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SINGULAR HYPERBOLICITY OF C^1 GENERIC THREE DIMENSIONAL VECTOR FIELDS

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Abstract. In the paper, we show that for a generic C^1 vector field X on a closed three dimensional manifold M, any isolated transitive set of X is singular hyperbolic. It is a partial answer of the conjecture in [13].

1. Introduction

The transitivity is a symbol of chaotic property for differential dynamical systems. The C^1 robust transitivity for diffeomorphisms are well investigate in a series of works [2, 3, 5], and then we have a good characterization on isolated transitive sets of C^1 generic diffeomorphisms at the same time. From the main result of [1] we know that if every isolated transitive set of a C^1 generic diffeomorphism admit a nontrivial dominated splitting, then it is volume hyperbolic.

It is well known that a singularity-free flow, for an instance, a suspension of a diffeomorphism, will take similar phenomenona of diffeomorphisms. However, once the recurrent regular points can accumulates a singularity, such as the Lorenz-like systems, we will meet something new. For instance, in [14], one have to use a new notion of singular hyperbolicity to characterize the robustly transitive sets of a 3-dimensional flow. Here the singular hyperbolicity is a generalization of hyperbolicity so that we can give the Lorenz attractor and Smale's horseshoe a unified characterization. In this article, we will show that an isolated transitive set of C^1 generic vector field on 3-dimensional manifold will be singular hyperbolic. That means, every isolated transitive set of a C^1 generic vector field looks like a Lorenz attractor [6, 10].

Let us be precise now. Denote by M a compact $d(\geq 2)$-dimensional smooth Riemannian manifold without boundary and by $\mathfrak{X}^1(M)$ the set of C^1 vector fields on M endowed with the C^1 topology. Every $X \in \mathfrak{X}^1(M)$ generates a flow $X^t : M \times \mathbb{R} \to M$ that is a C^1 map such that $X^t : M \to M$ is a diffeomorphism for all $t \in \mathbb{R}$ and then $X^0(x) = x$ and $X^{t+s}(x) = X^t(X^s(x))$ for all $s, t \in \mathbb{R}$ and $x \in M$. An *orbit* of X corresponding a point $x \in M$ is the

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set $Orb(x) = \{X^t(x) : t \in \mathbb{R}\}$. A point $x \in M$ is called *singular* if $X^t(\sigma) = \sigma$ for all $t \in \mathbb{R}$, and $p \in M$ is called *periodic* if $X^T = p$ for some $T > 0$. Let $Sing(X)$ denotes the set of singularities of X and $Per(X)$ is the set of periodic orbits of X. Denote by $Crit(X) = Sing(X) \cup Per(X)$ the set of all critical points of X.

Let $\Lambda \subset M$ be a closed X^t-invariant set. We say that Λ is *a hyperbolic set* of X if there are constants $C > 0$, $\lambda > 0$ and a DX^t-invariant continuous splitting $T_\Lambda M = E^s \oplus \langle X \rangle \oplus E^u$ such that

$$\|DX^t|_{E^s_x}\| \leq Ce^{-\lambda t} \text{ and } \|DX^{-t}|_{E^u_x}\| \leq Ce^{-\lambda t}$$

for $t > 0$ and $x \in \Lambda$, where $\langle X(x) \rangle$ denotes the space spanned by $X(x)$, which is 0-dimensional if x is a singularity or 1-dimensional if x is not a singularity. For any critical point $x \in Crit(X)$, if its orbit is a hyperbolic set, we denote by $\text{index}(x) = \dim E^s_x$.

Now let us recall the singular hyperbolicity firstly given by Morale, Pacífico and Pujals [14] which is an extension of hyperbolicity. We say that a compact invariant set Λ is *positively singular hyperbolic* for X (see [16]) if there are constants $K \geq 1$ and $\lambda > 0$, and a continuous invariant $T_\Lambda M = E^s \oplus E^{cu}$ with respect to DX^t such that

(i) E^s is (K, λ)-dominated by E^{cu}, that is,

$$\|DX^t|_{E^s(x)}\| \cdot \|DX^{-t}|_{E^c(X^t(x))}\| \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda \text{ and } t \geq 0.$$

(ii) E^s is contracting, that is,

$$\|DX^t|_{E^s(x)}\| \leq Ke^{-\lambda t}, \quad \forall x \in \Lambda \text{ and } t \geq 0.$$

(iii) E^{cu} is sectional expanding, that is, for any $x \in \Lambda$ and any 2-dimensional subspace $L \subset E^c(x)$,

$$|\det(DX^t|_L)| \geq K^{-1}e^{\lambda t}, \quad \forall t \geq 0.$$

We say that Λ is *negatively singular hyperbolic* for X if Λ is positively singular hyperbolic for $-X$, and then say that Λ is *singular hyperbolic* for X if it is either positively singular hyperbolic for X, or negatively singular hyperbolic for X. Definitely, we can see that if Λ is singular hyperbolic for X and it does not contain singularities, then it is hyperbolic (see [14, Proposition 1.8] for a proof). In the paper, we consider the relation between transitivity and hyperbolicity for an isolated compact invariant set. We say that Λ is *transitive* if there is $x \in \Lambda$ such that $\omega(x) = \Lambda$, where $\omega(x)$ is the omega limit set of x. We say that a closed X^t-invariant set Λ is *isolated* (or *locally maximal*) if there exists a neighborhood U of Λ such that

$$\Lambda = \Lambda_X(U) = \bigcap_{t \in \mathbb{R}} X^t(U).$$

Here U is said to be *isolated neighborhood* of Λ.

For the 3-dimensional case, Morales, Pacífico and Pujals [14] proved that if Λ is a robustly transitive set containing singularities, then it is a singular hyperbolic set for X. Here we will consider C^1 generic vector fields. We say that a subset $\mathcal{G} \subset \mathfrak{X}^1(M)$ is *residual* if it contains a countable intersection of open and dense subsets of $\mathfrak{X}^1(M)$. A property is called C^1 *generic* if it holds in a residual subset of $\mathfrak{X}^1(M)$. We give the following characterization of the isolated transitive sets of a C^1 generic vector field on 3-dimensional Riemannian manifold.

Theorem A. *For C^1 generic $X \in \mathfrak{X}^1(M)$, an isolated transitive set Λ is singular hyperbolic.*

2. Transitivity and locally star condition

Let M be a three dimensional smooth Riemannian manifold and let $X \in \mathfrak{X}^1(M)$ be the set of C^1 vector fields on M endowed with the C^1 topology. Here we collect some known generic properties for C^1 vector fields.

