, reflecting this notation change:

$$\text{CM-dim}_R\, X = \inf\{\text{G–dim}_Q(X \otimes_R R') - \text{G–dim}_Q(R') \mid R \longrightarrow R' \longleftarrow Q \text{ is a}$$

CM-quasi-deformation}.

The following result presents some immediate properties related to the Cohen-Macaulay dimension of homologically finite complexes.

Properties 3.6. Let $(R, \mathfrak{m})$ be a local ring and X be a homologically finite complex of R-modules.

(i) $\text{CM-dim}_R X \in \{\infty\} \cup \mathbb{Z} \cup \{-\infty\}$.
(ii) $\text{CM-dim}_R X = -\infty \Longleftrightarrow X \simeq 0$.
(iii) If $X \simeq Y$, then $\text{CM-dim}_R X = \text{CM-dim}_R Y$.
(iv) For any integer n, $\text{CM-dim}_R \Sigma^n X = \text{CM-dim}_R X + n$.

The following proposition extends the Cohen-Macaulay analogue of Auslander-Bouchsbum formula [9, Theorem 3.8] to complexes.

Proposition 3.7. *Let $(R, \mathfrak{m})$ be a local ring and X be a homologically finite complex of R-modules. There exists an inequality* $\text{CM-dim}_R X \leq \text{G–dim}_R X$ *with equality if* $\text{G–dim}_R X$ *is finite. Moreover, if* $\text{CM-dim}_R X$ *is finite, then* $\text{CM-dim}_R X = \operatorname{depth} R - \operatorname{depth}_R X$.

Proof. For the first part see [12, Remark 3.2(2)] and [5, Proposition 3.8(b)], and the proof for the second part is similar to [9, Theorem 3.8]. □

In the following lemma, by considering [5, Proposition 3.8(b)], we extend [10, Lemma 5.5] to homologically finite complexes of R-modules and also we improve the first part of Proposition 3.7. We recall that a *semi-dualizing* R-module C is a finitely generated R-module such that the natural homothety morphism $R \longrightarrow \operatorname{Hom}_R(C, C)$ is an isomorphism and $\operatorname{Ext}^i_R(C, C) = 0$ for all $i > 0$. An example of a semi-dualizing R-module is R itself. Let C be a semi-dualizing R-module. The direct sum $R \oplus C$ can be equipped with the product $(r_1, c_1)(r_2, c_2) = (r_1 r_2, r_1 c_2 + r_2 c_1)$. This turns $R \oplus C$ into a ring which is called a *trivial extension* of R by C and denoted by $R \ltimes C$. Note that we have an epimorphism $R \ltimes C \longrightarrow R$ such that its kernel is G-perfect by [9, Lemma 3.6].

Lemma 3.8. *Let $(R, \mathfrak{m})$ be a local ring and X be a homologically finite complex of R-modules. Assume that C is a semi-dualizing R-module. If* $\text{G–dim}_{R \ltimes C} X$ *is finite, then* $\text{CM-dim}_R X = \text{G–dim}_{R \ltimes C} X$.

Proof. We consider the G-quasi-deformation $R \longrightarrow R \longleftarrow R \ltimes C$. Then $\text{CM-dim}_R X \leq \text{G–dim}_{R \ltimes C} X$ and so by the assumption $\text{CM-dim}_R X$ is finite. It is easy to see that $\operatorname{depth}_{R \ltimes C} R \oplus C = \operatorname{depth}_R R$ and $\operatorname{depth}_{R \ltimes C} X = \operatorname{depth}_R X$. So, Proposition 3.7 and [3, Theorem 2.3.13] yield that $\text{CM-dim}_R X = \operatorname{depth} R - \operatorname{depth}_R X = \text{G–dim}_{R \ltimes C} X$. □

We require the following lemma to prove Theorem 3.10. Recall that for a homologically bounded to the right complex X, its *projective dimension* is defined by

$$\operatorname{pd}_R X := \inf\{\sup\{i \in \mathbb{Z} \mid P_i \neq 0\} \mid P \text{ is a projective resolution of } X\}.$$

Lemma 3.9. *Let R be a Noetherian ring and X be a homologically bounded to the right complex with finitely generated homology modules. Assume that n is an integer and $n \geq \sup X$.*

(i) *If $\Omega^n(X) = 0$, then $\mathrm{pd}_R X = \text{G–dim}_R X < n$.*
(ii) *If $\Omega^n(X) \neq 0$, then $\text{G–dim}_R(\Omega^n(X)) = \max\{0, \text{G–dim}_R X - n\}$.*

Proof. Let P be a projective resolution of X. Since $n \geq \sup X$, $P_n \sqsupset \simeq \Sigma^n \Omega^n(P)$. Consider the following exact sequence

$$(\dagger) \qquad 0 \to \sqsubset_{n-1} P \to P \to P_n \sqsupset \to 0.$$

Since $\mathrm{pd}_R(\sqsubset_{n-1} P)$ is finite, $\text{G–dim}_R(\sqsubset_{n-1} P) = \mathrm{pd}_R(\sqsubset_{n-1} P) \leq n-1$ by [3, Proposition 2.3.10]. Now, we consider the following cases:

(i) If $\Omega^n(X) = 0$, then from ($\dagger$) we deduce that $\sqsubset_{n-1} P \simeq P$ and so $\mathrm{pd}_R X = \mathrm{pd}_R P = \mathrm{pd}_R(\sqsubset_{n-1} P) \leq n-1$ which implies that $\mathrm{pd}_R X = \text{G–dim}_R X < n$ by [3, Proposition 2.3.10].

