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$(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-GORENSTEIN COMPLEXES

Yanjie Li and Renyu Zhao

Abstract. Let $\mathcal{V}$, $\mathcal{W}$, $\mathcal{Y}$, $\mathcal{X}$ be four classes of left R-modules. The notion of $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein R-complexes is introduced, and it is shown that under certain mild technical assumptions on $\mathcal{V}$, $\mathcal{W}$, $\mathcal{Y}$, $\mathcal{X}$, an R-complex $\boldsymbol{M}$ is $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein if and only if the module in each degree of $\boldsymbol{M}$ is $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein and the total Hom complexes $\mathrm{Hom}_R(\boldsymbol{Y}, \boldsymbol{M})$, $\mathrm{Hom}_R(\boldsymbol{M}, \boldsymbol{X})$ are exact for any $\boldsymbol{Y} \in \widetilde{\mathcal{Y}}$ and any $\boldsymbol{X} \in \widetilde{\mathcal{X}}$. Many known results are recovered, and some new cases are also naturally generated.

1. Introduction

Inspired by Auslander and Bridger's notion of finitely generated modules of G-dimension zero over two-sided Noetherian rings [1], Enochs and collaborators [7, 9] introduced and studied Gorenstein projective, Gorenstein injective and Gorenstein flat modules over arbitrary rings that developed the so-called Gorenstein homological algebra for modules. Since then, these modules and their analogues were extensively studied, see, e.g. [2, 4, 5, 10, 14, 17, 18, 21, 25, 28, 29, 32, 35, 36, 43, 44, 49]. Let $\mathcal{V}$, $\mathcal{W}$, $\mathcal{Y}$, $\mathcal{X}$ be four classes of left R-modules. Recently, motivated by Yang's notion of $(\mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein modules [35], Zhao and Sun's notion of $\mathcal{V}\mathcal{W}$-Gorenstein modules [49], we introduced the notion of $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein modules [46] which proposes a unified approach to the various Gorenstein modules appeared in the above citations.

In 1998, Enochs and García Rozas [6] extended the notion of Gorenstein projective and Gorenstein injective left R-modules to the category of complexes of left R-modules, and proved that over an n-Gorenstein ring R, a complex $\boldsymbol{M}$ of left R-modules is Gorenstein projective (resp., injective) if and only if each M_n is a Gorenstein projective (resp., injective) left R-module. Yang [34], Yang and Liu [40] further showed independently that the same results hold over any

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ring R. Thereafter, Gorenstein projective (resp., injective) complexes with respect to a semidulizing module [38], Ding projective (resp., injective) complexes [41], Gorenstein AC-projective (resp., AC-injective) complexes [3], as well as $\mathcal{W}$-Gorenstein complexes [24, 33], $\mathcal{VW}$-Gorenstein complexes [48], $(\mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein complexes [45], Gorenstein projective complexes with respect to a complete and hereditary cotorsion pair [47] etc. have been studied, which led to many papers with parallel results.

Given four classes $\mathcal{V}$, $\mathcal{W}$, $\mathcal{Y}$, $\mathcal{X}$ of left R-modules, following the idea of [46], in this paper, we introduce a notion of $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein complexes, see Definition 3.1, which proposes a unified approach to all the above-mentioned classes of Gorenstein complexes as well as some other particular Gorenstein complexes that is not explored. The main theorem of this paper is the following result, contained in Theorem 3.7. It showcases the benefit of our approach to studying $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein complexes, as it not only enables us to study $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein complexes via $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein modules, see Proposition 3.8-Corollary 3.12, but also it simultaneously encompasses the results of [40, Theorem 2.2, Proposition 2.8], [34, Theorems 1 and 2], [38, Theorems 4.6 and 4.7], [41, Theorems 3.7 and 3.20], [3, Theorems 3.2 and 4.13], [24, Corollary 4.8], [33, Theorem 4.6] and our own results [48, Theorem 3.8], [45, Theorem 3.9] and [47, Theorem 3.5], and generates some new results naturally, see Section 4.

Theorem 1.1. *Let $\mathcal{V}$, $\mathcal{W}$, $\mathcal{Y}$, $\mathcal{X}$ be four classes of left R-modules and $\boldsymbol{M}$ be a complex of left R-modules. If $\mathcal{V}$, $\mathcal{W}$ are self-orthogonal, closed under isomorphisms and finite direct sums, $\mathcal{Y}$, $\mathcal{X}$ are closed under extensions, $\mathcal{V} \subseteq \mathcal{Y}$, $\mathcal{W} \subseteq \mathcal{X}$, $\mathcal{V} \perp \mathcal{X}$, $\mathcal{Y} \perp \mathcal{W}$, and $\mathcal{V},\mathcal{W} \subseteq \mathcal{G}(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$, then $\boldsymbol{M}$ is $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein if and only if each M_n is a $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein module and $\mathrm{Hom}_R(\boldsymbol{Y}, \boldsymbol{M})$, $\mathrm{Hom}_R(\boldsymbol{M}, \boldsymbol{X})$ are exact for any $\boldsymbol{Y} \in \widetilde{\mathcal{Y}}$, $\boldsymbol{X} \in \widetilde{\mathcal{X}}$.*

The outline of the paper is as follows. Preliminary notions and notation are laid out in Section 2. In Section 3, we first give the definition of $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein complexes and prove Theorem 1.1. Then, we apply Theorem 1.1 to deduce some properties of $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein complexes from that of $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein modules. In Section 4, the particular cases of $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein complexes that fits to Theorem 1.1 are exhibited.

2. Preliminaries

Throughout this article, R denotes an associative ring with unit. By a module, unless otherwise explicitly stated, we mean a left R-module. The category of left R-modules is denoted by R-Mod. By a complex $\boldsymbol{M}$, we mean a complex of left R-modules as follows:

$$\cdots \to M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \to \cdots ,$$

i.e., the differentials lower the degree. The category of complexes of left R-modules is denoted by $\mathrm{Ch}(R)$. For an $\boldsymbol{M} \in \mathrm{Ch}(R)$, the nth cycle (resp., boundary, homology) of $\boldsymbol{M}$ is denoted by $\mathrm{Z}_n(\boldsymbol{M})$ (resp., $\mathrm{B}_n(\boldsymbol{M})$, $\mathrm{H}_n(\boldsymbol{M})$). For an $m\in\mathbb{Z}$, the symbol $\boldsymbol{M}[m]$ denotes the complex with $(\boldsymbol{M}[m])_n=M_{n-m}$ and $d_n^{\boldsymbol{M}[m]}=(-1)^m d_{n-m}^{\boldsymbol{M}}$ for all $n\in\mathbb{Z}$. Given a module M, we use $\overline{M}$ to denote the complex

$$\cdots\to 0\to M\xrightarrow{id} M\to 0\to\cdots$$

with M in the 1st and 0th degree.

Given $\boldsymbol{M},\boldsymbol{N}\in\mathrm{Ch}(R)$. The total Hom complex $\mathrm{Hom}_R(\boldsymbol{M},\boldsymbol{N})$ is given by

$$\mathrm{Hom}_R(\boldsymbol{M},\boldsymbol{N})_n=\prod_{k\in\mathbb{Z}}\mathrm{Hom}_R(M_k,N_{n+k})$$

and

$$d_n^{\mathrm{Hom}_R(\boldsymbol{M},\boldsymbol{N})}((f_k)_{k\in\mathbb{Z}})=(d_{n+k}^{\boldsymbol{N}}f_k-(-1)^n f_{k-1}d_k^{\boldsymbol{M}})_{k\in\mathbb{Z}}.$$

We use $\mathrm{Hom}_{\mathrm{Ch}(R)}(\boldsymbol{M},\boldsymbol{N})$ for $\mathrm{Z}_0(\mathrm{Hom}_R(\boldsymbol{M},\boldsymbol{N}))$, the group of all morphisms from $\boldsymbol{M}$ to $\boldsymbol{N}$, and $\mathrm{Ext}^i_{\mathrm{Ch}(R)}(\boldsymbol{M},\boldsymbol{N})$ to denote the groups one gets from the right-derived functor of $\mathrm{Hom}_{\mathrm{Ch}(R)}(-,-)$ for $i\geq 0$. Recall that $\mathrm{Ext}^1_{\mathrm{Ch}(R)}(\boldsymbol{M},\boldsymbol{N})$ is the group of (equivalence classes) of short exact sequences $0\to\boldsymbol{N}\to\boldsymbol{L}\to\boldsymbol{M}\to 0$ under the Baer sum. We let $\mathrm{Ext}^1_{dw}(\boldsymbol{M},\boldsymbol{N})$ be the subgroup of $\mathrm{Ext}^1_{\mathrm{Ch}(R)}(\boldsymbol{M},\boldsymbol{N})$ consisting of those short exact sequences which are split in each degree. We often make use of the following standard fact.

Lemma 2.1 ([15, Lemma 2.1])**.** *For any $\boldsymbol{M},\boldsymbol{N}\in\mathrm{Ch}(R)$ and any $n\in\mathbb{Z}$, we have*

$$\mathrm{Ext}^1_{dw}(\boldsymbol{M},\boldsymbol{N}[-n-1])\cong\mathrm{H}_n(\mathrm{Hom}_R(\boldsymbol{M},\boldsymbol{N}))=\mathrm{Hom}_{\mathrm{K}(R)}(\boldsymbol{M},\boldsymbol{N}[-n]),$$

where $\mathrm{K}(R)$ *denotes the homotopy category of complexes of left R-modules. In particular,* $\mathrm{Hom}_R(\boldsymbol{M},\boldsymbol{N})$ *is exact if and only if for any $n\in\mathbb{Z}$, any morphism $f:\boldsymbol{M}[n]\to\boldsymbol{N}$ is homotopic to* 0.

Let $\mathcal{Z}$ be a class of modules. We use $\widetilde{\mathcal{Z}}$ to denote the class of all exact complexes $\boldsymbol{M}$ with each cycle $\mathrm{Z}_n(\boldsymbol{M})\in\mathcal{Z}$, and use $dw\widetilde{\mathcal{Z}}$ to denote the class of all complexes $\boldsymbol{M}$ with each degree $M_n\in\mathcal{Z}$.

Let $\mathcal{D}$ be an abelian category, and $\mathcal{B}$ be a subcategory of $\mathcal{D}$. Recall that a sequence $\mathbb{U}$ in $\mathcal{D}$ is $\mathrm{Hom}_{\mathcal{D}}(\mathcal{B},-)$-exact (resp., $\mathrm{Hom}_{\mathcal{D}}(-,\mathcal{B})$-exact) if the sequence $\mathrm{Hom}_{\mathcal{D}}(B,\mathbb{U})$ (resp., $\mathrm{Hom}_{\mathcal{D}}(\mathbb{U},B)$) is exact for any $B\in\mathcal{B}$. For two subcategories $\mathcal{X},\mathcal{Y}$ of $\mathcal{D}$, we say $\mathcal{X}\perp\mathcal{Y}$ if $\mathrm{Ext}^{\geq 1}_{\mathcal{D}}(X,Y)=0$ for any $X\in\mathcal{X}$ and any $Y\in\mathcal{Y}$.