Proposition 2.1. *There is a residual set $\mathcal{G}_1 \subset \mathfrak{X}^1(M)$ such that for any $X \in \mathcal{G}_1$, X satisfies the following properties*:

1. *X is a Kupka-Samle system, that is, every periodic orbits and singularity of X is hyperbolic, and the corresponding invariant manifolds intersect transversely.*
2. *if there is a sequence of vector fields $\{X_n\}$ with critical orbit $\{P_n\}$ of X_n such that $X_n \to X$, $\mathrm{index}(P_n) = i$ and $P_n \to_H \Lambda$, then there is a sequence of critical orbit $\{Q_n\}$ of X such that $\mathrm{index}(Q_n) = i$ and $Q_n \to_H \Lambda$, where $\to_H$ is the Hausdorff limit.*

The item 1 is from the famous Kupka-Smale theorem (see [15]) and item 2 is a vector field version of [18, Lemma 3.5]

From item 1 of Proposition 2.1, we can see that if Λ is a trivial transitive set, that is, Λ is a periodic orbit or a singularity, then it should be hyperbolic and automatically singular hyperbolic. To prove Theorem A, we just need to consider the nontrivial case. Hereafter, we assume that Λ is a nontrivial transitive set of X. One can see that if Λ is a nontrivial transitive set, then Λ contains no hyperbolic sinks or sources.

Let U be an isolated neighborhood of Λ. Then for Y C^1 close to X, denote by

$$\Lambda_Y(U) = \bigcap_{t \in \mathbb{R}} Y^t(U)$$

the maximal invariant set of Y in U.

Lemma 2.2. *Let $\mathcal{G}_1 \subset \mathfrak{X}^1(M)$ be the residual set given in Proposition 2.1. For any $X \in \mathcal{G}_1$, if Λ is an isolated nontrivial transitive set of X, then there are a C^1 neighborhood $\mathcal{U}(X)$ of X and a neighborhood U of Λ such that for any $Y \in \mathcal{U}(X)$, we have every $\gamma \in \Lambda_Y(U) \cap Per(Y)$ is hyperbolic and* $\mathrm{index}(\gamma) = 1$.

Proof. Let $\mathcal{G}_1$ be the residual set in Proposition 2.1 and let Λ be an isolated transitive set of $X \in \mathcal{G}_1$. Arguing by contradiction, we assume that for any C^1

neighborhood $\mathcal{U}(X)$ of X and any neighborhood U of Λ, there is $Y \in \mathcal{U}(X)$ such that Y has a periodic orbit Q whose index is not 1. Then we have three cases: (i) Q is not hyperbolic, (ii) Q is hyperbolic but index$(Q) = 0$ or (iii) index$(Q) = 2$. Note that if the periodic orbit Q is not hyperbolic for Y, then we can take a vector field Z C^1 arbitrary close to Y such that either Q is a sink for Z or Q is a source for Z. Then we also have the case cases (ii) or (iii) happening. Thus we can take sequences $Y_n \to X$ and a periodic orbit P_n of Y_n such that index$(P_n) = 0$ or 2 and

$$\lim_{n\to\infty} P_n = \Gamma \subset \Lambda.$$

Then we can take a sequence of vector fields X_n tends to X and periodic orbits $\{Q_n\}$ of X_n with index$(Q_n) = 0$ or 2 such that

$$\lim_{n\to\infty} Q_n = \Gamma \subset \Lambda.$$

Without loss of generality, we can assume that all Q_n have the same index 0 or 2 once we take a subsequence. By the item 2 of Proposition 2.1, we know that there is a sequence P_n of periodic orbit of X with index 0 or 2 converging into Λ. Since Λ is isolated, for sufficiently large n, we have $P_n \subset \Lambda$. This is a contradiction since Λ is a nontrivial transitive set. $\square$

Let Λ be a closed X^t-invariant set. We say Λ is *locally star* if there are a C^1 neighborhood $\mathcal{U}(X)$ of $X \in \mathfrak{X}^1(M)$ and a neighborhood U of Λ such that for any $Y \in \mathcal{U}(X)$, every periodic orbit of Y in $\Lambda_Y(U) = \bigcap_{t\in\mathbb{R}} Y^t(U)$ is hyperbolic and has same indices.

Corollary 2.3. *There is a residual set $\mathcal{R} \subset \mathfrak{X}^1(M)$ such that for any $X \in \mathcal{R}$, if Λ is an isolated transitive set of X which is not an orbit, then Λ is a local star.*

Proof. Let $X \in \mathcal{R} = \mathcal{G}_1$ and let Λ be an isolated transitive set. By Lemma 2.2, there are a C^1 neighborhood $\mathcal{U}(X)$ of X and a neighborhood U of Λ such that for any $Y \in \mathcal{U}(X)$, every periodic orbit $\gamma \in \Lambda_Y(U) \cap Per(Y)$ is hyperbolic and index$(\gamma) = 1$. Thus Λ is a local star. $\square$

3. Transitivity and Lyapunov stability

Suppose $\sigma \in Sing(X)$ is hyperbolic. Then we denote by

$$W^s(\sigma) = W^s(\sigma, X) = \{y \in M : d(X^t(\sigma), X^t(y)) \to 0 \text{ as } t \to \infty\},$$

$$W^u(\sigma) = W^u(\sigma, X) = \{y \in M : d(X^t(\sigma), X^t(y)) \to 0 \text{ as } t \to -\infty\},$$

where $W^s(\sigma, X)$ is said to be the *stable manifold* of σ and $W^u(\sigma, X)$ is said to be the *unstable manifold* of σ. It is known that index$(\sigma) = \dim W^s(\sigma)$.

If X is a Kupka-Smale vector field, then X contains finitely many singularities and every singularity is hyperbolic. Thus by the structurally stability of hyperbolic singularity we know that there are a C^1 neighborhood

$\mathcal{U}(X)$ of X and a neighborhood U of Λ such that for any $Y \in \mathcal{U}(X)$, every $\sigma \in \Lambda_Y(U) \cap Sing(Y) \subset U$ is hyperbolic.

