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ON A SPECIAL CLASS OF MATRIX RINGS

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Abstract. In this paper, we continue to explore an idea presented in [3] and introduce a new class of matrix rings called *staircase* matrix rings which has applications in noncommutative ring theory. We show that these rings preserve the notions of reduced, symmetric, reversible, IFP, reflexive, abelian rings, etc.

1. Introduction

Throughout this paper, all rings are associative with identity unless otherwise mentioned. For a ring R and a positive integer n, $Z(R)$ denotes the center of R, $\mathcal{U}(R)$ denotes the set of units of R, $M_n(R)$ denotes the ring of $n \times n$ full matrices over R, and I_n denotes the $n \times n$ identity matrix. The set of positive integers is denoted by $\mathbb{N}$.

A ring R is called *reduced* if it has no nonzero nilpotent elements. A reduced ring may or may not be commutative. Lambek [13] called a ring R *symmetric* if for a, b, c in R, $abc = 0$ implies $acb = 0$ (equivalently, $abc = 0$ implies $bac = 0$). Due to Cohn [6], a ring R is called *reversible* if for a, b in R, $ab = 0$ implies $ba = 0$. It is easy to see that both commutative and reduced rings are symmetric, and symmetric rings are reversible. Bell [2] introduced the notion of *insertion-of-factors-property* (in short, *IFP*) for rings. A ring R is called *IFP* [2] if for a, b in R, $ab = 0$ implies $aRb = 0$. IFP rings are also called *semicommutative* [8] in the literature. Due to Mason [14], a ring R is called *reflexive* if for a, b in R, $aRb = 0$ implies $bRa = 0$. Interestingly, a ring R is reversible if and only if it is both IFP and reflexive [12, Proposition 2.2]. However, there exist non-reflexive IFP rings by [12, Example 2.3(1)]. A ring R is called *abelian* if each idempotent in R is central. IFP rings are abelian [8], but a reflexive ring need not be so [12].

A quick review of the literature (see [7, 11, 12]) reveals that none of the above mentioned properties (except the reflexive property) is preserved by the full matrix rings. As a matter of fact these properties (including the reflexive

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property) are not even preserved by any of the known (non-trivial) subrings of the full matrix rings, including the (upper) triangular matrix rings. Thus it is natural to search for a class of matrix rings which preserves all the aforementioned properties of its base ring. This serves as our main motivation to explore conditions under which certain (non-trivial) subrings of the full matrix rings preserve the properties of its base rings, and thereby to introduce a new class of matrix rings called *staircase* matrix rings which does so. These rings also help us to generate examples and counter-examples of many existing classes of rings.

2. Staircase matrix rings

For a ring R and for $n \geq 1$, assume that $p_k \in Z(R)$, where $1 \leq k < 2n$, and that $p_{2n} = 1$. Consider the following subset of $M_{2n}(R)$:

$$\mathcal{S}_{2n}(R) = \left\{ (x_{ij}) : x_{ij} = \begin{cases} a_i & \text{if } i = 2k-1, j = 2k, 1 \leq k \leq n \\ a_j & \text{if } i = 2k+1, j = 2k, 1 \leq k \leq n-1 \\ \sum\limits_{k=i}^{2n} p_k a_k & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}, \quad a_i \in R \text{ for } 1 \leq i \leq 2n \right\}.$$

A typical element of $\mathcal{S}_{2n}(R)$ has the form

$$\begin{pmatrix} \sum\limits_{k=1}^{2n} p_k a_k & a_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \sum\limits_{k=2}^{2n} p_k a_k & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_2 & \sum\limits_{k=3}^{2n} p_k a_k & a_3 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \sum\limits_{k=4}^{2n} p_k a_k & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \sum\limits_{k=2n-2}^{2n} p_k a_k & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & a_{2n-2} & \sum\limits_{k=2n-1}^{2n} p_k a_k & a_{2n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{2n} \end{pmatrix},$$

where $a_i \in R$ for $1 \leq i \leq 2n$.