Definition 2.2 ([46, Definition 3.1])**.** Let $\mathcal{V}$, $\mathcal{W}$, $\mathcal{Y}$, $\mathcal{X}$ be four classes of modules. A module M is called $(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein if there exists a both $\mathrm{Hom}_R(\mathcal{Y},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{X})$-exact exact sequence

$$\cdots\to V_1\to V_0\to W^0\to W^1\to\cdots$$

with $V_i\in\mathcal{V}$ and $W^i\in\mathcal{W}$ for all $i\geq 0$ such that $M\cong\mathrm{Im}(V_0\to W^0)$. The exact sequence is called a complete $(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$-resolution of M.

We denote the class of all $(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein modules by $\mathcal{G}(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$. This notion unifies and generalizes a number of known Gorenstein modules, see [46, Remark 3.2] for details.

3. $(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein complexes

In this section, we will extend the notion of $(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein modules to that of complexes and prove Theorem 1.1.

Definition 3.1. Let $\mathcal{V}$, $\mathcal{W}$, $\mathcal{Y}$, $\mathcal{X}$ be four classes of modules. A complex $\boldsymbol{M}$ is called $(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein if there exists a both $\mathrm{Hom}_{\mathrm{Ch}(R)}(\widetilde{\mathcal{Y}},-)$-exact and $\mathrm{Hom}_{\mathrm{Ch}(R)}(-,\widetilde{\mathcal{X}})$-exact exact sequence of complexes

$$\mathbb{U}:\cdots\to\boldsymbol{V}_1\to\boldsymbol{V}_0\to\boldsymbol{W}^0\to\boldsymbol{W}^1\to\cdots$$

with each $\boldsymbol{V}_i\in\widetilde{\mathcal{V}}$ and $\boldsymbol{W}^i\in\widetilde{\mathcal{W}}$ such that $\boldsymbol{M}\cong\mathrm{Im}(\boldsymbol{V}_0\to\boldsymbol{W}^0)$. In this case, we say $\mathbb{U}$ is a complete $(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$-resolution of $\boldsymbol{M}$.

We denote by $\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$ the class of all $(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein complexes. To characterize $(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein complexes, we need the following preparations.

Lemma 3.2 ([15, Lemma 3.1])**.** *Given a complex $\boldsymbol{M}$ and a module N, we have the following natural isomorphisms for any $n\in\mathbb{Z}$:*

(1) $\mathrm{Hom}_{\mathrm{Ch}(R)}(\overline{N}[n],\boldsymbol{M})\cong\mathrm{Hom}_R(N,M_{n+1})$.
(2) $\mathrm{Hom}_{\mathrm{Ch}(R)}(\boldsymbol{M},\overline{N}[n])\cong\mathrm{Hom}_R(M_n,N)$.
(3) $\mathrm{Ext}^1_{\mathrm{Ch}(R)}(\overline{N}[n],\boldsymbol{M})\cong\mathrm{Ext}^1_R(N,M_{n+1})$.
(4) $\mathrm{Ext}^1_{\mathrm{Ch}(R)}(\boldsymbol{M},\overline{N}[n])\cong\mathrm{Ext}^1_R(M_n,N)$.

Lemma 3.3 ([24, Lemma 4.4])**.** *Let $\mathcal{U},\mathcal{Z}$ be two classes of modules. If $\mathcal{U}\perp\mathcal{U}$, then the following statements hold:*

(1) *$\mathcal{Z}\perp\mathcal{U}$ if and only if $dw\widetilde{\mathcal{Z}}\perp\widetilde{\mathcal{U}}$.*
(2) *$\mathcal{U}\perp\mathcal{Z}$ if and only if $\widetilde{\mathcal{U}}\perp dw\widetilde{\mathcal{Z}}$.*

Corollary 3.4. *Let $\mathcal{U},\mathcal{Z}$ be two classes of modules such that $\mathcal{Z}$ is closed under extensions and $\mathcal{U}\perp\mathcal{U}$. Then the following statements hold:*

(1) *If $\mathcal{Z}\perp\mathcal{U}$, then $\widetilde{\mathcal{Z}}\perp\widetilde{\mathcal{U}}$.*
(2) *If $\mathcal{U}\perp\mathcal{Z}$, then $\widetilde{\mathcal{U}}\perp\widetilde{\mathcal{Z}}$.*

Proof. As noticed that $\widetilde{\mathcal{Z}}\subseteq dw\widetilde{\mathcal{Z}}$ whenever the class $\mathcal{Z}$ of modules is closed under extensions, the assertions follow immediate from Lemma 3.3. □

Lemma 3.5. *Let $\mathcal{U}$ be a class of modules such that $\mathcal{U}\perp\mathcal{U}$. Then $\mathrm{Hom}_R(\boldsymbol{U},\boldsymbol{M})$ and $\mathrm{Hom}_R(\boldsymbol{M},\boldsymbol{U})$ are exact for any $\boldsymbol{M}\in\mathrm{Ch}(R)$ and any $\boldsymbol{U}\in\widetilde{\mathcal{U}}$.*

Proof. The assumption $\mathcal{U}\perp\mathcal{U}$ implies that each $\boldsymbol{U}\in\widetilde{\mathcal{U}}$ is a zero object in $\mathrm{K}(R)$. Hence, Lemma 2.1 yields the desired conclusion. □

Lemma 3.6. *Let $\mathcal{Y}, \mathcal{X}$ be two classes of modules and*

$$\cdots \to \boldsymbol{M}_1 \xrightarrow{\sigma_1} \boldsymbol{M}_0 \xrightarrow{\sigma_0} \boldsymbol{M}_{-1} \to \cdots$$

be a $\mathrm{Hom}_{\mathrm{Ch}(R)}(\widetilde{\mathcal{Y}}, -)$*-exact and* $\mathrm{Hom}_{\mathrm{Ch}(R)}(-, \widetilde{\mathcal{X}})$*-exact exact sequence of complexes. Then for any $n \in \mathbb{Z}$, the exact sequence of modules*

$$\cdots \to (M_1)_n \xrightarrow{(\sigma_1)_n} (M_0)_n \xrightarrow{(\sigma_0)_n} (M_{-1})_n \to \cdots$$

is $\mathrm{Hom}_R(\mathcal{Y}, -)$*-exact and* $\mathrm{Hom}_R(-, \mathcal{X})$*-exact.*

Proof. Let $Y \in \mathcal{Y}$, $X \in \mathcal{X}$ and $n \in \mathbb{Z}$. Then $\overline{Y}[n-1] \in \widetilde{\mathcal{Y}}$ and $\overline{X}[n] \in \widetilde{\mathcal{X}}$. Thus we have the following exact sequences

$$\cdots \to \mathrm{Hom}_{\mathrm{Ch}(R)}(\overline{Y}[n-1], \boldsymbol{M}_1) \to \mathrm{Hom}_{\mathrm{Ch}(R)}(\overline{Y}[n-1], \boldsymbol{M}_0) \\ \to \mathrm{Hom}_{\mathrm{Ch}(R)}(\overline{Y}[n-1], \boldsymbol{M}_{-1}) \to \cdots,$$

$$\cdots \to \mathrm{Hom}_{\mathrm{Ch}(R)}(\boldsymbol{M}_{-1}, \overline{X}[n]) \to \mathrm{Hom}_{\mathrm{Ch}(R)}(\boldsymbol{M}_0, \overline{X}[n]) \\ \to \mathrm{Hom}_{\mathrm{Ch}(R)}(\boldsymbol{M}_1, \overline{X}[n]) \to \cdots.$$

By Lemma 3.2(1) and (2), we have exact sequences

$$\cdots \to \mathrm{Hom}_R(Y, (M_1)_n) \to \mathrm{Hom}_R(Y, (M_0)_n) \to \mathrm{Hom}_R(Y, (M_{-1})_n) \to \cdots,$$

$$\cdots \to \mathrm{Hom}_R((M_{-1})_n, X) \to \mathrm{Hom}_R((M_0)_n, X) \to \mathrm{Hom}_R((M_1)_n, X) \to \cdots.$$

So the result holds. □

In the rest of this section, we always assume that $\mathcal{V}$, $\mathcal{W}$, $\mathcal{Y}$, $\mathcal{X}$ are four classes of modules which satisfy the conditions: $\mathcal{V}$, $\mathcal{W}$ are self-orthogonal, closed under isomorphisms and finite direct sums, $\mathcal{Y}$, $\mathcal{X}$ are closed under extensions, $\mathcal{V} \subseteq \mathcal{Y}$, $\mathcal{W} \subseteq \mathcal{X}$, $\mathcal{V} \perp \mathcal{X}$, $\mathcal{Y} \perp \mathcal{W}$, and $\mathcal{V}$,$\mathcal{W} \subseteq \mathcal{G}(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$.

Now, we are in a position to prove Theorem 1.1.

Theorem 3.7. *Let $\boldsymbol{M}$ be a complex. Then $\boldsymbol{M}$ is $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein if and only if each M_n is a $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein module and* $\mathrm{Hom}_R(\boldsymbol{Y}, \boldsymbol{M})$, $\mathrm{Hom}_R(\boldsymbol{M}, \boldsymbol{X})$ *are exact for any $\boldsymbol{Y} \in \widetilde{\mathcal{Y}}$ and $\boldsymbol{X} \in \widetilde{\mathcal{X}}$.*

Proof. $\Rightarrow$) Assume that $\boldsymbol{M} \in \mathcal{G}(\widetilde{\mathcal{V}}, \widetilde{\mathcal{W}}, \widetilde{\mathcal{Y}}, \widetilde{\mathcal{X}})$ and let

$$\cdots \to \boldsymbol{V}_1 \to \boldsymbol{V}_0 \to \boldsymbol{W}^0 \to \boldsymbol{W}^1 \to \cdots$$

be a complete $(\widetilde{\mathcal{V}}, \widetilde{\mathcal{W}}, \widetilde{\mathcal{Y}}, \widetilde{\mathcal{X}})$-resolution of $\boldsymbol{M}$. The conditions $\mathcal{V}$, $\mathcal{W}$ are self-orthogonal, closed under isomorphisms and finite direct sums imply that each $(V_i)_n \in \mathcal{V}$ and each $(W^i)_n \in \mathcal{W}$. So, for any $n \in \mathbb{Z}$, it follows from Lemma 3.6 that

$$\cdots \to (V_1)_n \to (V_0)_n \to (W^0)_n \to (W^1)_n \to \cdots$$

is a complete $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-resolution of M_n, which yields that $M_n \in \mathcal{G}(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$ for any $n \in \mathbb{Z}$.