Lemma 3.1. *Let $\mathcal{G}_1 \subset \mathfrak{X}^1(M)$ be the residual set given in Proposition 2.1. For any $X \in \mathcal{G}_1$, if Λ is an isolated nontrivial transitive set of X, then there are a C^1 neighborhood $\mathcal{U}(X)$ of X and a neighborhood U of Λ such that for any $Y \in \mathcal{U}(X)$, every singularities in $\Lambda_Y(U)$ is saddles.*

Proof. We prove it by contradiction. Assume the contrary of the lemma. Then we can find a sequence of vector fields X_n tends to X and a sequence of singularity σ_n of X_n such that σ_n tends to a point σ such that the index of σ_n equals to 0 or 3. Without loss of generality, we assume that every σ_n has index 0, then we can see that σ is a singularity. Since $X \in \mathcal{G}_1$, we have σ is hyperbolic. By the structurally stability of σ we know σ have index 0 too. This contradicts with Λ is a nontrivial transitive set. □

Lemma 3.2. *Let Λ be a transitive set of a C^1 vector field X. If $\sigma \in \Lambda \cap Sing(X)$ is hyperbolic, then $(W^s(\sigma) \setminus \{\sigma\}) \cap \Lambda \neq \emptyset$ and $(W^u(\sigma) \setminus \{\sigma\}) \cap \Lambda \neq \emptyset$.*

Proof. We consider the case of $(W^s(\sigma) \setminus \{\sigma\}) \cap \Lambda \neq \emptyset$ (Other case is similar). Since $\sigma \in \Lambda = \omega(x)$ for some $x \in \Lambda$, there is $t_n \in \mathbb{R}^+$ with $t_n \to \infty$ such that $X^{t_n}(x) \to \sigma$. Since σ is hyperbolic, we can take $\epsilon > 0$ such that

$$\{x : X^t(x) \in B_\epsilon(\sigma) \text{ for all } t > 0\} \subset W^s(\sigma).$$

Denote by $x_n = X^{t_n}(x)$. For n large enough, $x_n \in B_\epsilon(\sigma)$. Let $\tau_n = \sup\{t : X^{(-t,0)}(x_n) \subset B_\epsilon(\sigma)\}$. Then we have $X^{-\tau_n}(x_n) \in B_\epsilon(\sigma)$. Let $y_n = X^{-\tau_n}(x_n)$. We can see that $\tau_n \to +\infty$ as $n \to \infty$. Take a subsequence if necessary, we can assume that $y_n \to y$ as $n \to \infty$. It is easy to see that $y \neq \sigma$. For every y_n, we have $X^{(0,\tau_n)}(y_n) \in \partial B_\epsilon(\sigma)$. By the continuity of the flow X^t, we have $X^{(0,+\infty)}(y) \subset B_\epsilon(\sigma)$, then $y \in W^s(\sigma) \setminus \{\sigma\}$. □

The following is the connecting lemma for C^1 vector fields.

Lemma 3.3 ([16])**.** *Let $X \in \mathfrak{X}^1(M)$ and $z \in M$ be neither periodic nor singular of X. For any C^1 neighborhood $\mathcal{U}(X) \subset \mathfrak{X}^1(M)$ of X, there exist three numbers $\rho > 1$, $L > 1$ and $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$ and any two points x, y outside the tube $\Delta = B_\delta(X^{[0,L]}(z))$ (or $\Delta = B_\delta(X^{[-L,0]}(z))$), if the positive X-orbit of x hits $B_{\delta/\rho}(z)$ and the negative X-orbit of y both hit $B_{\delta/\rho}(X^L(z))$, then there exists $Y \in \mathcal{U}(X)$ with $Y = X$ outside Δ such that y is on the positive Y-orbit of x.*

Lemma 3.4. *Let Λ be a transitive set for X and $\sigma \in \Lambda \cap Sing(X)$ be hyperbolic. Then for any C^1 neighborhood $\mathcal{U}(X)$ of X, any non-empty open set U in Λ, there is $Y \in \mathcal{U}(X)$ such that $W^s(\sigma, Y) \cap U \neq \emptyset$, where $W^s(\sigma, Y)$ is the stable manifold of σ with respect to Y.*

Proof. Let $\mathcal{U}(X)$ be fixed. By Lemma 3.2, there is a point $x \in (W^s(\sigma) \setminus \{\sigma\}) \cap \Lambda$. Then x is neither a singularity nor a periodic point. Let L, ρ and δ_0 be the constant given by Lemma 3.3. Take a point $X^T(x)$ with $T > L$ and $\delta > 0$ such that the tube

$$B_\delta(X^{[0,L]}(x)) \cap X^{[T,+\infty)}(x) = \emptyset.$$

Since Λ is transitive, there is $z \in \Lambda$ such that $\omega(z) = \Lambda$. For any small neighborhood U of y, we can find $0 < s < t$ such that $X^s(z) \in U$ and $X^t(z) \in B_{\delta/\rho}(x)$. Let $q = X^T(x)$ and $p = X^s(z)$. Then by Lemma 3.3, there is $Y \in \mathcal{U}(X)$ such that $Y^t(p) = q$ for some $t > 0$. Since $q = X^T(x) \in W^s(\sigma)$, we have $p \in W^s(\sigma, Y)$. □

From Lemma 3.1 we know that if $X \in \mathcal{G}_1$, and Λ is an isolated nontrivial transitive set of X, then every $\sigma \in \Lambda \cap Sing(X)$ has index 1 or 2.

Lemma 3.5. *There is a residual set $\mathcal{G}_2 \subset \mathfrak{X}^1(M)$ with the following property. For any $X \in \mathcal{G}_2$ and any isolated nontrivial transitive set Λ of X, if there is $\sigma \in \Lambda \cap Sing(X)$ with* index$(\sigma) = 2$*, then $\Lambda \subset \overline{W^u(\sigma)}$. Symmetrically, if there is $\sigma \in \Lambda \cap Sing(X)$ with* index$(\sigma) = 1$*, then $\Lambda \subset \overline{W^s(\sigma)}$.*

Proof. Let $\mathcal{O} = \{O_1, O_2, \ldots, O_n, \ldots\}$ be a countable basis of M. For each $m, k \in \mathbb{N}$, let

$$\begin{aligned}\mathcal{H}_{m,k} = \{X \in \mathfrak{X}^1(M) : &\text{ there is a } C^1 \text{ neighborhood } \mathcal{U}(X) \text{ of } X \text{ such that}\\ &\text{for any } Y \in \mathcal{U}(X),\ Y \text{ has a singularity } \sigma \in O_m \text{ with}\\ &\text{index}(\sigma) = 2 \text{ such that } W^u(\sigma, Y) \cap O_k \neq \emptyset\}.\end{aligned}$$

Then $\mathcal{H}_{m,k}$ is an open in $\mathfrak{X}^1(M)$. Let

$$\mathcal{N}_{m,k} = \mathfrak{X}^1(M) \setminus \overline{\mathcal{H}_{m,k}}.$$

Then $\mathcal{H}_{m,k} \cup \mathcal{N}_{m,k}$ is open and dense in $\mathfrak{X}^1(M)$. Let

$$\mathcal{G}_2 = \bigcap_{m,k \in \mathbb{N}} (\mathcal{H}_{m,k} \cup \mathcal{N}_{m,k}).$$

We will show that the residual set $\mathcal{G}_2$ satisfies the request of lemma. Let $X \in \mathcal{G}_2$ and Λ be an isolated transitive set and let $\sigma \in \Lambda \cap Sing(X)$ with index$(\sigma) = 2$. Since σ is hyperbolic, we can take O_m such that O_m is an isolated neighborhood of σ. By the structurally stability of hyperbolic singularity, there is a C^1 neighborhood $\mathcal{U}(X)$ of X such that for any $Y \in \mathcal{U}(X)$, Y has a unique hyperbolic singularity in O_m. For any $y \in \Lambda$ and any neighborhood U of y, we can choose $O_k \in \mathcal{O}$ such that $y \in O_k \subset U$.