Lemma 1. *Let R be a ring. For $n \geq 1$, assume that $p_k \in Z(R)$, where $1 \leq k < 2n$, and that $p_{2n} = 1$. Then $\mathcal{S}_{2n}(R)$ is a subring of $M_{2n}(R)$.*

Proof. We have $I_{2n} \in \mathcal{S}_{2n}(R)$. Let $A = (x_{ij})$ and $B = (y_{ij}) \in \mathcal{S}_{2n}(R)$, where

$$x_{ij} = \begin{cases} a_i & \text{if } i = 2k-1, j = 2k, 1 \le k \le n \\ a_j & \text{if } i = 2k+1, j = 2k, 1 \le k \le n-1 \\ \sum\limits_{k=i}^{2n} p_k a_k & \text{if } i = j \\ 0 & \text{otherwise} \end{cases},$$

$a_i \in R$ for $1 \le i \le 2n$, and

$$y_{ij} = \begin{cases} b_i & \text{if } i = 2k-1, j = 2k, 1 \le k \le n \\ b_j & \text{if } i = 2k+1, j = 2k, 1 \le k \le n-1 \\ \sum\limits_{k=i}^{2n} p_k b_k & \text{if } i = j \\ 0 & \text{otherwise} \end{cases},$$

$b_i \in R$ for $1 \le i \le 2n$. Then

$$x_{ij} - y_{ij} = \begin{cases} a_i - b_i & \text{if } i = 2k-1, j = 2k, 1 \le k \le n \\ a_j - b_j & \text{if } i = 2k+1, j = 2k, 1 \le k \le n-1 \\ \sum\limits_{k=i}^{2n} p_k (a_k - b_k) & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

and so $A - B \in \mathcal{S}_{2n}(R)$. For $i = 2k-1$, $j = 2k$, where $1 \le k \le n$, we have

$$\begin{aligned} \sum_{\ell=1}^{2n} x_{i\ell} y_{\ell j} &= \left(\sum_{m=i}^{2n} p_m a_m \right) b_i + a_i \left(\sum_{m=i+1}^{2n} p_m b_m \right) \\ &= d_i \text{ (say)}. \end{aligned}$$

For $i = 2k+1$, $j = 2k$, where $1 \le k \le n-1$, we have

$$\begin{aligned} \sum_{\ell=1}^{2n} x_{i\ell} y_{\ell j} &= a_j \left(\sum_{m=j}^{2n} p_m b_m \right) + \left(\sum_{m=j+1}^{2n} p_m b_m \right) b_j \\ &= d_j \text{ (say)}. \end{aligned}$$

For $i = j = 2k-1$, where $1 \le k \le n$, we have

$$\begin{aligned} \sum_{\ell=1}^{2n} x_{i\ell} y_{\ell i} &= \left(\sum_{m=i}^{2n} p_m a_m \right) \left(\sum_{m=i}^{2n} p_m b_m \right) \\ &= \left(p_i a_i + \sum_{m=i+1}^{2n} p_m a_m \right) \left(p_i b_i + \sum_{m=i+1}^{2n} p_m b_m \right) \end{aligned}$$

$$
\begin{aligned}
&= p_i \left[\left(p_i a_i + \sum_{m=i+1}^{2n} p_m a_m \right) b_i + a_i \left(\sum_{m=i+1}^{2n} p_m b_m \right) \right] \\
&\quad + \left(\sum_{m=i+1}^{2n} p_m a_m \right) \left(\sum_{m=i+1}^{2n} p_m b_m \right) \\
&= p_i \left[\left(\sum_{m=i}^{2n} p_m a_m \right) b_i + a_i \left(\sum_{m=i+1}^{2n} p_m b_m \right) \right] \\
&\quad + \left(\sum_{m=i+1}^{2n} p_m a_m \right) \left(\sum_{m=i+1}^{2n} p_m b_m \right) \\
&= p_i d_i + \left(\sum_{m=i+1}^{2n} p_m a_m \right) \left(\sum_{m=i+1}^{2n} p_m b_m \right)
\end{aligned}
$$

as $p_k \in Z(R)$ for $1 \leq k < 2n$ and $p_{2n} = 1$. Similarly, for $i = j = 2k$, where $1 \leq k \leq n-1$, we have

$$
\begin{aligned}
\sum_{\ell=1}^{2n} x_{j\ell} y_{\ell j} &= \left(\sum_{m=j}^{2n} p_m a_m \right) \left(\sum_{m=j}^{2n} p_m b_m \right) \\
&= p_j \left[a_j \left(\sum_{m=j}^{2n} p_m b_m \right) + \left(\sum_{m=j+1}^{2n} p_m b_m \right) b_j \right] \\
&\quad + \left(\sum_{m=j+1}^{2n} p_m a_m \right) \left(\sum_{m=j+1}^{2n} p_m b_m \right) \\
&= p_j d_j + \left(\sum_{m=j+1}^{2n} p_m a_m \right) \left(\sum_{m=j+1}^{2n} p_m b_m \right).
\end{aligned}
$$