Let $\boldsymbol{Y} \in \widetilde{\mathcal{Y}}$, $\boldsymbol{X} \in \widetilde{\mathcal{X}}$. To show that $\operatorname{Hom}_R(\boldsymbol{Y}, \boldsymbol{M})$ and $\operatorname{Hom}_R(\boldsymbol{M}, \boldsymbol{X})$ are exact, we set $\boldsymbol{K}_0 = \operatorname{Im}(\boldsymbol{V}_1 \to \boldsymbol{V}_0)$ and $\boldsymbol{K}^1 = \operatorname{Im}(\boldsymbol{W}^0 \to \boldsymbol{W}^1)$. Consider exact sequences

$$\operatorname{Hom}_{\operatorname{Ch}(R)}(\boldsymbol{Y}, \boldsymbol{W}^0) \to \operatorname{Hom}_{\operatorname{Ch}(R)}(\boldsymbol{Y}, \boldsymbol{K}^1) \to \operatorname{Ext}^1_{\operatorname{Ch}(R)}(\boldsymbol{Y}, \boldsymbol{M}) \to \operatorname{Ext}^1_{\operatorname{Ch}(R)}(\boldsymbol{Y}, \boldsymbol{W}^0)$$

and

$$\operatorname{Hom}_{\operatorname{Ch}(R)}(\boldsymbol{V}_0, \boldsymbol{X}) \to \operatorname{Hom}_{\operatorname{Ch}(R)}(\boldsymbol{K}_0, \boldsymbol{X}) \to \operatorname{Ext}^1_{\operatorname{Ch}(R)}(\boldsymbol{M}, \boldsymbol{X}) \to \operatorname{Ext}^1_{\operatorname{Ch}(R)}(\boldsymbol{V}_0, \boldsymbol{X}).$$

By the assumptions on $\mathcal{V}$, $\mathcal{W}$, $\mathcal{Y}$, $\mathcal{X}$, Corollary 3.4 applies to yield that $\operatorname{Ext}^1_{\operatorname{Ch}(R)}$ $(\boldsymbol{Y}, \boldsymbol{W}^0) = 0$ and $\operatorname{Ext}^1_{\operatorname{Ch}(R)}(\boldsymbol{V}_0, \boldsymbol{X}) = 0$. The $\operatorname{Hom}_{\operatorname{Ch}(R)}(\widetilde{\mathcal{Y}}, -)$-exactness of $0 \to \boldsymbol{M} \to \boldsymbol{W}^0 \to \boldsymbol{K}^1 \to 0$ and the $\operatorname{Hom}_{\operatorname{Ch}(R)}(-, \widetilde{\mathcal{X}})$-exactness of $0 \to \boldsymbol{K}_0 \to \boldsymbol{V}_0 \to \boldsymbol{M} \to 0$ now yield that $\operatorname{Ext}^1_{\operatorname{Ch}(R)}(\boldsymbol{Y}, \boldsymbol{M}) = 0$ and $\operatorname{Ext}^1_{\operatorname{Ch}(R)}(\boldsymbol{M}, \boldsymbol{X}) = 0$. It then follows from [46, Corollary 3.4] and Lemma 2.1 that $\operatorname{Hom}_R(\boldsymbol{Y}, \boldsymbol{M})$ and $\operatorname{Hom}_R(\boldsymbol{M}, \boldsymbol{X})$ are exact.

$\Leftarrow$) Suppose that each M_n is a $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein module, both $\operatorname{Hom}_R(\boldsymbol{Y}, \boldsymbol{M})$ and $\operatorname{Hom}_R(\boldsymbol{M}, \boldsymbol{X})$ are exact for any $\boldsymbol{Y} \in \widetilde{\mathcal{Y}}$ and $\boldsymbol{X} \in \widetilde{\mathcal{X}}$. To prove that $\boldsymbol{M}$ is $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein, by definition, it suffices to show that $\boldsymbol{M}$ has a complete $(\widetilde{\mathcal{V}}, \widetilde{\mathcal{W}}, \widetilde{\mathcal{Y}}, \widetilde{\mathcal{X}})$-resolution, that is, there is a $\operatorname{Hom}_{\operatorname{Ch}(R)}(\widetilde{\mathcal{Y}}, -)$-exact and $\operatorname{Hom}_{\operatorname{Ch}(R)}(-, \widetilde{\mathcal{X}})$-exact exact sequence of complexes

$$\cdots \to \boldsymbol{V}_1 \to \boldsymbol{V}_0 \to \boldsymbol{W}^0 \to \boldsymbol{W}^1 \to \cdots$$

with each $\boldsymbol{V}_i \in \widetilde{\mathcal{V}}$ and $\boldsymbol{W}^i \in \widetilde{\mathcal{W}}$ such that $\boldsymbol{M} \cong \operatorname{Im}(\boldsymbol{V}_0 \to \boldsymbol{W}^0)$. We only build the left part of such an exact sequence, i.e., to construct a both $\operatorname{Hom}_{\operatorname{Ch}(R)}(\widetilde{\mathcal{Y}}, -)$-exact and $\operatorname{Hom}_{\operatorname{Ch}(R)}(-, \widetilde{\mathcal{X}})$-exact exact sequence of complexes

$$(\P) \qquad \cdots \to \boldsymbol{V}_2 \to \boldsymbol{V}_1 \to \boldsymbol{V}_0 \to \boldsymbol{M} \to 0$$

with each $\boldsymbol{V}_i \in \widetilde{\mathcal{V}}$. The construction of the right half is dual.

For any $n \in \mathbb{Z}$, as M_n is $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein, it follows that there is an exact sequence of modules

$$0 \to G_n \to V_n \xrightarrow{g_n} M_n \to 0,$$

where $V_n \in \mathcal{V}$ and $G_n \in \mathcal{G}(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$ by [46, Corollary 3.11]. One then gets an exact sequence of complexes

$$0 \to \bigoplus_{n\in\mathbb{Z}} \overline{G_n}[n-1] \to \bigoplus_{n\in\mathbb{Z}} \overline{V_n}[n-1] \xrightarrow{\ g\ } \bigoplus_{n\in\mathbb{Z}} \overline{M_n}[n-1] \to 0,$$

where $g = \bigoplus_{n\in\mathbb{Z}} \overline{g_n}[n-1]$. Put $\boldsymbol{V}_0 = \bigoplus_{n\in\mathbb{Z}} \overline{V_n}[n-1]$. It is clear that $\boldsymbol{V}_0 \in \widetilde{\mathcal{V}}$. On the other hand, for $\boldsymbol{M}$, there is always a (degreewise split) short exact sequence

$$0 \to \boldsymbol{M}[-1] \xrightarrow{\binom{1}{-d}} \bigoplus_{n\in\mathbb{Z}} \overline{M_n}[n-1] \xrightarrow{(d,1)} \boldsymbol{M} \to 0,$$

where d is the differential of $\boldsymbol{M}$. Let $\beta : \boldsymbol{V}_0 \to \boldsymbol{M}$ be the composite

$$\bigoplus_{n\in\mathbb{Z}} \overline{V_n}[n-1] \xrightarrow{g} \bigoplus_{n\in\mathbb{Z}} \overline{M_n}[n-1] \xrightarrow{(d,1)} \boldsymbol{M}.$$

Then β is epic since it is the composite of two epimorphisms. Setting $\boldsymbol{K}_0 = \mathrm{Ker}\beta$ yields an exact sequence of complexes

$$(\P_0) \qquad 0 \to \boldsymbol{K}_0 \to \boldsymbol{V}_0 \to \boldsymbol{M} \to 0.$$

To construct the exact sequence (¶), it is now sufficient to show that $\boldsymbol{K}_0$ has the same properties as $\boldsymbol{M}$, and that the exact sequence $(\P_0)$ is both $\mathrm{Hom}_{\mathrm{Ch}(R)}(\widetilde{\mathcal{Y}}, -)$-exact and $\mathrm{Hom}_{\mathrm{Ch}(R)}(-, \widetilde{\mathcal{X}})$-exact. To this end, consider the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \boldsymbol{K}_0 & \longrightarrow & \boldsymbol{M}[-1] & & \\
 & & & & \downarrow & & \downarrow{\scriptstyle \binom{1}{-d}} & & \\
0 & \longrightarrow & \bigoplus_{n\in\mathbb{Z}} \overline{G_n}[n-1] & \longrightarrow & \boldsymbol{V}_0 & \xrightarrow{g} & \bigoplus_{n\in\mathbb{Z}} \overline{M_n}[n-1] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle (d,1)} & & \\
0 & \longrightarrow & 0 & \longrightarrow & \boldsymbol{M} & = & \boldsymbol{M} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Apply the Snake Lemma to this diagram to get the exact sequence

$$0 \to \bigoplus_{n\in\mathbb{Z}} \overline{G_n}[n-1] \to \boldsymbol{K}_0 \to \boldsymbol{M}[-1] \to 0.$$

As both $\bigoplus_{n\in\mathbb{Z}} \overline{G^n}[n-1]$ and $\boldsymbol{M}[-1]$ are complexes of $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein modules, it follows from [46, Theorem 3.6] that each degree of $\boldsymbol{K}_0$ is $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein.