Claim. $X \notin \mathcal{N}_{m,k}$.

Proof of Claim. For any neighborhood $\mathcal{V}(X) \subset \mathcal{U}(X)$, by Lemma 3.4, there is $Y \in \mathcal{V}(X)$ such that Y has a singularity $\sigma \in O_m$ with index$(\sigma) = 2$ and $W^u(\sigma, Y) \cap O_k \neq \emptyset$. Note that σ may not be a singularity of $Z \in \mathcal{U}(Y)$. By the persistence of hyperbolic singularity σ, there is a singularity σ_Z of Z such that $W^u(\sigma, Z) \cap O_k \neq \emptyset$. Thus we have $Y \in \mathcal{H}_{m,k}$. Hence $X \in \overline{\mathcal{H}_{m,k}}$. This ends the proof of claim. □

Then by claim, since $X \in \mathcal{G}_2$, we have $X \in \mathcal{H}_{m,k}$. Note that O_m is an isolated neighborhood of σ, by the definition of $\mathcal{H}_{m,k}$, we know that $W^u(\sigma) \cap O_k \neq \emptyset$. This prove that for every neighborhood U of y, we know that $W^u(\sigma) \cap U \neq \emptyset$. This means that $\Lambda \subset \overline{W^u(\sigma)}$. □

We say that a closed X^t-invariant set Λ is *Lyapunov stable* for X if for every neighborhood U of Λ there is a neighborhood $V \subset U$ of Λ such that $X^t(V) \subset U$ for every $t \geq 0$. Let σ be a hyperbolic singularity of X with $\dim W^u(\sigma) = 1$. Then $W^u(\sigma) \setminus \{\sigma\}$ can be divided into two connected branches Γ_1, Γ_2, that is, $W^u(\sigma) = \{\sigma\} \cup \Gamma_1 \cup \Gamma_2$.

Lemma 3.6. *Let $X \in \mathfrak{X}^1(M)$ and Λ be a transitive set of X. Assume $\sigma \in \Lambda$ is a hyperbolic singularity of X with* $\dim W^u(\sigma) = 1$. *Let $\Gamma_1 = Orb(x_1)$ and $\Gamma_2 = Orb(x_2)$ be the two branches of $W^u(\sigma) \setminus \{\sigma\}$. If $x_1 \in \Lambda$, then for any neighborhood $\mathcal{U}(X)$ of X, and any neighborhood V of x_2, there is $Y \in \mathcal{U}(X)$ such that x_1 is still in the unstable manifold of σ and the positive orbit of x_1 will cross V with respect to Y.*

Proof. We prove this lemma by a standard application of the connecting lemma. By Lemma 3.2 we know that there is a point $z \in (W^s(\sigma) \setminus \{\sigma\}) \cap \Lambda$. Then we have two triple of $\rho > 1$, $L > 1$ and δ_0 with the properties stated as in Lemma 3.3 with respect to the point x_1 and z and the neighborhood $\mathcal{U}(X)$ of X. By taking the larger ρ, L, and smaller δ_0, we get a triple, still denoted by ρ, L and δ_0, works both for x_1 and z.

Now we can take $\delta > 0$ small enough such that the two tubes $\Delta_1 = B_\delta(X^{[0,L]}(x_1))$ and $\Delta_2 = B_\delta(X^{[-L,0]}(z))$ are disjoint. For any neighborhood V of x_2 and any neighborhood V' of z, by the inclination lemma we know that there are a point $y \in V$ and $T > 0$ such that $X^{-T}(y) \in V'$. If $\delta > 0$ is choosing small enough, we can take y and T such that $X^{[-T,0]}(y)$ does not touch Δ_1.

Since Λ is transitive, we can find a point $x \in \Lambda$ such that $\Lambda = \omega(x)$. Then we can find $t_1 < t_2$ such that $X^{t_1}(x) \in B_{\delta/\rho}(X^L(x_1))$ and $X^{t_2}(x) \in B_{\delta/\rho}(X^{-L}(z))$ and a point $y \in V$ with $X^{-T}(y) \in B_{\delta/\rho}(z)$. Then apply Lemma 3.3, we can find a vector filed $Y \in \mathcal{U}(X)$ differs from X at tubes Δ_1 and Δ_2 such that the negative orbit of x_1 is not changed and y is contained in the positive orbit of x_1. It is easy to see that Y satisfies the request of lemma. □

Lemma 3.7. *Let $\mathcal{G}_2 \subset \mathfrak{X}^1(M)$ be the residual set chosen as in Lemma 3.5. Then for any $X \in \mathcal{G}_2$ and any isolated nontrivial transitive set Λ of X, if there is a singularity $\sigma \in \Lambda$ with* $\mathrm{index}(\sigma) = 2$, *then we have $\overline{W^u(\sigma)} \subset \Lambda$.*

Proof. Let $\mathcal{O} = \{O_1, O_2, \ldots, O_n, \ldots\}$ be a countable basis of M. Recall that for each $m, k \in \mathbb{N}$, we take

$$\begin{aligned}\mathcal{H}_{m,k} = \{X \in \mathfrak{X}^1(M) :\ & \text{there is a } C^1 \text{ neighborhood } \mathcal{U}(X) \text{ of } X \text{ such that} \\ & \text{for any } Y \in \mathcal{U}(X), Y \text{ has a singularity } \sigma \in O_m \text{ with} \\ & \mathrm{index}(\sigma) = 2 \text{ such that } W^u(\sigma, Y) \cap O_k \neq \emptyset\}.\end{aligned}$$

Then take $\mathcal{N}_{m,k} = \mathfrak{X}^1(M) \setminus \overline{\mathcal{H}_{m,k}}$ and

$$\mathcal{G}_2 = \bigcap_{m,k\in\mathbb{N}} (\mathcal{H}_{m,k} \cup \mathcal{N}_{m,k}).$$

We will see that this $\mathcal{G}_2$ satisfies the request of lemma.