Thus

$$
\sum_{\ell=1}^{2n} x_{i\ell} y_{\ell j} = \begin{cases} d_i & \text{if } i = 2k-1, j = 2k, 1 \leq k \leq n \\ d_j & \text{if } i = 2k+1, j = 2k, 1 \leq k \leq n-1 \\ \sum\limits_{k=i}^{2n} p_k d_k & \text{if } i = j \\ 0 & \text{otherwise} \end{cases},
$$

where $d_{2n} = a_{2n} b_{2n}$. This shows that $AB \in \mathcal{S}_{2n}(R)$ and hence $\mathcal{S}_{2n}(R)$ is a subring of $M_{2n}(R)$. $\square$

Theorem 2. *Let R be a ring. For $n \geq 1$, assume that $p_k \in Z(R) \cap \mathcal{U}(R)$, where $1 \leq k < 2n$, and that $p_{2n} = 1$. Let $A = (x_{ij}) \in \mathcal{S}_{2n}(R)$, where*

$$x_{ij} = \begin{cases} a_i & \text{if } i = 2k-1, j = 2k, 1 \leq k \leq n \\ a_j & \text{if } i = 2k+1, j = 2k, 1 \leq k \leq n-1 \\ \sum\limits_{k=i}^{2n} p_k a_k & \text{if } i = j \\ 0 & \text{otherwise} \end{cases},$$

$a_i \in R$ for $1 \leq i \leq 2n$. Then we have the following.

(1) *$A = 0$ if and only if $x_{ii} = 0$ for all $1 \leq i \leq 2n$.*

(2) *The following conditions are equivalent*:

  (a) *$a_i \in Z(R)$ for all $1 \leq i \leq 2n$.*

  (b) *$x_{ii} \in Z(R)$ for all $1 \leq i \leq 2n$.*

  (c) *$A \in Z(\mathcal{S}_{2n}(R))$.*

Proof. (1) If $A = 0$, then $a_i = 0$ for all $1 \leq i \leq 2n$ and so $x_{ii} = 0$ for all $1 \leq i \leq 2n$. Conversely, let $x_{ii} = 0$ for all $1 \leq i \leq 2n$. Since $p_{2n} = 1$, $a_{2n} = 0$. For $1 < k \leq 2n$, assume that $a_i = 0$ for all $k \leq i \leq 2n$. Then $x_{jj} = 0$, where $j = k-1$, yields $p_{k-1}a_{k-1} = 0$. By assumption, $p_{k-1} \in \mathcal{U}(R)$ and so $a_{k-1} = 0$. By induction, $a_i = 0$ for all $1 \leq i \leq 2n$ and hence $A = 0$.

(2) (a) $\Rightarrow$ (b) is straightforward as $Z(R)$ is a subring of R.

(b) $\Rightarrow$ (a) Let $x_{ii} \in Z(R)$ for all $1 \leq i \leq 2n$. Since $p_{2n} = 1$, $a_{2n} \in Z(R)$. For $1 < k \leq 2n$, assume that $a_i \in Z(R)$ for all $k \leq i \leq 2n$. Then $x_{jj} \in Z(R)$, where $j = k-1$, yields $p_{k-1}a_{k-1} \in Z(R)$ as $Z(R)$ is a subring of R and $p_i \in Z(R)$ for all $1 \leq i < 2n$. Since $p_{k-1} \in Z(R) \cap \mathcal{U}(R)$, for any $r \in R$, we have $p_{k-1}r = rp_{k-1}$, which yields $rp_{k-1}^{-1} = p_{k-1}^{-1}r$. This shows that $p_{k-1}^{-1} \in Z(R)$ and consequently, $a_{k-1} = p_{k-1}^{-1}(p_{k-1}a_{k-1}) \in Z(R)$. Hence by induction, $a_i \in Z(R)$ for all $1 \leq i \leq 2n$.