Let $\boldsymbol{Y} \in \widetilde{\mathcal{Y}}$. As $\mathcal{Y}$ is closed under extensions, it follows that $Y_i \in \mathcal{Y}$ for each $i \in \mathbb{Z}$. Thus $\mathrm{Ext}^1_R(Y_i, (K_0)_{n+i}) = 0$ by [46, Corollary 3.4] for any $n, i \in \mathbb{Z}$. So we have the following exact sequence

$$0 \to \mathrm{Hom}_R(Y_i, (K_0)_{n+i}) \to \mathrm{Hom}_R(Y_i, (V_0)_{n+i}) \to \mathrm{Hom}_R(Y_i, M_{n+i}) \to 0$$

for all $n, i \in \mathbb{Z}$. One thus gets the following exact sequence of complexes

$$0 \to \mathrm{Hom}_R(\boldsymbol{Y}, \boldsymbol{K}_0) \to \mathrm{Hom}_R(\boldsymbol{Y}, \boldsymbol{V}_0) \to \mathrm{Hom}_R(\boldsymbol{Y}, \boldsymbol{M}) \to 0.$$

As the complex $\boldsymbol{V}_0$ is a zero object in $\mathrm{K}(R)$, the homotopy category over R, it follows from Lemma 2.1 that the complex $\mathrm{Hom}_R(\boldsymbol{Y}, \boldsymbol{V}_0)$ is exact. Since $\mathrm{Hom}_R(\boldsymbol{Y}, \boldsymbol{M})$ is exact, it follows that $\mathrm{Hom}_R(\boldsymbol{Y}, \boldsymbol{K}_0)$ is exact. Similarly, one can show that $\mathrm{Hom}_R(\boldsymbol{K}_0, \boldsymbol{X})$ is exact for any $\boldsymbol{X} \in \widetilde{\mathcal{X}}$. As $\boldsymbol{K}_0$ and $\boldsymbol{M}$ are both degreewise $(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein, it follows from [46, Corollary 3.4] and Lemma 2.1 that

$$\mathrm{Ext}^1_{\mathrm{Ch}(R)}(\boldsymbol{Y},\boldsymbol{K}_0)=\mathrm{Ext}^1_{dw}(\boldsymbol{Y},\boldsymbol{K}_0)\cong \mathrm{H}_{-1}(\mathrm{Hom}_R\,(\boldsymbol{Y},\boldsymbol{K}_0))=0$$

and

$$\mathrm{Ext}^1_{\mathrm{Ch}(R)}(\boldsymbol{M},\boldsymbol{X})=\mathrm{Ext}^1_{dw}(\boldsymbol{M},\boldsymbol{X})\cong \mathrm{H}_{-1}(\mathrm{Hom}_R\,(\boldsymbol{M},\boldsymbol{X}))=0$$

for any $\boldsymbol{Y}\in\widetilde{\mathcal{Y}}$ and any $\boldsymbol{X}\in\widetilde{\mathcal{X}}$. This implies that the sequence

$$0\to\boldsymbol{K}_0\to\boldsymbol{V}_0\to\boldsymbol{M}\to 0$$

is both $\mathrm{Hom}_{\mathrm{Ch}(R)}(\widetilde{\mathcal{Y}},-)$-exact and $\mathrm{Hom}_{\mathrm{Ch}(R)}(-,\widetilde{\mathcal{X}})$-exact. This establishes the desired conclusion. □

As applications of Theorem 3.7, in the rest of this section, we deduce some properties of $(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein complexes in terms of those of $(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein modules which were established in [46].

Proposition 3.8. *Let*

$$0\to\boldsymbol{M}'\to\boldsymbol{M}\to\boldsymbol{M}''\to 0 \tag{3.1}$$

be a short exact sequence of complexes.

(1) *If* $\boldsymbol{M}',\boldsymbol{M}''\in\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$*, then* $\boldsymbol{M}\in\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$.

(2) *If* $\boldsymbol{M}',\boldsymbol{M}\in\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$*, then* $\boldsymbol{M}''\in\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$ *if and only if* $\mathrm{Ext}^1_{\mathrm{Ch}(R)}(\boldsymbol{M}'',\boldsymbol{X})=0$ *for any* $\boldsymbol{X}\in\widetilde{\mathcal{X}}$.

(3) *If* $\boldsymbol{M},\boldsymbol{M}''\in\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$*, then* $\boldsymbol{M}'\in\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$ *if and only if* $\mathrm{Ext}^1_{\mathrm{Ch}(R)}(\boldsymbol{Y},\boldsymbol{M}')=0$ *for any* $\boldsymbol{Y}\in\widetilde{\mathcal{Y}}$.

Proof. (1) Assume that $\boldsymbol{M}',\boldsymbol{M}''\in\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$. Then it follows from Theorem 3.7 and [46, Theorem 3.6] that $M_n\in\mathcal{G}(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$ for any $n\in\mathbb{Z}$. Let $\boldsymbol{Y}\in\widetilde{\mathcal{Y}}$. In view of [46, Corollary 3.4], the sequence

$$0\to\mathrm{Hom}_R(\boldsymbol{Y},\boldsymbol{M}')\to\mathrm{Hom}_R(\boldsymbol{Y},\boldsymbol{M})\to\mathrm{Hom}_R(\boldsymbol{Y},\boldsymbol{M}'')\to 0$$

is exact. By Theorem 3.7, $\mathrm{Hom}_R(\boldsymbol{Y},\boldsymbol{M}')$ and $\mathrm{Hom}_R(\boldsymbol{Y},\boldsymbol{M}'')$ are exact, then so is $\mathrm{Hom}_R(\boldsymbol{Y},\boldsymbol{M})$. Similarly, one can show that $\mathrm{Hom}_R(\boldsymbol{M},\boldsymbol{X})$ is exact for any $\boldsymbol{X}\in\widetilde{\mathcal{X}}$. Thus, $\boldsymbol{M}\in\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$ by Theorem 3.7.

(2) Let $\boldsymbol{M}''\in\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$. As $\mathcal{V}\perp\mathcal{X}$, it follows from Theorem 3.7 and [46, Corollary 3.4] that $\mathrm{Ext}^1_R(M''_n,X)=0$ for any $X\in\mathcal{X}$ and any $n\in\mathbb{Z}$. Hence, Lemma 2.1 and Theorem 3.7 imply that $\mathrm{Ext}^1_{\mathrm{Ch}(R)}(\boldsymbol{M}'',\boldsymbol{X})=\mathrm{Ext}^1_{dw}(\boldsymbol{M}'',\boldsymbol{X})\cong \mathrm{H}_{-1}(\mathrm{Hom}_R\,(\boldsymbol{M}'',\boldsymbol{X}))=0$ for any $\boldsymbol{X}\in\widetilde{\mathcal{X}}$.

Conversely, assume that $\boldsymbol{M}',\boldsymbol{M}$ are in $\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$ and $\mathrm{Ext}^1_{\mathrm{Ch}(R)}(\boldsymbol{M}'',\boldsymbol{X})=0$ for any $\boldsymbol{X}\in\widetilde{\mathcal{X}}$. By Theorem 3.7, $M'_n\in\mathcal{G}(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$ for any $n\in\mathbb{Z}$. Thus for any $\boldsymbol{Y}\in\widetilde{\mathcal{Y}}$, it follows [46, Corollary 3.4] that the sequence

$$0\to\mathrm{Hom}_R(\boldsymbol{Y},\boldsymbol{M}')\to\mathrm{Hom}_R(\boldsymbol{Y},\boldsymbol{M})\to\mathrm{Hom}_R(\boldsymbol{Y},\boldsymbol{M}'')\to 0$$

is exact. Since $\mathrm{Hom}_R(\boldsymbol{Y},\boldsymbol{M}')$ and $\mathrm{Hom}_R(\boldsymbol{Y},\boldsymbol{M})$ are exact by Theorem 3.7, so is $\mathrm{Hom}_R(\boldsymbol{Y},\boldsymbol{M}'')$. For any $X\in\mathcal{X}$, by the assumption on $\boldsymbol{M}''$, Lemma 3.2 applies to yield $\mathrm{Ext}^1_R(M''_n,X)=0$ for any $n\in\mathbb{Z}$. It is now immediate from Theorem 3.7 and [46, Theorem 3.6] that $M''_n\in\mathcal{G}(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$ for every $n\in\mathbb{Z}$. Moreover, for any $\boldsymbol{X}\in\widetilde{\mathcal{X}}$, one has the following exact sequence

$$0\to\mathrm{Hom}_R(\boldsymbol{M}'',\boldsymbol{X})\to\mathrm{Hom}_R(\boldsymbol{M},\boldsymbol{X})\to\mathrm{Hom}_R(\boldsymbol{M}',\boldsymbol{X})\to 0.$$

As $\mathrm{Hom}_R(\boldsymbol{M}',\boldsymbol{X})$ and $\mathrm{Hom}_R(\boldsymbol{M},\boldsymbol{X})$ are exact, the displayed sequence yields that $\mathrm{Hom}_R(\boldsymbol{M}'',\boldsymbol{X})$ is exact. Thus, $\boldsymbol{M}''\in\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$ by Theorem 3.7.

(3) It is dual to the proof of (2). □

Proposition 3.9. *$\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$ is closed under direct summands.*

Proof. It is an immediate consequence of Theorem 3.7 and the fact that $\mathcal{G}(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$ is closed under direct summands, see [46, Theorem 3.8]. □

Learned from [30, Corollary 4.1], [22, Theorem 4.1] and [23, Theorem 3.1], we showed that $\mathcal{G}\left(\mathcal{G}(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X}),\mathcal{G}(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X}),\mathcal{Y},\mathcal{X}\right)=\mathcal{G}(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$, see [46, Theorem 3.9], which shows that the class $\mathcal{G}(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$ has a good stability. Now, using of Theorem 3.7, we can extend this result to the category of complexes.

Proposition 3.10. *Let $\boldsymbol{M}$ be a complex. Then $\boldsymbol{M}\in\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$ if and only if there exists a both $\mathrm{Hom}_{\mathrm{Ch}(R)}(\widetilde{\mathcal{Y}},-)$-exact and $\mathrm{Hom}_{\mathrm{Ch}(R)}(-,\widetilde{\mathcal{X}})$-exact exact sequence $\cdots\to\boldsymbol{G}_1\to\boldsymbol{G}_0\to\boldsymbol{G}^0\to\boldsymbol{G}^1\to\cdots$ of $(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein complexes such that $\boldsymbol{M}\cong\mathrm{Im}(\boldsymbol{G}_0\to\boldsymbol{G}^0)$.*

Proof. The "only if" part is trivial. For the "if" part, assume that there is a $\mathrm{Hom}_{\mathrm{Ch}(R)}(\widetilde{\mathcal{Y}},-)$-exact and $\mathrm{Hom}_{\mathrm{Ch}(R)}(-,\widetilde{\mathcal{X}})$-exact exact sequence of $(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein complexes

$$\cdots\to\boldsymbol{G}_1\to\boldsymbol{G}_0\to\boldsymbol{G}^0\to\boldsymbol{G}^1\to\cdots$$

such that $\boldsymbol{M}\cong\mathrm{Im}(\boldsymbol{G}_0\to\boldsymbol{G}^0)$. Then for any $n\in\mathbb{Z}$, it follows from Lemma 3.6 and Theorem 3.7 that there is a $\mathrm{Hom}_R(\mathcal{Y},-)$-exact and $\mathrm{Hom}_R(-,\mathcal{X})$-exact exact sequence of $(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein modules

$$\cdots\to(G_1)_n\to(G_0)_n\to(G^0)_n\to(G^1)_n\to\cdots$$

such that $M_n\cong\mathrm{Im}((G_0)_n\to(G^0)_n)$. It thus follows from [46, Theorem 3.9] that M_n belongs to $\mathcal{G}(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$ for each $n\in\mathbb{Z}$. By Theorem 3.7, to prove that $\boldsymbol{M}\in\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$, it is now sufficient to show that $\mathrm{Hom}_R(\boldsymbol{Y},\boldsymbol{M})$ and $\mathrm{Hom}_R(\boldsymbol{M},\boldsymbol{X})$ are exact for all $\boldsymbol{Y}\in\widetilde{\mathcal{Y}}$ and $\boldsymbol{X}\in\widetilde{\mathcal{X}}$.