(ii) Assume that $\Omega^n(X) \neq 0$. Since $\text{G–dim}_R(\sqsubset_{n-1} P)$ is finite, from ($\dagger$) and [3, Corollary 2.3.8 and Lemma 2.1.12] we deduce that $\text{G–dim}_R P$ is finite if and only if $\text{G–dim}_R(P_n \sqsupset)$ is finite. As $\text{G–dim}_R(\Omega^n(P)) = \text{G–dim}_R(P_n \sqsupset) - n$, we conclude that $\text{G–dim}_R(P_n \sqsupset)$ is finite if and only if $\text{G–dim}_R(\Omega^n(P))$ is finite. So, we assume that $\text{G–dim}_R(\Omega^n(P))$ is finite. Then, by [3, Corollary 2.3.8 and A.2.1.3], we have

$$\begin{aligned}
\text{G–dim}_R(\Omega^n(P)) &= -\inf \mathbf{R}\mathrm{Hom}_R(\Omega^n(P), R) \\
&= -\inf \mathbf{R}\mathrm{Hom}_R(\Sigma^{-n}(P_n \sqsupset), R) \\
&= -\inf \Sigma^n \mathbf{R}\mathrm{Hom}_R(P_n \sqsupset, R) \\
&= -(\inf \mathbf{R}\mathrm{Hom}_R(P_n \sqsupset, R) + n) \\
&= -\inf \sqsubset_{-n} \mathbf{R}\mathrm{Hom}_R(P, R) - n \\
&= -\min\{-n, \inf \mathbf{R}\mathrm{Hom}_R(P, R)\} - n \\
&= \max\{n, -\inf \mathbf{R}\mathrm{Hom}_R(P, R)\} - n \\
&= \max\{0, \text{G–dim}_R X - n\}.
\end{aligned}$$

□

In the following theorem, we determine the Cohen-Macaulay dimension of a homologically finite complex of R-modules based on the Cohen-Macaulay dimension of its syzygies.

Theorem 3.10. *Let $(R, \mathfrak{m})$ be a local ring and X be a homologically finite complex of R-modules. Assume that G is a bounded to the right complex of totally reflexive R-modules such that $G \simeq X$. Fix an integer $n \geq \sup X$.*

(i) *If $\Omega^n(G) = 0$, then $\text{CM-dim}_R X = \text{G–dim}_R X < n$.*
(ii) *If $\Omega^n(G) \neq 0$, then $\text{CM-dim}_R(\Omega^n(G)) = \max\{0, \text{CM-dim}_R X - n\}$.*

Proof. Since $n \geq \sup X$, $G_n \sqsupset \simeq \Sigma^n \Omega^n(G)$. So, from the following exact sequence

$$0 \to \sqsubset_{n-1} G \to G \to G_n \sqsupset \to 0,$$

and [11, Proposition 5.1], we have

$$\begin{aligned} \operatorname{depth}_R(\Omega^n(G)) &= \operatorname{depth}_R(G_n \sqsupset) + n \\ &\geq \min\{\operatorname{depth}_R(\sqsubset_{n-1} G) - 1, \operatorname{depth}_R G\} + n. \end{aligned} \tag{✠}$$

Now, we consider the following cases:

(i) If $\Omega^n(G) = 0$, then $\sqsubset_{n-1} G \simeq G$ and so $\text{G–dim}_R X = \text{G–dim}_R G = \text{G–dim}_R(\sqsubset_{n-1} G) \leq n-1$ which implies that $\text{CM-dim}_R X = \text{G–dim}_R X < n$ by Proposition 3.7.