Let $X \in \mathcal{G}_2$ and Λ be an isolated transitive set of X. Assume there is singularity $\sigma \in \Lambda$ with index 2. Let $\Gamma_1 = Orb(x_1)$ and $\Gamma_2 = Orb(x_2)$ be the two branches of $W^u(\sigma) \setminus \sigma$. By Lemma 3.2, we know that either x_1 or x_2 is contained in Λ. Without loss of generality, we assume that $x_1 \in \Lambda$. To prove $\overline{W^u(\sigma)} \subset \Lambda$, we just need to prove that x_2 is also contained in Λ. By the compactness of Λ, we just need to prove that for any neighborhood U of x_2, one has $U \cap \Lambda \neq \emptyset$. For a given arbitrarily small neighborhood U of x, we can find k such that $O_k \subset U$. Let O_m be an isolated neighborhood of σ. Then we have:

Claim. $X \notin \mathcal{N}_{m,k}$.

Proof of Claim. For any neighborhood $\mathcal{V}(X) \subset \mathcal{U}(X)$, by Lemma 3.6, there is $Y \in \mathcal{V}(X)$ such that Y has a singularity $\sigma \in O_m$ with index$(\sigma) = 2$ and $W^u(\sigma, Y) \cap O_k \neq \emptyset$. By the continuity of the unstable manifold we know that there is a C^1 neighborhood $\mathcal{U}(Y)$ of Y such that for any $Z \in \mathcal{U}(Y)$, $W^u(\sigma, Z) \cap O_k \neq \emptyset$. Thus we have $Y \in \mathcal{H}_{m,k}$. Hence $X \in \overline{\mathcal{H}_{m,k}}$. This ends the proof of claim. □

Since $X \in \mathcal{G}_2$ and $X \notin \mathcal{N}_{m,k}$, we have $X \in \mathcal{H}_{m,k}$. Since σ is the only singularity of X in O_m, by the definition of $\mathcal{H}_{m,k}$ we can see that $W^u(\sigma) \cap O_k \neq \emptyset$. Hence for any neighborhood U of x_2, there is a point contained in $W^u(\sigma)$. This ends the proof of Lemma 3.7. □

The following lemma is collected from [4].

Lemma 3.8 ([4, Proposition 4.1])**.** *There is a residual set $\mathcal{G}_3 \subset \mathfrak{X}^1(M)$ such that for any $X \in \mathcal{G}_3$, $\overline{W^u(\sigma)}$ is Lyapunov stable for X and $\overline{W^s(\sigma)}$ is Lyapunov stable for $-X$ for all $\sigma \in Sing(X)$.*

Proposition 3.9. *There is a residual set $\mathcal{S} \subset \mathfrak{X}^1(M)$ such that for any $X \in \mathcal{S}$, and any isolated nontrivial transitive set Λ of X, if there is a singularity $\sigma \in \Lambda \cap Sing(X)$ with* index$(\sigma) = 2$*, then Λ is Lyapunov stable for X. Symmetrically, if there is $\sigma \in \Lambda \cap Sing(X)$ with* index$(\sigma) = 1$*, then Λ is Lyapunov stable for $-X$.*

Proof. Let $X \in \mathcal{S} = \mathcal{G}_2 \cap \mathcal{G}_3$ and Λ be an isolated transitive set of X. Suppose that $\sigma \in \Lambda \cap Sing(X)$ with index$(\sigma) = 2$. Then by Proposition 3.5 and Lemma 3.7, we have $\overline{W^u(\sigma)} = \Lambda$. By Lemma 3.8, Λ is Lyapunov stable for X. □

A point $\sigma \in Sing(X)$ of X is called *Lorenz-like* if $DX(\sigma)$ has three real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ such that $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$. Let $\sigma \in Sing(X)$ be a Lorenz-like singularity. Then we use E^{ss}_σ, E^{cs}_σ, E^u_σ to denote the eigenspaces of $DX(\sigma)$ corresponding the eigenspaces λ_2, λ_3, λ_1, respectively. Denoted by $W^{ss}_X(\sigma)$ the one-dimensional invariant manifold of X associated to the eigenvalue λ_2. We have the following lemma was proved in [13].

Lemma 3.10 ([13, Lemma A.4]). *There is a residual set $\mathcal{G}_4 \subset \mathfrak{X}^1(M)$ such that for any $X \in \mathcal{R}$, if Λ is a Lyapunov stable nontrivial transitive set of X, then every singularity $\sigma \in \Lambda$ is Lorenz-like and one has $W^{ss}_X(\sigma) \cap \Lambda = \{\sigma\}$.*

Here is the main conclusion in this section.

Proposition 3.11. *There is a residual set $\mathcal{T} \subset \mathfrak{X}^1(M)$ with the following properties. Let $X \in \mathcal{T}$ and Λ be an isolated transitive set of X. If there is a singularity with index 2, then for all singularity $\sigma \in \Lambda$, one has (1) $\mathrm{index}(\sigma) = 2$, (2) σ is Lorenz-like, and (3) $W^{ss}_X(\sigma) \cap \Lambda = \{\sigma\}$. Symmetrically, if there is a singularity with index 1, then for all singularity $\sigma \in \Lambda$, one has (1) $\mathrm{index}(\sigma) = 1$, (2) σ is Lorenz-like for $-X$, and (3) $W^{uu}_X(\sigma) \cap \Lambda = \{\sigma\}$.*

Proof. Let $X \in \mathcal{T} = \mathcal{S} \cap \mathcal{G}_4$ and Λ be an isolated transitive set of X. Suppose that there is $\eta \in \Lambda \cap Sing(X)$ such that $\mathrm{index}(\eta) = 2$. By Proposition 3.9, Λ is Lyapunov stable for X. On the other hand, since $X \in \mathcal{G}_4$, according to Lemma 3.11, σ is Lorenz-like, and $W^{ss}_X(\sigma) \cap \Lambda = \{\sigma\}$. We directly obtained $\mathrm{index}(\sigma) = 2$ for all $\sigma \in \Lambda \cap Sing(X)$. □

4. Proof of Theorem A

To prove Theorem A, we prepare two techniques here. One is the extended linear Poincaré flow given by Li, Gan and Wen [7], and another one is the ergodic closing lemma given by Mañé [11,12].