(a) $\Rightarrow$ (c) Let $a_i \in Z(R)$ for all $1 \leq i \leq 2n$. To show $A \in Z(\mathcal{S}_{2n}(R))$. From (b), we have $x_{ii} \in Z(R)$ for all $1 \leq i \leq 2n$. Let $B = (y_{ij}) \in \mathcal{S}_{2n}(R)$, where

$$y_{ij} = \begin{cases} b_i & \text{if } i = 2k-1, j = 2k, 1 \leq k \leq n \\ b_j & \text{if } i = 2k+1, j = 2k, 1 \leq k \leq n-1 \\ \sum\limits_{k=i}^{2n} p_k b_k & \text{if } i = j \\ 0 & \text{otherwise} \end{cases},$$

$b_i \in R$ for $1 \leq i \leq 2n$. For $i = 2k-1$, $j = 2k$, where $1 \leq k \leq n$, we have

$$\begin{aligned} \sum_{\ell=1}^{2n} y_{i\ell}x_{\ell j} &= \left(\sum_{m=i}^{2n} p_m b_m\right) a_i + b_i \left(\sum_{m=i+1}^{2n} p_m a_m\right) \\ &= p_i b_i a_i + \left(\sum_{m=i+1}^{2n} p_m b_m\right) a_i + b_i \left(\sum_{m=i+1}^{2n} p_m a_m\right) \end{aligned}$$

$$= \left(\sum_{m=i}^{2n} p_m a_m\right) b_i + a_i \left(\sum_{m=i+1}^{2n} p_m b_m\right) = \sum_{\ell=1}^{2n} x_{i\ell} y_{\ell j}$$

as a_i, $p_i \in Z(R)$ for all $1 \leq i \leq 2n$. Similarly, for $i = 2k+1$, $j = 2k$, where $1 \leq k \leq n-1$, we have

$$\begin{aligned} \sum_{\ell=1}^{2n} y_{i\ell} x_{\ell j} &= b_j \left(\sum_{m=j}^{2n} p_m a_m\right) + \left(\sum_{m=j+1}^{2n} p_m b_m\right) a_j \\ &= a_j \left(\sum_{m=j}^{2n} p_m b_m\right) + \left(\sum_{m=j+1}^{2n} p_m b_m\right) b_j = \sum_{\ell=1}^{2n} x_{i\ell} y_{\ell j}. \end{aligned}$$

Therefore $AB = BA$ and so $A \in Z(\mathcal{S}_{2n}(R))$.
(c) $\Rightarrow$ (a) Let $A \in Z(\mathcal{S}_{2n}(R))$. For $r \in R$, we have $A(rI_{2n}) = (rI_{2n})A$, yielding $x_{ii} r = r x_{ii}$ for all $1 \leq i \leq 2n$. Thus $x_{ii} \in Z(R)$ and hence $a_i \in Z(R)$ for all $1 \leq i \leq 2n$. $\square$

For a ring R and for $n \geq 1$, assume that $p_k \in Z(R)$, where $1 \leq k \leq 2n$, and that $p_{2n+1} = 1$. Consider the following subset of $M_{2n+1}(R)$:

$$\mathcal{S}_{2n+1}(R) = \left\{ (x_{ij}) : x_{ij} = \begin{cases} a_i & \text{if } i = 2k, j = 2k+1, 1 \leq k \leq n \\ a_j & \text{if } i = 2k, j = 2k-1, 1 \leq k \leq n \\ \sum\limits_{k=i}^{2n+1} p_k a_k & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}, \right.$$
$$\left. a_i \in R \text{ for } 1 \leq i \leq 2n+1 \right\}.$$

A typical element of $\mathcal{S}_{2n+1}(R)$ has the form

$$\begin{pmatrix} \sum\limits_{k=1}^{2n+1} p_k a_k & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_1 & \sum\limits_{k=2}^{2n+1} p_k a_k & a_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \sum\limits_{k=3}^{2n+1} p_k a_k & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_3 & \sum\limits_{k=4}^{2n+1} p_k a_k & a_4 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sum\limits_{k=5}^{2n+1} p_k a_k & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \sum\limits_{k=2n-1}^{2n+1} p_k a_k & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & a_{2n-1} & \sum\limits_{k=2n}^{2n+1} p_k a_k & a_{2n} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{2n+1} \end{pmatrix},$$

where $a_i \in R$ for $1 \le i \le 2n+1$.

Lemma 3. *Let R be a ring. For $n \ge 1$, assume that $p_k \in Z(R)$, where $1 \le k \le 2n$, and that $p_{2n+1} = 1$. Then $\mathcal{S}_{2n+1}(R)$ is a subring of $M_{2n+1}(R)$.*

Proof. It is similar to the proof of Lemma 1. □

Proposition 4. *Let R be a ring. For $n \ge 1$, assume that $p_k \in Z(R) \cap \mathcal{U}(R)$, where $1 \le k \le 2n$, and that $p_{2n+1} = 1$. Let $A = (x_{ij}) \in \mathcal{S}_{2n+1}(R)$, where*

$$x_{ij} = \begin{cases} a_i & \text{if } i = 2k, j = 2k+1, 1 \le k \le n \\ a_j & \text{if } i = 2k, j = 2k-1, 1 \le k \le n \\ \sum_{k=i}^{2n+1} p_k a_k & \text{if } i = j \\ 0 & \text{otherwise} \end{cases},$$

$a_i \in R$ for $1 \le i \le 2n+1$. Then we have the following.