(ii) Assume that $\Omega^n(G) \neq 0$. We show that $\text{CM-dim}_R X$ is finite if and only if $\text{CM-dim}_R(\Omega^n(G))$ is finite. First, we assume that $\text{CM-dim}_R X$ is finite. Then there exists a CM-quasi-deformation $R \longrightarrow R' \longleftarrow Q$ such that $\text{CM-dim}_R X = \text{G–dim}_Q(X \otimes_R R') - \text{G–dim}_Q R'$. Therefore $\text{G–dim}_Q(X \otimes_R R')$ is finite and so

$$\text{G–dim}_Q(\Omega^n(X \otimes_R R')) = \max\{0, \text{G–dim}_Q(X \otimes_R R') - n\}$$

is finite by Lemma 3.9. Note that $n \geq \sup(X \otimes_R R')$ and $\Omega^n(X \otimes_R R') \neq 0$. Since $\text{G–dim}_Q(\Omega^n(X \otimes_R R')) = \text{G–dim}_Q(\Omega^n(X) \otimes_R R')$, we deduce that $\text{G–dim}_Q(\Omega^n(X) \otimes_R R')$ is finite which implies that $\text{CM-dim}_R(\Omega^n(G))$ is finite. Similarly, one can see that if $\text{CM-dim}_R(\Omega^n(G))$ is finite, then $\text{CM-dim}_R X$ is also finite. Let us assume that $t := \text{CM-dim}_R(\Omega^n(G))$ is finite. Then there exists a CM-quasi-deformation $R \longrightarrow R' \longleftarrow Q$ such that

$$t = \text{G–dim}_Q\,(\Omega^n(G) \otimes_R R') - \text{G–dim}_Q R'.$$

So

$$\begin{aligned} \text{CM-dim}_R X - n &\leq \text{G–dim}_Q(X \otimes_R R') - \text{G–dim}_Q R' - n \\ &\leq \max\{0, \text{G–dim}_Q(X \otimes_R R') - n\} - \text{G–dim}_Q R' \\ &= \text{G–dim}_Q\,(\Omega^n(X \otimes_R R')) - \text{G–dim}_Q R' \\ &= \text{G–dim}_Q\,(\Omega^n(X) \otimes_R R') - \text{G–dim}_Q R' \\ &= t, \end{aligned}$$

which implies that $\max\{0, \text{CM-dim}_R X - n\} \leq t$. On the other hand, by Proposition 3.7, (✠), and [3, Theorem 2.3.13], we have

$$\begin{aligned} t &= \operatorname{depth} R - \operatorname{depth}_R(\Omega^n(G)) \\ &\leq \operatorname{depth} R - \min\{\operatorname{depth}_R(\sqsubset_{n-1} G) - 1, \operatorname{depth}_R G\} - n \\ &= \operatorname{depth} R + \max\{1 - \operatorname{depth}_R(\sqsubset_{n-1} G), - \operatorname{depth}_R G\} - n \\ &= \max\{1 - \operatorname{depth}_R(\sqsubset_{n-1} G) + \operatorname{depth} R, \operatorname{depth} R - \operatorname{depth}_R G\} - n \\ &= \max\{1 + \text{G–dim}_R(\sqsubset_{n-1} G), \text{CM-dim}_R G\} - n \\ &\leq \max\{n, \text{CM-dim}_R G\} - n \\ &= \max\{0, \text{CM-dim}_R G - n\} \\ &= \max\{0, \text{CM-dim}_R X - n\}, \end{aligned}$$

and this completes the proof. □

We record the following immediate corollary.

Corollary 3.11. *Let $(R, \mathfrak{m})$ be a local ring and X be a homologically finite complex of R-modules. The following are equivalent.*

(i) $\text{CM-dim}_R X < \infty$.

(ii) *For any integer $n \geq \sup X$ and any bounded to the right complex G of totally reflexive R-modules such that $G \simeq X$, we have* $\text{CM-dim}_R(\Omega^n(G)) < \infty$.

(iii) *For some integer $n \geq \sup X$ and any bounded to the right complex G of totally reflexive R-modules such that $G \simeq X$, we have* $\text{CM-dim}_R(\Omega^n(G)) < \infty$.

As an interesting application of Theorem 3.10, we prove that any homologically finite complex X with a finite Cohen-Macaulay dimension has a finite *CM-resolution*. This means that there exists a bounded complex G of finitely generated R-modules, isomorphic to X, such that each nonzero module G_i in the complex has zero Cohen-Macaulay dimension.

Corollary 3.12. *Let $(R, \mathfrak{m})$ be a local ring and X be a homologically finite complex of R-modules. If* $\text{CM-dim}_R X$ *is finite, then there exists a bounded complex of finitely generated R-modules G such that $G \simeq X$ and each nonzero G_i has zero Cohen-Macaulay dimension.*

Proof. By [8, Theorem 2.6 L)], there exists a bounded to the right complex of finitely generated free R-modules G such that $X \simeq G$. Let $s := \text{CM-dim}_R X$. Then $\sup X \leq s$ by [5, Proposition 3.8], [12, Corollary 4.2], and [4, Proposition 2.2], and so $X \simeq G \simeq \subset_s G$. Then $(\subset_s G)_i = 0$ for each $i > s$ and $(\subset_s G)_i$ is finitely generated free R-module for each $i \neq s$. Also, $(\subset_s G)_s \cong \Omega^s(G)$ has zero Cohen-Macaulay dimension by Theorem 3.10. So, $\subset_s G$ is a complex that satisfies in the conditions of our claim. □

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