Firstly we recall the notion of linear Poincaré flow firstly given by Liao [8,9]. For any regular point $x \in M \setminus Sing(X)$, we can put a normal space

$$N_x = \{v \in T_xM : v \perp X(x)\}.$$

Then we have a normal bundle

$$N = N(X) = \bigcup_{x \in M \setminus Sing(X)} N_x.$$

Denote by π_x the orthogonal projection from T_xM to N_x for any $x \in M \setminus Sing(X)$. From the tangent flow, we can define the *linear Poincaré flow*

$$P^X_t : N(X) \to N(X)$$

$$P^X_t(v) = \pi_{X^t(x)}(DX^t(v)) \text{ for all } v \in N_x, \text{ and } x \in M \setminus Sing(X).$$

Note that the linear Poincaré flow is defined on the normal bundle over a non compact set. We consider a compactification for P^X_t as following.

Let

$$G^1 = \{L : L \text{ is a one dimensional subspace in } T_xM,\ x \in M\}$$

be the Grassmannian manifold of M. Then for any $L \in G^1$, assuming $L \subset T_xM$ for some $x \in M$, we can define a normal space associated to L as follows:

$$N_L = \{v \in T_xM : v \perp L\}.$$

Now we can take a normal bundle

$$N = N_{G^1} = \bigcup_{L \in G^1} N_L.$$

Note that G^1 is a compact manifold, so N_{G^1} is a bundle over a compact space.

For any $L \in G^1$ contained in T_xM, denoted by π_L the orthogonal projection from T_xM to N_L along L. Let X be a C^1 vector field. Similar to the linear Poincaré flow, we can define the *extended linear Poincaré flow*

$$\tilde{P}_t^X : N_{G^1} \to N_{G^1}$$
$$\tilde{P}_t^X(v) = \pi_{DX^t(L)}(DX^t(v))$$

for all $L \in G^1$ and $v \in N_L$. One can check that for any $x \in M \setminus Sing(X)$, we have $N_x = N_{\langle X(x) \rangle}$ and $P_t^X|_{N_x} = \tilde{P}_t^X|_{N_{\langle X(x) \rangle}}$. Here, $\tilde{P}_t^X$ is said to be the *extended linear Poincaré flow.*

For any compact invariant set Λ of the vector fields X, we use $\tilde{\Lambda}$ to denote the closure of

$$\{\langle X(x) \rangle : x \in \Lambda \setminus Sing(X)\}$$

in the space of G^1. Let $\sigma \in \Lambda$ be a singularity, denote by

$$\tilde{\Lambda}_\sigma = \{L \in \tilde{\Lambda} : L \subset T_\sigma M\}.$$

From the facts we got from Proposition 3.11, we have the following characterization of $\tilde{\Lambda}_\sigma$.

Lemma 4.1. *Let $X \in \mathcal{T}$ and Λ be an isolated transitive set of X. Suppose there is a singularity with index 2. Then for all singularity $\sigma \in \Lambda$, we have $L \subset E_\sigma^{cs} \oplus E_\sigma^u$ for all $L \in \tilde{\Lambda}_\sigma$.*

Proof. Let $X \in \mathcal{T}$ and Λ be an isolated transitive set of X. Suppose on the contrary, that is, there is $L \in \tilde{\Lambda}_\sigma$ such that L is not a subspace in $E_\sigma^{cs} \oplus E_\sigma^u$. Note that $DX^t(L)$ is contained in $\tilde{\Lambda}_\sigma$ for all $t \in \mathbb{R}$ and $\tilde{\Lambda}_\sigma$ is a closed set. By taking a limit line of $DX^t(L)$ as $t \to -\infty$, we know that there is $L \in \tilde{\Lambda}_\sigma$ such that $L \subset E_\sigma^{ss}$. From now on, we assume that $L \in \tilde{\Lambda}$ and $L \subset E_\sigma^{ss}$. By the definition of $\tilde{\Lambda}$, we know that there exist $x_n \in \Lambda \setminus Sing(X)$ such that $\langle X(x_n) \rangle \to L \subset E_\sigma^{ss}$. For the simplicity of notations, we assume everything happens in a local chart containing σ. For any $0 < \eta \le 1$, denote by $E_\sigma^{cu} = E_\sigma^{cs} \oplus E_\sigma^u$ and

$$C_\eta^{cu}(\sigma) = \{v = v^{ss} + v^{cu} \in T_\sigma M : |v^{ss}| < \eta |v^{cu}|,\ v^{ss} \in E_\sigma^{ss},\ v^{cu} \in E_\sigma^{cu}\}$$

the *cu*-cone at the singularity σ. These cones can be parallel translated to x who is close to σ. Since $E_\sigma^{ss} \oplus E_\sigma^{cu}$ is a dominated splitting for the tangent flow DX^t, there are two constants $T > 0$ and $0 < \lambda < 1$ such that

$$DX^t(C_1^{cu}(\sigma)) \subset C_\lambda^{cu}(\sigma)$$

for any $t \in [T, 2T]$. By the continuous property of the cone to a cone field in a small neighborhood U_σ of σ, for any $t \in [T, 2T]$, $X^{[0,t]}(x) \subset U_\sigma$ then we have $DX^t(C_1^{cu}(x)) \subset C_1^{cu}(X^t(x))$. Now let $t_n = \sup\{t > 0 : X^{[-t,0]}(x_n) \subset U_\sigma\}$. We know that $t_n \to +\infty$ as $n \to \infty$ because $x_n \to \sigma$ as $n \to \infty$. Denote by $y_n = X^{-t_n}(x_n)$. Then we can take $q = \lim_{n\to\infty} y_n \in \partial U_\sigma$ by taking the subsequence if necessary. We know that for $t > 0$, $X^t(q) \in U_\sigma$ and so, $q \in W^s(\sigma)$. Since $y_n \in \Lambda$ we know $q \in \Lambda$. If $q \in W^{ss}(\sigma) \cap \Lambda$, because we have already $q \in \partial U_\sigma$, hence $q \neq \sigma$, then from the fact that $X \in \mathcal{T}_1$ and Λ is an isolated nontrivial transitive set, this is a contradiction by Proposition 3.11. Now we assume that $q \in W^s(\sigma) \setminus W^{ss}(\sigma)$. We have $\langle X(X^t(q))\rangle \to E_\sigma^{cs}$ as $t \to +\infty$. Thus there is $T_1 > 0$ big enough such that $X(X^{T_1}(q)) \in C_1^{cu}(X^{T_1}(q))$. For n big enough we have $X(X^{T_1}(y_n)) \in C_1^{cu}(X^{T_1}(y_n))$. Since $t_n \to \infty$, we assume that $t_n - T_1 > T$. Since $X^{[T_1,t_n]}(y_n) \subset U_\sigma$, we know that

$$\begin{aligned} X(x_n) &= X(X^{t_n}(y_n)) = DX^{t_n - T_1}(X(X^{T_1}(y_n))) \\ &\in DX^{t_n - T_1}(C_1^{cu}(X^{T_1}(y_n))) \\ &\subset C_1^{cu}(X^{t_n}(y_n)) = C_1^{cu}(x_n). \end{aligned}$$

This is a contradiction with the assumption $\langle X(x_n)\rangle \to L \subset E_\sigma^{ss}$. □

It is proved in Section 2 that generically, if Λ is an isolated transitive set, then it is locally star. By some well know results from the proof of stability conjecture, we have the following proposition.