(1) *$A = 0$ if and only if $x_{ii} = 0$ for all $1 \le i \le 2n+1$.*
(2) *The following conditions are equivalent*:
 (a) *$a_i \in Z(R)$ for all $1 \le i \le 2n+1$.*
 (b) *$x_{ii} \in Z(R)$ for all $1 \le i \le 2n+1$.*
 (c) *$A \in Z(\mathcal{S}_{2n+1}(R))$.*

Proof. It is similar to the proof of Theorem 2. □

Definition 5. Based on Lemmas 1 and 3, for a ring R and for $p_k \in Z(R)$, where $1 \le k < n$ and $p_n = 1$, we call $\mathcal{S}_n(R)$ as the $n \times n$ *staircase* matrix ring over R *generated* by p_1, p_2, …, p_{n-1}, where $n \ge 2$.

Theorem 2 and Proposition 4 lead us to the following proposition.

Proposition 6. *Let R be a ring. For $n \ge 2$, assume that $p_k \in Z(R) \cap \mathcal{U}(R)$, where $1 \le k < n$, and that $p_n = 1$. Suppose that $A = (x_{ij}) \in \mathcal{S}_n(R)$. Then we have the following.*

(1) *$A = 0$ if and only if $x_{ii} = 0$ for all $1 \le i \le n$.*
(2) *$A \in Z(\mathcal{S}_n(R))$ if and only if $x_{ii} \in Z(R)$ for all $1 \le i \le n$.*

3. Staircase matrix rings over some classes of rings

In this section, we study the staircase matrix rings over several existing classes of rings.

Theorem 7. *Let R be a ring. For $n \ge 2$, assume that $p_k \in Z(R) \cap \mathcal{U}(R)$, where $1 \le k < n$, and that $p_n = 1$. Then*

(1) *R is commutative if and only if $\mathcal{S}_n(R)$ is commutative.*
(2) *R is reduced if and only if $\mathcal{S}_n(R)$ is reduced.*
(3) *R is symmetric if and only if $\mathcal{S}_n(R)$ is symmetric.*
(4) *R is reversible if and only if $\mathcal{S}_n(R)$ is reversible.*
(5) *R is IFP if and only if $\mathcal{S}_n(R)$ is IFP.*

(6) *R is reflexive if and only if $\mathcal{S}_n(R)$ is reflexive.*
(7) *R is abelian if and only if $\mathcal{S}_n(R)$ is abelian.*

Proof. (1) It is clear from Theorem 2(2) and Proposition 4(2).

(2) Let R be a reduced ring and let $A = (x_{ij}) \in \mathcal{S}_n(R)$ such that $A^k = 0$ for some $k \in \mathbb{N}$. Then $x_{ii}^k = 0$ for all $1 \leq i \leq n$. Since R is reduced, $x_{ii} = 0$ for all $1 \leq i \leq n$ and thus by Proposition 6(1), $A = 0$. Hence $\mathcal{S}_n(R)$ is reduced. Converse is straightforward as a subring of a reduced ring is reduced.

(3) Let R be a symmetric ring and let $A = (x_{ij})$, $B = (y_{ij})$, $C = (z_{ij}) \in \mathcal{S}_n(R)$ such that $ABC = 0$. Then $x_{ii}y_{ii}z_{ii} = 0$ for all $1 \leq i \leq n$ and as R is symmetric, $x_{ii}z_{ii}y_{ii} = 0$ for all $1 \leq i \leq n$. Thus by Proposition 6(1), $ACB = 0$ and hence $\mathcal{S}_n(R)$ is symmetric. Converse is straightforward as a subring of a symmetric ring is symmetric.

(4) It is similar to (3).