Let $\boldsymbol{X}\in\widetilde{\mathcal{X}}$. Consider the exact sequence $0\to\boldsymbol{K}_1\to\boldsymbol{G}_0\to\boldsymbol{M}\to 0$, where $\boldsymbol{K}_1=\mathrm{Im}(\boldsymbol{G}_1\to\boldsymbol{G}_0)$. Notice that $\boldsymbol{G}_0\in\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$, it follows from Theorem 3.7 that $\mathrm{Hom}_R(\boldsymbol{G}_0,\boldsymbol{X})$ is exact and each $(G_0)_n\in\mathcal{G}(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$, so [46, Corollary 3.4] and Lemma 2.1 yield $\mathrm{Ext}^1_{\mathrm{Ch}(R)}(\boldsymbol{G}_0,\boldsymbol{X}[-n-1])=0$ for any $n\in\mathbb{Z}$. As $\mathrm{Hom}_{\mathrm{Ch}(R)}(-,\widetilde{\mathcal{X}})$ leaves $0\to\boldsymbol{K}_1\to\boldsymbol{G}_0\to\boldsymbol{M}\to 0$ exact, it follows that $\mathrm{Ext}^1_{\mathrm{Ch}(R)}(\boldsymbol{M}, \boldsymbol{X}[-n-1]) = 0$ for any $n \in \mathbb{Z}$. Thus $\mathrm{Hom}_R(\boldsymbol{M}, \boldsymbol{X})$ is exact by Lemma 2.1. Dually, one can show that $\mathrm{Hom}_R(\boldsymbol{Y}, \boldsymbol{M})$ is exact for any $\boldsymbol{Y} \in \widetilde{\mathcal{Y}}$. Now, the result follows. □

The following result shows that $(\widetilde{\mathcal{V}}, \widetilde{\mathcal{W}}, \widetilde{\mathcal{Y}}, \widetilde{\mathcal{X}})$-Gorenstein complexes possess a kind of symmetry.

Corollary 3.11. *Let $\boldsymbol{M}$ be a complex. Then $\boldsymbol{M} \in \mathcal{G}(\widetilde{\mathcal{V}}, \widetilde{\mathcal{W}}, \widetilde{\mathcal{Y}}, \widetilde{\mathcal{X}})$ if and only if there exists a both $\mathrm{Hom}_{\mathrm{Ch}(R)}(\widetilde{\mathcal{Y}}, -)$-exact and $\mathrm{Hom}_{\mathrm{Ch}(R)}(-, \widetilde{\mathcal{X}})$-exact exact sequence*

$$\cdots \to \boldsymbol{U}_1 \to \boldsymbol{U}_0 \to \boldsymbol{U}^0 \to \boldsymbol{U}^1 \to \cdots$$

with all $\boldsymbol{U}_i, \boldsymbol{U}^i \in \widetilde{\mathcal{V}} \cup \widetilde{\mathcal{W}}$ such that $\boldsymbol{M} \cong \mathrm{Im}(\boldsymbol{U}_0 \to \boldsymbol{U}^0)$.

Proof. The "only if" part is clear. For the "if" part, as $\mathcal{V}$, $\mathcal{W}$ are self-orthogonal, closed under isomorphisms and finite direct sums, and $\mathcal{V}, \mathcal{W} \subseteq \mathcal{G}(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$, it follows from Lemma 2.1 and Theorem 3.7 that $\widetilde{\mathcal{V}}, \widetilde{\mathcal{W}} \subseteq \mathcal{G}(\widetilde{\mathcal{V}}, \widetilde{\mathcal{W}}, \widetilde{\mathcal{Y}}, \widetilde{\mathcal{X}})$. Thus $\boldsymbol{M} \in \mathcal{G}(\widetilde{\mathcal{V}}, \widetilde{\mathcal{W}}, \widetilde{\mathcal{Y}}, \widetilde{\mathcal{X}})$ by Proposition 3.10. □

By Corollary 3.11, we have the following result.

Corollary 3.12. *Every kernel in a complete $(\widetilde{\mathcal{V}}, \widetilde{\mathcal{W}}, \widetilde{\mathcal{Y}}, \widetilde{\mathcal{X}})$-resolution is $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein.*

4. Some examples and applications

In this section, we provide some specific examples of $(\mathcal{V}, \mathcal{W}, \mathcal{Y}, \mathcal{X})$-Gorenstein complexes and apply Theorem 3.7 to these special cases. Some results existing in the literature are recovered and some new results are generated. We begin with some notions and notation.

According to [31], a module M is called *FP-injective* (or *absolutely pure* [26]) if $\mathrm{Ext}^1_R(N, M) = 0$ for all finitely presented modules N. A module Q is called *FP-projective* [27] if $\mathrm{Ext}^1_R(Q, M) = 0$ for any FP-injective module M. Recall that a module F is said to be *super finitely presented* [11] (or *type* FP_∞ [4]) if it has a projective resolution by finitely generated projective modules. A module M is *weak injective* [11] (or *absolutely clean* [4]) provided that $\mathrm{Ext}^1_R(F, M) = 0$ for any super finitely presented module F. A module N is *weak flat* [11] (or *level* [4]) provided that $\mathrm{Tor}^R_1(F, N) = 0$ for any super finitely presented right R-module F. Recall that a module P is *weak projective* [10] if $\mathrm{Ext}^1_R(P, M) = 0$ for all weak injective modules M. We write $\mathcal{P}$ (resp., $\mathcal{I}$, $\mathcal{F}$, $\mathcal{FI}$, $\mathcal{FP}$, $\mathcal{WI}$, $\mathcal{WF}$, $\mathcal{WP}$) for the class of projective (resp., injective, flat, FP-injective, FP-projective, weak injective, weak flat, weak projective) modules, and set $\mathcal{FI}^{\mathrm{fp}} = \mathcal{FI} \cap \mathcal{FP}$, $\mathcal{WI}^{\mathrm{wp}} = \mathcal{WI} \cap \mathcal{WP}$.

Let R be a commutative ring. Following [32], an R-module C is called *semidualizing* if

(1) C admits a degreewise finite projective resolution.

(2) The homothety map ${}_R R_R \xrightarrow{\gamma_R} \mathrm{Hom}_R(C, C)$ is an isomorphism.

(3) $\mathrm{Ext}_R^{\geqslant 1}(C,C)=0$.

In the remainder of the paper, let C be an arbitrary but fixed semidualizing module over a commutative ring R.

Following [32], the *Auslander class* $\mathcal{A}_C$ with respect to C consists of all modules M satisfying:

(1) $\mathrm{Tor}^R_{\geqslant 1}(C,M)=0=\mathrm{Ext}_R^{\geqslant 1}(C,C\otimes_R M)$ and
(2) The natural evaluation homomorphism $\mu_M: M\to \mathrm{Hom}_R(C,C\otimes_R M)$ is an isomorphism.

The *Bass class* $\mathcal{B}_C$ with respect to C consists of all modules N satisfying:

(1) $\mathrm{Ext}_R^{\geqslant 1}(C,N)=0=\mathrm{Tor}^R_{\geqslant 1}(C,\mathrm{Hom}_R(C,N))$ and
(2) The natural evaluation homomorphism $v_N: C\otimes_R \mathrm{Hom}_R(C,N)\to N$ is an isomorphism.

Let $\mathcal{X},\mathcal{Y}$ be two classes of modules over a commutative ring R. Set

$$\mathcal{X}_C^T=\{M\mid M\cong C\otimes_R X \text{ for some } X\in\mathcal{X}\}$$

and

$$\mathcal{Y}_C^H=\{M\mid M\cong \mathrm{Hom}_R(C,Y) \text{ for some } Y\in\mathcal{Y}\}.$$

In particular, if $\mathcal{X}=\mathcal{P}$ (resp., $\mathcal{F}$, $\mathcal{WF}$), then we denote the class $\mathcal{X}_C^T$ by $\mathcal{P}_C$ (resp., $\mathcal{F}_C$, $\mathcal{WF}_C$) and call modules in $\mathcal{P}_C$ (resp., $\mathcal{F}_C$, $\mathcal{WF}_C$) *C-projective* (resp., *C-flat*, *C-weak flat*) as usual; if $\mathcal{Y}=\mathcal{I}$ (resp., $\mathcal{FI}$, $\mathcal{WI}$, $\mathcal{FI}^{\mathrm{fp}}$, $\mathcal{WI}^{\mathrm{wp}}$), then we denote the class $\mathcal{Y}_C^H$ by $\mathcal{I}_C$ (resp., $\mathcal{FI}_C$, $\mathcal{WI}_C$, $\mathcal{FI}_C^{\mathrm{fp}}$, $\mathcal{WI}_C^{\mathrm{wp}}$) and call modules in $\mathcal{I}_C$ (resp., $\mathcal{FI}_C$, $\mathcal{WI}_C$, $\mathcal{FI}_C^{\mathrm{fp}}$, $\mathcal{WI}_C^{\mathrm{wp}}$) *C-injective* (resp., *C-FP-injective*, *C-weak injective*, *C-FP-injective C-FP-projective*, *C-weak injective C-weak projective*).

We now give some special cases of $(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein complexes and outline the consequences of Theorem 3.7 for these special cases.

4.1. $(\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein complexes

Let $\mathcal{W}$, $\mathcal{Y}$, $\mathcal{X}$ be three classes of modules. Then $(\mathcal{W},\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein modules are just $(\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein modules [35], which unify $\mathcal{X}$-Gorenstein projective modules [2, 29], $\mathcal{Y}$-Gorenstein injective modules [29], $\mathcal{W}$-Gorenstein modules [14, 30], Gorenstein projective (injective) modules [7], Ding projective (injective) modules [5] ([28]) and Gorenstein AC-projective (injective) modules [4]. Accordingly, $(\mathcal{W},\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein complexes are exactly $(\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein complexes [45], which contain $\mathcal{X}$-Gorenstein projective complexes [45], $\mathcal{Y}$-Gorenstein injective complexes [45], $\mathcal{W}$-Gorenstein complexes [24, 33], Gorenstein projective (injective) complexes [6, 40], Ding projective (injective) complexes [41] and Gorenstein AC-projective (injective) complexes [3] as their special cases, see [45] for more details. By Theorem 3.7, we have:

Corollary 4.1 ([45, Theorem 3.9])**.** *Let $\mathcal{W}$, $\mathcal{Y}$, $\mathcal{X}$ be classes of modules. If $\mathcal{Y}$, $\mathcal{X}$ are closed under extensions, $\mathcal{W}$ is closed under isomorphisms and finite direct sums, $\mathcal{W}\subseteq\mathcal{Y}$, $\mathcal{W}\subseteq\mathcal{X}$ and $\mathcal{Y}\perp\mathcal{W}$, $\mathcal{W}\perp\mathcal{X}$, then for any $\boldsymbol{M}\in\mathrm{Ch}(R)$,*

$\boldsymbol{M}$ is $(\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein if and only if each M_n is a $(\mathcal{W},\mathcal{Y},\mathcal{X})$-Gorenstein module and $\mathrm{Hom}_R(\boldsymbol{Y},\boldsymbol{M}),\mathrm{Hom}_R(\boldsymbol{M},\boldsymbol{X})$ *are exact for any* $\boldsymbol{Y}\in\widetilde{\mathcal{Y}},\boldsymbol{X}\in\widetilde{\mathcal{X}}$.