Proposition 4.2 ([9, 11])**.** *Let Λ be a locally star set for $X \in \mathfrak{X}^1(M)$ and let $\mathcal{U}(X), U$ be the neighborhoods in the definition of local star. Then there are constants $0 < \lambda_0 < 1$, $T_0 > 0$ such that for any $Y \in \mathcal{U}(X)$ and any $p \in \Lambda_Y(U) \cap Per(Y)$, the following properties hold*:

(a) *$\Delta^s \oplus \Delta^u$ is a dominated splitting with respect to the linear Poincaré flow. Precisely, for any $t \geq T_0$ and any $x \in Orb(p)$,*

$$\|P_t^Y|_{\Delta^s(x)}\| \cdot \|P_{-t}^Y|_{\Delta^u(Y^t(x))}\| \leq e^{-2\lambda_0 t};$$

(b) *if τ is the period of p and m is any positive integer, and if $0 = t_0 < t_1 < \cdots < t_k = m\tau$ is any partition of the time interval $[0, m\tau]$ with $t_{i+1} - t_i \geq T_0$, then*

$$\frac{1}{m\tau} \sum_{i=0}^{k-1} \log \|P_{t_{i+1}-t_i}^Y|_{\Delta^s(Y^{t_i}(p))}\| < -\lambda_0,$$

and

$$\frac{1}{m\tau} \sum_{i=0}^{k-1} \log \|P_{-(t_{i+1}-t_i)}^Y|_{\Delta^u(Y^{t_{i+1}}(p))}\| < -\lambda_0,$$

where $\Delta^s \oplus \Delta^u$ is the hyperbolic splitting with respect to $P_\tau^X|_{N_{Orb(p)}}$.

Now we assume that Λ is an isolated transitive set of a C^1-generic vector field X. By the closing lemma we know that for any $x \in \Lambda \setminus Sing(X)$, one can find a sequence of periodic points p_n of X such that $p_n \to x$ as $n \to \infty$. Consequently, for any $L \in \tilde{\Lambda}$, we can find a sequence of periodic points p_n of X, such that L is the limit of $\langle X(p_n)\rangle$. Since Λ is locally star, from item (a) of Proposition 4.2 we can see that for any $L \in \tilde{\Lambda}$, we can get two one dimensional subspaces $\Delta^1(L) = \lim_{n\to\infty} \Delta^s(p_n)$ and $\Delta^2(L) = \lim_{n\to\infty} \Delta^u(p_n)$ with the property: for any $t \geq T_0$,

$$\|\tilde{P}_t^Y|_{\Delta^1(L)}\| \cdot \|\tilde{P}_{-t}^Y|_{\Delta^2(DX^t(L))}\| \leq e^{-2\lambda_0 t}.$$

This implies that there is a dominated splitting $N_{\tilde{\Lambda}} = \Delta^1 \oplus \Delta^2$ for the extended linear Poincaré flow $\tilde{P}_t^X$. For any $x \in \Lambda \setminus Sing(X)$, we can put $\Delta^i(x) = \Delta^i(\langle X(x)\rangle)$ for $i = 1, 2$, then we can get a dominated splitting $N_{\Lambda\setminus Sing(X)} = \Delta^1 \oplus \Delta^2$ for the linear Poincaré flow P_t^X.

If $X \in \mathcal{T}$ and Λ be an isolated transitive set of X, then we have only finitely many singularity in Λ. Without loss of generality, after a change of equivalent Riemmanian structure, we can assume that for any $\sigma \in \Lambda$ with index 2, the subspaces E_σ^{ss}, E_σ^{cs}, E_σ^u are mutually orthogonal. From Lemma 4.1 we know that every $L \in \tilde{\Lambda}_\sigma$ is orthogonal to E_σ^{ss}, this fact derives the following lemma.

Lemma 4.3. *Let $X \in \mathcal{T}$ and Λ be an isolated transitive set of X. Suppose there is a singularity with index 2. Then for all singularity $\sigma \in \Lambda$ with mutually orthogonal E_σ^{ss}, E_σ^{cs}, E_σ^u, we have $\Delta^1(L) = E_\sigma^{ss}$ and $\tilde{P}_S^X|_{\Delta^1(L)} = DX^S|_{E_\sigma^{ss}}$ for any $L \in \tilde{\Lambda}_\sigma$.*

Proof. We denote by $E_\sigma^{cu} := E_\sigma^{cs} \oplus E_\sigma^u$ for any given singularity $\sigma \in \Lambda$. For any $L \in \tilde{\Lambda}_\sigma$, we set $N^1(L) = E_\sigma^{ss}$ and $N^2(L) = E_\sigma^{cu} \cap N_L$. By the fact that L is orthogonal to E_σ^{ss} we know that $N^1(L) \subset N_L$ for any $L \in \tilde{\Lambda}_\sigma$. Now we have two subbundles

$$N_{\tilde{\Lambda}_\sigma}^1 = \bigcup_{L\in\tilde{\Lambda}_\sigma} N^1(L), \qquad N_{\tilde{\Lambda}_\sigma}^2 = \bigcup_{L\in\tilde{\Lambda}_\sigma} N^2(L).$$

These two subbundles are $\tilde{P}_t^X$-invariant by the fact that $L \subset E_\sigma^{cu}$ for any $L \in \tilde{\Lambda}_\sigma$ and both E_σ^{ss} and E_σ^{cu} are DX^t-invariant.