(5) Let R be an IFP ring and let $A = (x_{ij})$, $B = (y_{ij}) \in \mathcal{S}_n(R)$ such that $AB = 0$. Then $x_{ii}y_{ii} = 0$ for all $1 \leq i \leq n$ and as R is IFP, $x_{ii}Ry_{ii} = 0$ for all $1 \leq i \leq n$. Thus for any $(r_{ij}) \in \mathcal{S}_n(R)$, $x_{ii}r_{ii}y_{ii} = 0$ and so by Proposition 6(1), $A(r_{ij})B = 0$, yielding $A\mathcal{S}_n(R)B = 0$. Hence $\mathcal{S}_n(R)$ is IFP. Converse is straightforward as a subring of an IFP ring is IFP.

(6) Let R be a reflexive ring and let $A = (x_{ij})$, $B = (y_{ij}) \in \mathcal{S}_n(R)$ such that $A\mathcal{S}_n(R)B = 0$. For any $r \in R$, we have $rI_n \in \mathcal{S}_n(R)$ and so $A(rI_n)B = 0$, yielding $x_{ii}ry_{ii} = 0$ for all $1 \leq i \leq n$. Thus for each i with $1 \leq i \leq n$, $x_{ii}Ry_{ii} = 0$ and as R is reflexive, $y_{ii}Rx_{ii} = 0$. Then for any $(r_{ij}) \in \mathcal{S}_n(R)$, $y_{ii}r_{ii}x_{ii} = 0$ for all $1 \leq i \leq n$ and so by Proposition 6(1), $B(r_{ij})A = 0$. Thus $B\mathcal{S}_n(R)A = 0$ and hence $\mathcal{S}_n(R)$ is reflexive. Conversely, let $\mathcal{S}_n(R)$ be a reflexive ring and let $a, b \in R$ such that $aRb = 0$. Then for any $(r_{ij}) \in \mathcal{S}_n(R)$, we have $ar_{ii}b = 0$ for all $1 \leq i \leq n$ and so by Proposition 6(1), $(aI_n)(r_{ij})(bI_n) = 0$, yielding $(aI_n)\mathcal{S}_n(R)(bI_n) = 0$. Since $\mathcal{S}_n(R)$ is reflexive, $(bI_n)\mathcal{S}_n(R)(aI_n) = 0$. Then for any $r \in R$, we have $(bI_n)(rI_n)(aI_n) = 0$, yielding $bra = 0$. Thus $bRa = 0$ and hence R is reflexive.

(7) Let R be an abelian ring and let $E = (e_{ij}) \in \mathcal{S}_n(R)$ such that $E^2 = E$. Then $e_{ii}^2 = e_{ii}$ for all $1 \leq i \leq n$. Since R is abelian, $e_{ii} \in Z(R)$ for all $1 \leq i \leq n$ and thus by Proposition 6(2), $E \in Z(\mathcal{S}_n(R))$. Hence $\mathcal{S}_n(R)$ is abelian. Converse is straightforward as a subring of an abelian ring is abelian. ☐

Definition 8. Following the literature, a ring R is called

(1) *central reduced* [1] if every nilpotent in R is central,
(2) *right* (resp., *left*) *central symmetric* [9, 10] if for a, b, c in R, $abc = 0$ implies $acb \in Z(R)$ (resp., $bac \in Z(R)$),
(3) *central reversible* [10] if for a, b in R, $ab = 0$ implies $ba \in Z(R)$,
(4) *central IFP* [15] if for a, b in R, $ab = 0$ implies $aRb \subseteq Z(R)$,
(5) *central reflexive* [4] if for a, b in R, $aRb = 0$ implies $bRa \subseteq Z(R)$.

We change over from "a central semicommutative ring" in [15] to "a central IFP ring", so as to keep consistency with other related definitions.

Note that a central reduced ring is both right and left central symmetric, by [9, Lemma 2.5] and [10, Lemma 2.3], and every right (left) central symmetric ring is central reversible [10, Lemma 2.7] as well as central IFP [9, Lemma 2.9], however, a central reversible ring need not be central IFP and vice versa, by [5, Example 2.1] and [9, Example 2.6]. Interestingly, both central reversible and central IFP rings are abelian, by [9, Lemma 2.7] and [15, Lemma 2.6]. Also every central reversible ring is central reflexive [4, Proposition 2.3], however, an IFP ring need not be central reflexive by a simple computation.

Applying the method used in the proof of Proposition 7, upon using Proposition 6(2), we have the following.