We note that Corollary 4.1 generalizes both [41, Theorems 3.7, 3.20] and [3, Theorems 3.2, 4.13]. It also recovers [24, Corollary 4.8], [33, Theorem 3.12], [34, Theorems 1, 2] and [40, Theorem 2.2, Proposition 2.8] by Lemma 3.5.

4.2. $\mathcal{VW}$-Gorenstein complexes

Let $\mathcal{V}$, $\mathcal{W}$ be two classes of modules. Then $\mathcal{G}(\mathcal{V},\mathcal{W},\mathcal{V},\mathcal{W})$ is just the class of all $\mathcal{VW}$-Gorenstein modules [49] and $\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{V}},\widetilde{\mathcal{W}})$ is exactly the class of all $\mathcal{VW}$-Gorenstein complexes [48]. By Theorem 3.7 and Lemma 3.5, one can obtain:

Corollary 4.2 ([48, Theorem 3.8])**.** *Let $\mathcal{V}$, $\mathcal{W}$ be two classes of modules and $\boldsymbol{M}\in\mathrm{Ch}(R)$. If $\mathcal{V}$, $\mathcal{W}$ are closed under extensions, isomorphisms and finite direct sums, and $\mathcal{V}\perp\mathcal{V}$, $\mathcal{W}\perp\mathcal{W}$, $\mathcal{V}\perp\mathcal{W}$, $\mathcal{V},\mathcal{W}\subseteq\mathcal{G}(\mathcal{V},\mathcal{W},\mathcal{V},\mathcal{W})$, then $\boldsymbol{M}$ is $\mathcal{VW}$-Gorenstein if and only if M_n is a $\mathcal{VW}$-Gorenstein module for any $n\in\mathbb{Z}$.*

Corollary 4.2 contains [24, Corollary 4.8] and [33, Theorem 3.12] as well as the following results as its special cases.

Corollary 4.3 ([38, Theorems 4.6 and 4.7])**.** *Let R be a commutative ring, C a semidualizing R-module and $\boldsymbol{M}\in\mathrm{Ch}(R)$. Then $\boldsymbol{M}$ is G_C-projective (resp., G_C-injective) if and only if M_n is G_C-projective (resp., G_C-injective) for any $n\in\mathbb{Z}$.*

Proof. Note that $\mathcal{G}(\mathcal{P},\mathcal{P}_C,\mathcal{P},\mathcal{P}_C)$ (resp., $\mathcal{G}(\mathcal{I}_C,\mathcal{I},\mathcal{I}_C,\mathcal{I})$) agrees with the class of all G_C-projective modules [18,32] (resp., the class of all G_C-injective modules [18]), and $\mathcal{G}(\widetilde{\mathcal{P}},\widetilde{\mathcal{P}_C},\widetilde{\mathcal{P}},\widetilde{\mathcal{P}_C})$ (resp., $\mathcal{G}(\widetilde{\mathcal{I}_C},\widetilde{\mathcal{I}},\widetilde{\mathcal{I}_C},\widetilde{\mathcal{I}})$) is precisely the class of G_C-projective (resp., G_C-injective) complexes [38]. Moreover, $\mathcal{P}_C$ (resp., $\mathcal{I}_C$) is self-orthogonal, closed under extensions, isomorphisms and finite direct sums, and $\mathcal{P},\mathcal{P}_C\subseteq\mathcal{G}(\mathcal{P},\mathcal{P}_C,\mathcal{P},\mathcal{P}_C)$ (resp., $\mathcal{I},\mathcal{I}_C\subseteq\mathcal{G}(\mathcal{I}_C,\mathcal{I},\mathcal{I}_C,\mathcal{I})$), see [14, Remark 2.3(4)], [19, Proposition 5.2], [32, Proposition 2.6] and [38, Lemma 3.1]. Then the result follows immediately from Corollary 4.2. □

By virtue of [19, Lemma 6.1, Theorems 2 and 6.1], $\mathcal{A}_C=\mathcal{G}(\mathcal{P},\mathcal{I}_C,\mathcal{P},\mathcal{I}_C)$ and $\mathcal{B}_C=\mathcal{G}(\mathcal{P}_C,\mathcal{I},\mathcal{P}_C,\mathcal{I})$, so application of Corollary 4.2, along with [14, Remark 2.3(4)] and [19, Proposition 5.2, Lemma 4.1, Corollary 6.1] yield the following result.

Corollary 4.4. *Let R be a commutative ring, C a semidualizing R-module and $\boldsymbol{M}\in\mathrm{Ch}(R)$. Then $\boldsymbol{M}\in\mathcal{G}(\widetilde{\mathcal{P}},\widetilde{\mathcal{I}_C},\widetilde{\mathcal{P}},\widetilde{\mathcal{I}_C})$ (resp., $\mathcal{G}(\widetilde{\mathcal{P}_C},\widetilde{\mathcal{I}},\widetilde{\mathcal{P}_C},\widetilde{\mathcal{I}})$) if and only if $M_n\in\mathcal{A}_C$ (resp., $\mathcal{B}_C$) for any $n\in\mathbb{Z}$.*

According to [10, Definition 3.6 and Proposition 3.9], $\mathcal{G}(\mathcal{WI}_C^{\mathrm{wp}},\mathcal{I},\mathcal{WI}_C^{\mathrm{wp}},\mathcal{I})$ is exactly the class of G_C-weak injective modules. Correspondingly, we call complexes in $\mathcal{G}(\widetilde{\mathcal{WI}_C^{\mathrm{wp}}},\widetilde{\mathcal{I}},\widetilde{\mathcal{WI}_C^{\mathrm{wp}}},\widetilde{\mathcal{I}})$ G_C-weak injective complexes. By definition and [10, Proposition 3.2, Lemma 5.3], $\mathcal{WI}_C^{\mathrm{wp}}$ is self-orthogonal, closed under extensions, isomorphisms and finite direct sums. In addition, using [19, Lemma 1.2] and [12, Theorem 2.2], one can show in the same way as [38, Lemma 3.1] that injective modules and C-weak injective C-weak projective modules are G_C-weak injective. So, by Corollary 4.2, one has:

Corollary 4.5. *Let R be a commutative ring, C be a semidualizing R-module and $\boldsymbol{M} \in \mathrm{Ch}(R)$. Then $\boldsymbol{M}$ is a G_C-weak injective complex if and only if each M_n is a G_C-weak injective module.*

In view of [21, Definition 3.3, Theorem 3.7], $\mathcal{G}(\mathcal{FI}_C^{\mathrm{fp}},\mathcal{I},\mathcal{FI}_C^{\mathrm{fp}},\mathcal{I})$ is precisely the class of G_C-FP-injective modules. Correspondingly, complexes in $\mathcal{G}(\widetilde{\mathcal{FI}_C^{\mathrm{fp}}},\widetilde{\mathcal{I}},\widetilde{\mathcal{FI}_C^{\mathrm{fp}}},\widetilde{\mathcal{I}})$ are called G_C-FP-injective complexes. Note that, if R is a left coherent ring, then a weak injective (resp., weak projective) left R-module is the same as an FP-injective (resp., FP-projective) left R-module, and so a G_C-weak injective module (resp., complex) is exactly a G_C-FP-injective module (resp., complex). Thus, by Corollary 4.5, one has:

Corollary 4.6. *Let R be a commutative coherent ring, C be a semidualizing R-module and $\boldsymbol{M} \in \mathrm{Ch}(R)$. Then $\boldsymbol{M}$ is a G_C-FP-injective complex if and only if each M_n is a G_C-FP-injective module.*

4.3. $\boldsymbol{D_C}$-projective (resp., $\boldsymbol{D_C}$-injective) complexes

Let R be a commutative ring. Set $\mathcal{V}=\mathcal{Y}=\mathcal{P}$, $\mathcal{W}=\mathcal{P}_C$ and $\mathcal{X}=\mathcal{F}_C$, then $\mathcal{G}(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$ is just the class of all D_C-projective modules, see [43]. In this case, we call complexes in $\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$ D_C-projective complexes. By Theorem 3.7 and Lemma 3.5, along with [14, Remark 2.3(4)], [19, Proposition 5.2] and [43, Proposition 1.8], one can get:

Corollary 4.7. *Let R be a commutative ring, C be a semidualizing R-module and $\boldsymbol{M} \in \mathrm{Ch}(R)$. Then $\boldsymbol{M}$ is D_C-projective if and only if each M_n is a D_C-projective module and* $\mathrm{Hom}_R(\boldsymbol{M},\boldsymbol{F})$ *is exact for any $\boldsymbol{F} \in \widetilde{\mathcal{F}_C}$.*

If we take $\mathcal{W}=\mathcal{X}=\mathcal{I}$, $\mathcal{V}=\mathcal{I}_C$ and $\mathcal{Y}=\mathcal{FI}_C$, then $\mathcal{G}(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$ is exactly the class of all D_C-injective modules [43]. In this situation, we call complexes in $\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$ D_C-injective complexes. Using Theorem 3.7 and Lemma 3.5, together with [14, Remark 2.3(4)], [42, Proposition 2.5] and the dual of [43, Proposition 1.8], one can obtain:

Corollary 4.8. *Let R be a commutative ring, C be a semidualizing R-module and $\boldsymbol{M} \in \mathrm{Ch}(R)$. Then $\boldsymbol{M}$ is D_C-injective if and only if each M_n is a D_C-injective module and* $\mathrm{Hom}_R(\boldsymbol{E},\boldsymbol{M})$ *is exact for any $\boldsymbol{E} \in \widetilde{\mathcal{FI}_C}$.*

4.4. $\boldsymbol{GAC_C}$-projective (resp., $\boldsymbol{GAC_C}$-injective) complexes

Let R be a commutative ring. Set $\mathcal{V}=\mathcal{Y}=\mathcal{P}$, $\mathcal{W}=\mathcal{P}_C$ and $\mathcal{X}=\mathcal{WF}_C$, then modules in $\mathcal{G}(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$ are called GAC_C-projective modules, and complexes in $\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$ are called GAC_C-projective complexes. A proof similar to that of [32, Proposition 2.6] shows that projective modules and C-projective modules are GAC_C-projective. So, by Theorem 3.7 and Lemma 3.5, along with [14, Remark 2.3(4)], [12, Proposition 2.5], one can get:

Corollary 4.9. *Let R be a commutative ring, C be a semidualizing R-module and $\boldsymbol{M} \in \mathrm{Ch}(R)$. Then $\boldsymbol{M}$ is GAC_C-projective if and only if each M_n is a GAC_C-projective module and $\mathrm{Hom}_R(\boldsymbol{M}, \boldsymbol{F})$ is exact for any $\boldsymbol{F} \in \widetilde{\mathcal{WF}_C}$.*

If we take $\mathcal{W}=\mathcal{X}=\mathcal{I}$, $\mathcal{V}=\mathcal{I}_C$ and $\mathcal{Y}=\mathcal{WI}_C$, then modules in $\mathcal{G}(\mathcal{V},\mathcal{W},\mathcal{Y},\mathcal{X})$ are called GAC_C-injective R-modules, and complexes in $\mathcal{G}(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}},\widetilde{\mathcal{Y}},\widetilde{\mathcal{X}})$ are called GAC_C-injective complexes. Using the proof of [38, Lemma 3.1] shows that injective modules and C-injective modules are GAC_C-injective. Thus, using Theorem 3.7 and Lemma 3.5, together with [14, Remark 2.3(4)], [12, Proposition 2.5], one can obtain:

Corollary 4.10. *Let R be a commutative ring, C be a semidualizing R-module and $\boldsymbol{M} \in \mathrm{Ch}(R)$. Then $\boldsymbol{M}$ is GAC_C-injective if and only if each M_n is a GAC_C-injective module and $\mathrm{Hom}_R(\boldsymbol{E}, \boldsymbol{M})$ is exact for any $\boldsymbol{E} \in \widetilde{\mathcal{WI}_C}$.*

4.5. Gorenstein complexes with respect to cotorsion pairs

Let $\mathcal{D}$ be an abelian category. Recall that a pair of classes $(\mathcal{A},\mathcal{B})$ of objects in $\mathcal{D}$ is said to be a cotorsion pair if $\mathcal{A} = \{A \in \mathcal{D} \mid \mathrm{Ext}^1_{\mathcal{D}}(A,B) = 0 \text{ for all } B \in \mathcal{B}\}$ and $\mathcal{B} = \{B \in \mathcal{D} \mid \mathrm{Ext}^1_{\mathcal{D}}(A,B) = 0 \text{ for all } A \in \mathcal{A}\}$. A cotorsion pair $(\mathcal{A},\mathcal{B})$ is said to be hereditary if $\mathrm{Ext}^i_{\mathcal{D}}(A,B) = 0$ for all $i \geq 1$, all $A \in \mathcal{A}$, and all $B \in \mathcal{B}$. The cotorsion pair is called complete if for any object $M \in \mathcal{D}$, there are exact sequences $0 \to B \to A \to M \to 0$ and respectively $0 \to M \to B' \to A' \to 0$ with $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$. For more details on cotorsion pairs, the reader can consult [8, 16].

Given a cotorsion pair $(\mathcal{A},\mathcal{B})$ in R-Mod. Following [15, Definition 3.3], there are two classes of complexes in $\mathrm{Ch}(R)$:

$$\mathrm{dg}\widetilde{\mathcal{A}} = \{\boldsymbol{A} \in \mathrm{Ch}(R) \mid A_n \in \mathcal{A} \text{ and } \mathrm{Hom}_R(\boldsymbol{A},\boldsymbol{B}) \text{ is exact whenever } \boldsymbol{B} \in \widetilde{\mathcal{B}}\},$$

$$\mathrm{dg}\widetilde{\mathcal{B}} = \{\boldsymbol{B} \in \mathrm{Ch}(R) \mid B_n \in \mathcal{B} \text{ and } \mathrm{Hom}_R(\boldsymbol{A},\boldsymbol{B}) \text{ is exact whenever } \boldsymbol{A} \in \widetilde{\mathcal{A}}\}.$$

By virtue of [15, Proposition 3.6, Corollary 3.13] and [39, Theorem 3.5] or [37, Theorem 2.4], if $(\mathcal{A},\mathcal{B})$ is a complete hereditary cotorsion pair in R-Mod, then $(\widetilde{\mathcal{A}},\mathrm{dg}\widetilde{\mathcal{B}})$ and $(\mathrm{dg}\widetilde{\mathcal{A}},\widetilde{\mathcal{B}})$ are complete hereditary cotorsion pairs in $\mathrm{Ch}(R)$.

Let $(\mathcal{A},\mathcal{B})$ be a complete hereditary cotorsion pair in R-Mod. Then $\mathcal{G}(\mathcal{A},\mathcal{A},\mathcal{P},\mathcal{A}\cap\mathcal{B})$ is the class of all Gorenstein $\mathcal{A}$-modules with respect to the cotorsion pair $(\mathcal{A},\mathcal{B})$, see [36], and dually, $\mathcal{G}(\mathcal{B},\mathcal{B},\mathcal{A}\cap\mathcal{B},\mathcal{I})$ is the class of all Gorenstein $\mathcal{B}$-modules with respect to the cotorsion pair $(\mathcal{A},\mathcal{B})$. Correspondingly, complexes in $\mathcal{G}(\widetilde{\mathcal{A}},\widetilde{\mathcal{A}},\widetilde{\mathcal{P}},\widetilde{\mathcal{A}}\cap\mathrm{dg}\widetilde{\mathcal{B}})$ is called Gorenstein $\widetilde{\mathcal{A}}$-complexes with respect to the cotorsion pair $(\mathcal{A},\mathcal{B})$, see [47], and dually, complexes in $\mathcal{G}(\widetilde{\mathcal{B}},\widetilde{\mathcal{B}},\mathrm{dg}\widetilde{\mathcal{A}}\cap\widetilde{\mathcal{B}},\widetilde{\mathcal{I}})$ is called Gorenstein $\widetilde{\mathcal{B}}$-complexes with respect to the cotorsion pair $(\mathcal{A},\mathcal{B})$. Note that $\mathcal{G}(\mathcal{A},\mathcal{A},\mathcal{P},\mathcal{A}\cap\mathcal{B}) = \mathcal{G}(\mathcal{P},\mathcal{A}\cap\mathcal{B},\mathcal{P},\mathcal{A}\cap\mathcal{B})$, $\mathcal{G}(\mathcal{B},\mathcal{B},\mathcal{A}\cap\mathcal{B},\mathcal{I}) =$

$\mathcal{G}(\mathcal{A}\cap\mathcal{B},\mathcal{I},\mathcal{A}\cap\mathcal{B},\mathcal{I})$, $\mathcal{G}(\widetilde{\mathcal{A}},\widetilde{\mathcal{A}},\widetilde{\mathcal{P}},\widetilde{\mathcal{A}}\cap \mathrm{dg}\widetilde{\mathcal{B}}) = \mathcal{G}(\widetilde{\mathcal{P}(R)},\widetilde{\mathcal{A}}\cap \mathrm{dg}\widetilde{\mathcal{B}},\widetilde{\mathcal{P}(R)},\widetilde{\mathcal{A}}\cap \mathrm{dg}\widetilde{\mathcal{B}})$, $\mathcal{G}(\widetilde{\mathcal{B}},\widetilde{\mathcal{B}},\mathrm{dg}\widetilde{\mathcal{A}}\cap\widetilde{\mathcal{B}},\widetilde{\mathcal{I}}) = \mathcal{G}(\mathrm{dg}\widetilde{\mathcal{A}}\cap\widetilde{\mathcal{B}},\widetilde{\mathcal{I}},\mathrm{dg}\widetilde{\mathcal{A}}\cap\widetilde{\mathcal{B}},\widetilde{\mathcal{I}}))$, see e.g. [36, Lemma 3.2] and [46, Proposition 3.5], and that $\widetilde{\mathcal{A}}\cap\mathrm{dg}\widetilde{\mathcal{B}}=\widetilde{\mathcal{A}\cap\mathcal{B}} = \mathrm{dg}\widetilde{\mathcal{A}}\cap\widetilde{\mathcal{B}}$, see e.g. [15, Theorem 3.12]. Via Corollary 4.2, one can obtain the next result, which is [47, Theorem 3.5] and its dual.

Corollary 4.11. *Let $(\mathcal{A},\mathcal{B})$ be a complete hereditary cotorsion pair in R-Mod and $\boldsymbol{M}\in \mathrm{Ch}(R)$. Then $\boldsymbol{M}$ is a Gorenstein $\widetilde{\mathcal{A}}$-complex (resp., Gorenstein $\widetilde{\mathcal{B}}$-complex) with respect to the cotorsion pair $(\mathcal{A},\mathcal{B})$ if and only if each M_n is a Gorenstein $\mathcal{A}$-module (resp., Gorenstein $\mathcal{B}$-module) with respect to the cotorsion pair $(\mathcal{A},\mathcal{B})$.*

Recall from [8] that a module M is called cotorsion if $\mathrm{Ext}^1_R(F,M)=0$ for any flat module F. Denote by $\mathcal{C}$ the collection of all cotorsion modules. It is well-known that $(\mathcal{F},\mathcal{C})$ is a complete hereditary cotorsion pair. When $(\mathcal{A},\mathcal{B}) = (\mathcal{F},\mathcal{C})$, Gorenstein $\mathcal{F}$-modules with respect to the cotorsion pair $(\mathcal{F},\mathcal{C})$ are exactly **F**-Gorenstein flat modules, and Gorenstein $\widetilde{\mathcal{F}}$-complexes with respect to the cotorsion pair $(\mathcal{F},\mathcal{C})$ are just **F**-Gorenstein flat complexes, see [20]. So, by Corollary 4.11, one has:

Corollary 4.12 ([20, Theorem 4.7])**.** *Let $\boldsymbol{M}\in\mathrm{Ch}(R)$. Then $\boldsymbol{M}$ is* **F***-Gorenstein flat if and only if each M_n is* **F***-Gorenstein flat.*

Recall that a module is Gorenstein flat [9, 17] if it is a syzygy in an exact sequence consisting of flat modules which remains exact after tensoring by arbitrary injective right R-module, and recall that a complex is Gorenstein flat [13] if it is a syzygy in an exact sequence of flat complexes which remains exact after tensoring by arbitrary injective complex of right R-modules. Here the tensor product of complexes is the modified tensor product of complexes in the sense of [13, p. 87]. Let R be a right coherent ring. Then **F**-Gorenstein flat left R-modules and the Gorenstein flat left R-modules coincide by [20, Lemma 3.2], an **F**-Gorenstein flat complex of left R-modules is the same as a Gorenstein flat complex of left R-modules by [47, Proposition 3.3]. So one has:

Corollary 4.13 ([40, Theorem 3.1], [34, Theorem 5])**.** *Let $\boldsymbol{M}\in\mathrm{Ch}(R)$. If R is a right coherent ring, then $\boldsymbol{M}$ is Gorenstein flat if and only if each M_n is Gorenstein flat.*

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References

[1] M. Auslander and M. Bridger, *Stable module theory*, Memoirs of the American Mathematical Society, No. 94, Amer. Math. Soc., Providence, RI, 1969.