Proposition 9. *Let R be a ring. For $n \geq 2$, assume that $p_k \in Z(R) \cap \mathcal{U}(R)$, where $1 \leq k < n$, and that $p_n = 1$. Then*

(1) *R is central reduced if and only if $\mathcal{S}_n(R)$ is central reduced.*
(2) *R is right (left) central symmetric if and only if $\mathcal{S}_n(R)$ is right (left) central symmetric.*
(3) *R is central reversible if and only if $\mathcal{S}_n(R)$ is central reversible.*
(4) *R is central IFP if and only if $\mathcal{S}_n(R)$ is central IFP.*
(5) *R is central reflexive if and only if $\mathcal{S}_n(R)$ is central reflexive.*

4. Concluding remarks

For a ring R and for $n \geq 1$, consider the following subset of $M_{2n}(R)$:

$$\mathcal{S}^*_{2n}(R) = \left\{ (x_{ij}) : x_{ij} = \begin{cases} a_{2i} & \text{if } i = 2k-1, j = 2k, 1 \leq k \leq n \\ a_{2j} & \text{if } i = 2k+1, j = 2k, 1 \leq k \leq n-1 \\ a_{2i-1} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}, \right.$$

$$\left. a_i \in R \text{ for } 1 \leq i \leq 4n-1 \right\}.$$

A typical element of $\mathcal{S}^*_{2n}(R)$ has the form

$$\begin{pmatrix} a_1 & a_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_3 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_4 & a_5 & a_6 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & a_7 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & a_8 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{4n-5} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & a_{4n-4} & a_{4n-3} & a_{4n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{4n-1} \end{pmatrix},$$

where $a_i \in R$ for $1 \le i \le 4n-1$.

Lemma 10. *Let R be a ring and let $n \ge 1$. Then $\mathcal{S}^*_{2n}(R)$ is a subring of $M_{2n}(R)$.*

Proof. We have $I_{2n} \in \mathcal{S}^*_{2n}(R)$. Let $A = (x_{ij})$ and $B = (y_{ij}) \in \mathcal{S}^*_{2n}(R)$, where

$$x_{ij} = \begin{cases} a_{2i} & \text{if } i = 2k-1, j = 2k, 1 \le k \le n \\ a_{2j} & \text{if } i = 2k+1, j = 2k, 1 \le k \le n-1 \\ a_{2i-1} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases},$$

$a_i \in R$ for $1 \le i \le 4n-1$, and

$$y_{ij} = \begin{cases} b_{2i} & \text{if } i = 2k-1, j = 2k, 1 \le k \le n \\ b_{2j} & \text{if } i = 2k+1, j = 2k, 1 \le k \le n-1 \\ b_{2i-1} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases},$$

$b_i \in R$ for $1 \le i \le 4n-1$. Then

$$x_{ij} - y_{ij} = \begin{cases} a_{2i} - b_{2i} & \text{if } i = 2k-1, j = 2k, 1 \le k \le n \\ a_{2j} - b_{2j} & \text{if } i = 2k+1, j = 2k, 1 \le k \le n-1 \\ a_{2i-1} - b_{2i-1} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

and

$$\sum_{\ell=1}^{2n} x_{i\ell} y_{\ell j} = \begin{cases} a_{2i-1}b_{2i} + a_{2i}b_{2i+1} & \text{if } i = 2k-1, j = 2k, 1 \le k \le n \\ a_{2j}b_{2j-1} + a_{2j+1}b_{2j} & \text{if } i = 2k+1, j = 2k, 1 \le k \le n-1 \\ a_{2i-1}b_{2i-1} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

Thus $A - B,\ AB \in \mathcal{S}^*_{2n}(R)$ and hence $\mathcal{S}^*_{2n}(R)$ is a subring of $M_{2n}(R)$. □

For a ring R and for $n \ge 1$, if $p_k \in Z(R)$, where $1 \le k < 2n$, and if $p_{2n} = 1$, then it is clear from Lemmas 1 and 10 that $\mathcal{S}_{2n}(R)$ is a subring of $\mathcal{S}^*_{2n}(R)$. Next we show that the properties mentioned in Section 3 are not preserved by $\mathcal{S}^*_{2n}(R)$, in general.

Example 11. Let $\mathbb{Z}_2$ denote the ring of integers modulo 2. For $n \ge 1$, consider the ring $\mathcal{S}^*_{2n}(\mathbb{Z}_2)$. Let E_{ij} denote the matrix with (i,j)-th entry 1 and other entries 0. Clearly, $E_{11} \in \mathcal{S}^*_{2n}(\mathbb{Z}_2)$ such that $E_{11}^2 = E_{11}$, however, $E_{11}E_{12} = E_{12}$ and $E_{12}E_{11} = 0$, entailing $E_{11} \notin Z(\mathcal{S}^*_{2n}(\mathbb{Z}_2))$. Thus $\mathcal{S}^*_{2n}(\mathbb{Z}_2)$ is not abelian and hence it is not (central) reduced, (right or left central) symmetric, (central) reversible, and (central) IFP. Also for any $(r_{ij}) \in \mathcal{S}^*_{2n}(\mathbb{Z}_2)$, we have $E_{12}(r_{ij})E_{11} = 0$, yielding $E_{12}\mathcal{S}^*_{2n}(\mathbb{Z}_2)E_{11} = 0$, however, $E_{11}E_{12} = E_{12} \notin Z(\mathcal{S}^*_{2n}(\mathbb{Z}_2))$. This shows that $E_{11}\mathcal{S}^*_{2n}(\mathbb{Z}_2)E_{12} \nsubseteq Z(\mathcal{S}^*_{2n}(\mathbb{Z}_2))$, entailing $\mathcal{S}^*_{2n}(\mathbb{Z}_2)$ is not (central) reflexive.

For a ring R and for $n \geq 1$, consider the following subset of $M_{2n+1}(R)$:

$$\mathcal{S}^*_{2n+1}(R) = \left\{ (x_{ij}) : x_{ij} = \begin{cases} a_{2i} & \text{if } i = 2k, j = 2k+1, 1 \leq k \leq n \\ a_{2j} & \text{if } i = 2k, j = 2k-1, 1 \leq k \leq n \\ a_{2i-1} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}, \quad a_i \in R \text{ for } 1 \leq i \leq 4n+1 \right\}.$$

A typical element of $\mathcal{S}^*_{2n+1}(R)$ has the form

$$\begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_3 & a_4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_5 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_6 & a_7 & a_8 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_9 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a_{4n-3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & a_{4n-2} & a_{4n-1} & a_{4n} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{4n+1} \end{pmatrix},$$

where $a_i \in R$ for $1 \leq i \leq 4n+1$.

Lemma 12. *Let R be a ring and let $n \geq 1$. Then $\mathcal{S}^*_{2n+1}(R)$ is a subring of $M_{2n+1}(R)$.*

Proof. It is similar to the proof of Lemma 10. □

For a ring R and for $n \geq 1$, if $p_k \in Z(R)$, where $1 \leq k \leq 2n$, and if $p_{2n+1} = 1$, then it is clear from Lemmas 3 and 12 that $\mathcal{S}_{2n+1}(R)$ is a subring of $\mathcal{S}^*_{2n+1}(R)$. Next we show that the properties mentioned in Section 3 are not preserved by $\mathcal{S}^*_{2n+1}(R)$, in general.

Example 13. For $n \geq 1$, consider the ring $\mathcal{S}^*_{2n+1}(\mathbb{Z}_2)$. Clearly, $E_{11} \in \mathcal{S}^*_{2n+1}(\mathbb{Z}_2)$ such that $E_{11}^2 = E_{11}$, however, $E_{11} \notin Z(\mathcal{S}^*_{2n+1}(\mathbb{Z}_2))$ as $E_{11}E_{21} = 0$ and $E_{21}E_{11} = E_{21}$. Thus $\mathcal{S}^*_{2n+1}(\mathbb{Z}_2)$ is not abelian. Also $E_{11}\mathcal{S}^*_{2n+1}(\mathbb{Z}_2)E_{21} = 0$, but $E_{21}E_{11} = E_{21} \notin Z(\mathcal{S}^*_{2n+1}(\mathbb{Z}_2))$. This shows that $E_{21}\mathcal{S}^*_{2n+1}(\mathbb{Z}_2)E_{11} \nsubseteq Z(\mathcal{S}^*_{2n+1}(\mathbb{Z}_2))$ and so $\mathcal{S}^*_{2n+1}(\mathbb{Z}_2)$ is not central reflexive.

Remark 14. For a ring R and for $n \geq 2$, if $p_k \in Z(R)$, where $1 \leq k < n$, and if $p_n = 1$, then from Lemmas 1, 3, 10 and 12, we see that $\mathcal{S}_n(R)$ is a subring of $\mathcal{S}_n^*(R)$. Though the elements of both the rings are visually similar but structurally they are different as shown by Propositions 7, 9 and Examples 11, 13. Examples 11 and 13 also demonstrate that the conditions imposed in defining $\mathcal{S}_n(R)$ are not superfluous.

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