[2] D. Bennis and K. Ouarghi, *$\mathscr{X}$-Gorenstein projective modules*, Int. Math. Forum **5** (2010), no. 9-12, 487–491.

[3] D. Bravo and J. Gillespie, *Absolutely clean, level, and Gorenstein AC-injective complexes*, Comm. Algebra **44** (2016), no. 5, 2213–2233. https://doi.org/10.1080/00927872.2015.1044100

[4] D. Bravo, J. Gillespie, and M. Hovey, *The stable module category of a general ring*, arXiv:1405.5768, 2014.

[5] N. Ding, Y. Li, and L. Mao, *Strongly Gorenstein flat modules*, J. Aust. Math. Soc. **86** (2009), no. 3, 323–338. https://doi.org/10.1017/S1446788708000761

[6] E. E. Enochs and J. R. García Rozas, *Gorenstein injective and projective complexes*, Comm. Algebra **26** (1998), no. 5, 1657–1674. https://doi.org/10.1080/00927879808826229

[7] E. E. Enochs and O. M. G. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), no. 4, 611–633. https://doi.org/10.1007/BF02572634

[8] E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, De Gruyter Expositions in Mathematics, **30**, Walter de Gruyter & Co., Berlin, 2000.

[9] E. E. Enochs, O. M. G. Jenda, and B. Torrecillas, *Gorenstein flat modules*, Nanjing Daxue Xuebao Shuxue Bannian Kan **10** (1993), no. 1, 1–9.

[10] Z. Gao, X. Ma, and T. Zhao, *Gorenstein weak injective modules with respect to a semidualizing bimodule*, J. Korean Math. Soc. **55** (2018), no. 6, 1389–1421. https://doi.org/10.4134/JKMS.j170718

[11] Z. Gao and F. Wang, *Weak injective and weak flat modules*, Comm. Algebra **43** (2015), no. 9, 3857–3868. https://doi.org/10.1080/00927872.2014.924128

[12] Z. Gao and T. Zhao, *Foxby equivalence relative to C-weak injective and C-weak flat modules*, J. Korean Math. Soc. **54** (2017), no. 5, 1457–1482. https://doi.org/10.4134/JKMS.j160528

[13] J. R. García Rozas, *Covers and envelopes in the category of complexes of modules*, Chapman & Hall/CRC Research Notes in Mathematics, 407, Chapman & Hall/CRC, Boca Raton, FL, 1999.

[14] Y. Geng and N. Ding, *$\mathscr{W}$-Gorenstein modules*, J. Algebra **325** (2011), 132–146. https://doi.org/10.1016/j.jalgebra.2010.09.040

[15] J. Gillespie, *The flat model structure on* Ch(R), Trans. Amer. Math. Soc. **356** (2004), no. 8, 3369–3390. https://doi.org/10.1090/S0002-9947-04-03416-6

[16] R. Göbel and J. Trlifaj, *Approximations and endomorphism algebras of modules*, De Gruyter Expositions in Mathematics, 41, Walter de Gruyter, Berlin, 2006. https://doi.org/10.1515/9783110199727

[17] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 167–193. https://doi.org/10.1016/j.jpaa.2003.11.007

[18] H. Holm and P. Jørgensen, *Semi-dualizing modules and related Gorenstein homological dimensions*, J. Pure Appl. Algebra **205** (2006), no. 2, 423–445. https://doi.org/10.1016/j.jpaa.2005.07.010

[19] H. Holm and D. White, *Foxby equivalence over associative rings*, J. Math. Kyoto Univ. **47** (2007), no. 4, 781–808. https://doi.org/10.1215/kjm/1250692289

[20] J. Hu and A. Xu, *On stability of F-Gorenstein flat categories*, Algebra Colloq. **23** (2016), no. 2, 251–262. https://doi.org/10.1142/S1005386716000286

[21] J. Hu and D. Zhang, *Weak AB-context for FP-injective modules with respect to semidualizing modules*, J. Algebra Appl. **12** (2013), no. 7, 1350039, 17 pp. https://doi.org/10.1142/S0219498813500394

[22] Z. Huang, *Proper resolutions and Gorenstein categories*, J. Algebra **393** (2013), 142–169. https://doi.org/10.1016/j.jalgebra.2013.07.008

[23] F. Kong and P. Zhang, *From CM-finite to CM-free*, J. Pure Appl. Algebra **220** (2016), no. 2, 782–801. https://doi.org/10.1016/j.jpaa.2015.07.016

[24] L. Liang, N. Ding, and G. Yang, *Some remarks on projective generators and injective cogenerators*, Acta Math. Sin. (Engl. Ser.) **30** (2014), no. 12, 2063–2078. https://doi.org/10.1007/s10114-014-3227-z

[25] Z. Liu, Z. Huang, and A. Xu, *Gorenstein projective dimension relative to a semidualizing bimodule*, Comm. Algebra **41** (2013), no. 1, 1–18. https://doi.org/10.1080/00927872.2011.602782

[26] B. H. Maddox, *Absolutely pure modules*, Proc. Amer. Math. Soc. **18** (1967), 155–158. https://doi.org/10.2307/2035245

[27] L. Mao and N. Ding, *Relative FP-projective modules*, Comm. Algebra **33** (2005), no. 5, 1587–1602. https://doi.org/10.1081/AGB-200061047

[28] L. Mao and N. Ding, *Gorenstein FP-injective and Gorenstein flat modules*, J. Algebra Appl. **7** (2008), no. 4, 491–506. https://doi.org/10.1142/S0219498808002953

[29] F. Meng and Q. X. Pan, *$\mathscr{X}$-Gorenstein projective and $\mathscr{Y}$-Gorenstein injective modules*, Hacet. J. Math. Stat. **40** (2011), no. 4, 537–554.

[30] S. Sather-Wagstaff, T. Sharif, and D. White, *Stability of Gorenstein categories*, J. Lond. Math. Soc. (2) **77** (2008), no. 2, 481–502. https://doi.org/10.1112/jlms/jdm124

[31] B. Stenström, *Coherent rings and FP-injective modules*, J. London Math. Soc. (2) **2** (1970), 323–329. https://doi.org/10.1112/jlms/s2-2.2.323

[32] D. White, *Gorenstein projective dimension with respect to a semidualizing module*, J. Commut. Algebra **2** (2010), no. 1, 111–137. https://doi.org/10.1216/JCA-2010-2-1-111

[33] D. Xin, J. Chen, and X. Zhang, *Completely $\mathscr{W}$-resolved complexes*, Comm. Algebra **41** (2013), no. 3, 1094–1106. https://doi.org/10.1080/00927872.2011.630707

[34] G. Yang, *Gorenstein projective, injective and flat complexes*, Acta Math. Sinica (Chinese Ser.) **54** (2011), no. 3, 451–460.

[35] X. Yang, *Gorenstein categories $\mathcal{G}(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ and dimensions*, Rocky Mountain J. Math. **45** (2015), no. 6, 2043–2064. https://doi.org/10.1216/RMJ-2015-45-6-2043

[36] X. Yang and W. Chen, *Relative homological dimensions and Tate cohomology of complexes with respect to cotorsion pairs*, Comm. Algebra **45** (2017), no. 7, 2875–2888. https://doi.org/10.1080/00927872.2016.1233226

[37] X. Yang and N. Ding, *On a question of Gillespie*, Forum Math. **27** (2015), no. 6, 3205–3231. https://doi.org/10.1515/forum-2013-6014

[38] C. Yang and L. Liang, *Gorenstein injective and projective complexes with respect to a semidualizing module*, Comm. Algebra **40** (2012), no. 9, 3352–3364. https://doi.org/10.1080/00927872.2011.568030

[39] G. Yang and Z. Liu, *Cotorsion pairs and model structures on* $\mathrm{Ch}(R)$, Proc. Edinb. Math. Soc. (2) **54** (2011), no. 3, 783–797. https://doi.org/10.1017/S0013091510000489

[40] X. Yang and Z. Liu, *Gorenstein projective, injective, and flat complexes*, Comm. Algebra **39** (2011), no. 5, 1705–1721. https://doi.org/10.1080/00927871003741497

[41] G. Yang, Z. Liu, and L. Liang, *Model structures on categories of complexes over Ding-Chen rings*, Comm. Algebra **41** (2013), no. 1, 50–69. https://doi.org/10.1080/00927872.2011.622326

[42] D. Zhang and B. Ouyang, *Semidualizing modules and related modules*, J. Algebra Appl. **10** (2011), no. 6, 1261–1282. https://doi.org/10.1142/S0219498811005695

[43] C. Zhang, L. Wang, and Z. Liu, *Ding projective modules with respect to a semidualizing module*, Bull. Korean Math. Soc. **51** (2014), no. 2, 339–356. https://doi.org/10.4134/BKMS.2014.51.2.339

[44] C. Zhang, L. Wang, and Z. Liu, *Ding projective modules with respect to a semidualizing bimodule*, Rocky Mountain J. Math. **45** (2015), no. 4, 1389–1411. https://doi.org/10.1216/RMJ-2015-45-4-1389

[45] R. Zhao and N. Ding, *$(\mathscr{W}, \mathscr{Y}, \mathscr{X})$-Gorenstein complexes*, Comm. Algebra **45** (2017), no. 7, 3075–3090. https://doi.org/10.1080/00927872.2016.1235173

[46] R. Zhao and Y. Li, *A unified approach to various Gorenstein modules*, Rocky Mountain J. Math. **52** (2022), no. 6, 2229–2246. https://doi.org/10.1216/rmj.2022.52.2229

[47] R. Zhao and P. Ma, *Gorenstein projective complexes with respect to cotorsion pairs*, Czechoslovak Math. J. **69(144)** (2019), no. 1, 117–129. https://doi.org/10.21136/CMJ.2018.0194-17

[48] R. Zhao and W. Ren, *$\mathcal{VW}$-Gorenstein complexes*, Turkish J. Math. **41** (2017), no. 3, 537–547. https://doi.org/10.3906/mat-1603-88

[49] G. Zhao and J. Sun, *$\mathcal{VW}$-Gorenstein categories*, Turkish J. Math. **40** (2016), no. 2, 365–375. https://doi.org/10.3906/mat-1502-37

Yanjie Li
College of Mathematics and Statistics
Northwest Normal University
Lanzhou 730070, P. R. China
Email address: 18809403126@163.com

Renyu Zhao
College of Mathematics and Statistics
Northwest Normal University
Gansu Provincial Research Center for Basic Disciplines of Mathematics and Statistics
Lanzhou 730070, P. R. China
Email address: zhaory@nwnu.edu.cn