Since $E_\sigma^{ss} \oplus E_\sigma^{cu}$ is a dominated splitting for DX^t, we know that there are constants $C > 1$, $\lambda > 0$ such that

$$\frac{\|DX^{-t}(u)\|}{\|DX^{-t}(v)\|} \leq Ce^{-\lambda t}$$

for any unit vectors $u \in E_\sigma^{cu}$ and $v \in E_\sigma^{ss}$ and any $t > 0$. Then for any $L \in \tilde{\Lambda}_\sigma$ and any unit vectors $u \in N^2(L)$, $v \in N^1(L)$, we have

$$\frac{\|\tilde{P}_{-t}^X(u)\|}{\|\tilde{P}_{-t}^X(v)\|} \leq \frac{\|DX^{-t}(u)\|}{\|DX^{-t}(v)\|} \leq Ce^{-\lambda t}.$$

This says that $N_{\tilde{\Lambda}_\sigma} = N^1_{\tilde{\Lambda}_\sigma} \oplus N^2_{\tilde{\Lambda}_\sigma}$ is a dominated splitting on $\tilde{\Lambda}_\sigma$ with respect to the extended linear Poincaré flow $\tilde{P}^X_t$. By the uniqueness of dominated splitting we know that $N^1_{\tilde{\Lambda}_\sigma} = \Delta^1_{\tilde{\Lambda}_\sigma}$. Thus we have $\Delta^1(L) = E^{ss}_\sigma$ for all $L \in \tilde{\Lambda}_\sigma$. By the definition of extended linear Poincaré flow, we directly have the fact that $\tilde{P}^X_S|_{\Delta^1(L)} = DX^S|_{E^{ss}_\sigma}$ for all $L \in \tilde{\Lambda}_\sigma$. □

Now let us recall the ergodic closing lemma. A point $x \in M \setminus Sing(X)$ is called a *well closable point* of X if for any C^1 neighborhood $\mathcal{U}(X)$ of X and any $\delta > 0$, there are $Y \in \mathcal{U}(X)$, $z \in M$, $\tau > 0$ and $T > 0$ such that the following conditions are hold:

(a) $Y^\tau(z) = z$,
(b) $d(X^t(x), Y^t(z)) < \delta$ for any $0 \le t \le \tau$, and
(c) $X = Y$ on $M \setminus B(X^{[-T,0]}(x), \delta)$.

Denote by $\Sigma(X)$ the set of all well closable points of X. Here we will use the flow version of the ergodic closing lemma which was proved in [17].

Lemma 4.4 ([17])**.** *For any $X \in \mathfrak{X}^1(M)$, $\mu(\Sigma(X) \cup Sing(X)) = 1$ for every $T > 0$ and every X^T-invariant Borel probability measure μ.*

Assume $X \in \mathcal{T}$ and Λ is an isolated transitive set of X. From Proposition 4.2 we have already known that there is a dominated splitting $N_{\Lambda \setminus Sing(X)} = \Delta^1 \oplus \Delta^2$ with $\dim(\Delta^1) = \dim(\Delta^2) = 1$ with respect to the linear Poincaré flow P^X_t. By applying the ergodic closing lemma, we have the following lemma.

Lemma 4.5. *Let $X \in \mathcal{T}$ and Λ be an isolated transitive set of X. Suppose there is a singularity with index 2. Then there are constants $C > 1$ and $\lambda > 0$ such that*

$$\|DX^t|_{\langle X(x)\rangle}\|^{-1} \cdot \|P^X_t|_{\Delta^1(x)}\| < C\mathrm{e}^{-\lambda t},$$
$$\|DX^{-t}|_{\langle X(x)\rangle}\| \cdot \|P^X_{-t}|_{\Delta^2(x)}\| < C\mathrm{e}^{-\lambda t}$$

for all $x \in \Lambda \setminus Sing(X)$ and $t \ge 0$.

Proof. Let $X \in \mathcal{T}$ and Λ be an isolated transitive set of X. Then there is a $\tilde{P}^X_t$ invariant splitting $N_{\tilde{\Lambda}} = \Delta^1 \oplus \Delta^2$ with constants $T_0 > 0$ and $\lambda_0 > 0$ such that the followings are satisfied:

(1) if $L = \langle X(x)\rangle$ for some $x \in \Lambda \setminus Sing(X)$, then $\Delta^i(\langle X(x)\rangle) = \Delta^i(x)$ for $i = 1, 2$,
(2) $\|\tilde{P}^Y_t|_{\Delta^1(L)}\| \cdot \|\tilde{P}^Y_{-t}|_{\Delta^2(DX^t(L))}\| \le e^{-2\lambda_0 t}$ for any $t > T_0$, and
(3) $L \in \tilde{\Lambda}$.

To prove the lemma, we just need to prove that there are $C > 1$ and $\lambda > 0$ such that for any $L \in \tilde{\Lambda}$ and any $t > 0$, we have

$$\|DX^t|_L\|^{-1} \cdot \|\tilde{P}^X_t|_{\Delta^1(L)}\| < C\mathrm{e}^{-\lambda t},$$
$$\|DX^{-t}|_L\| \cdot \|\tilde{P}^X_{-t}|_{\Delta^2(L)}\| < C\mathrm{e}^{-\lambda t}.$$

Since $\tilde{\Lambda}$ is compact, we just need to show that for any $L \in \tilde{\Lambda}$, there is a $T > 0$ such that

$$\log \|\tilde{P}_T^X|_{\Delta^1(L)}\| - \log \|DX^T|_L\| < 0,$$
$$\log \|\tilde{P}_{-T}^X|_{\Delta^2(L)}\| + \log \|DX^{-T}|_L\| < 0.$$

Now let us prove these properties of $\Delta^1 \oplus \Delta^2$ by contradiction. Firstly we prove the first half part. Assume that for any $L \in \tilde{\Lambda}$ and any $t > 0$

$$\log \|\tilde{P}_t^X|_{\Delta^1(L)}\| - \log \|DX^t|_L\| \geq 0.$$

Similar to [12, Lemma I.5], by a typical application of Birkhoff ergodic theorem, for any $S > 0$ there is an ergodic DX^T-invariant measure $\tilde{\mu} \in \mathcal{M}(G^1)$ with $supp(\tilde{\mu}) \subset \tilde{\Lambda}$ such that

$$\int (\log \|\tilde{P}_S^X|_{\Delta^1(L)}\| - \log \|DX^S|_L\|) d\tilde{\mu}(L) \geq 0.$$

In the following, we will always choose S is big enough.

Claim. *If S is big enough, then for any singularity $\sigma \in \Lambda \cap Sing(X)$, one has $\tilde{\mu}(\tilde{\Lambda}_\sigma) = 0$.*

Proof of Claim. According to Lemma 4.1, for every $L \in \tilde{\Lambda}_\sigma$, $L \subset E_\sigma^{cs} \oplus E_\sigma^u := E_\sigma^{cu}$. Without loss of generality, we assume that E_σ^{ss} is orthogonal to E_σ^{cu}. Then by Lemma 4.3 we have $\tilde{P}_S^X|_{\Delta^1(L)} = DX^S|_{E_\sigma^{ss}}$ for any $L \in \tilde{\Lambda}_\sigma$. Since E_σ^{ss} is dominated by E_σ^{cu}, we can take S big enough such that

$$\log \|\tilde{P}_S^X|_{\Delta^1(L)}\| - \log \|DX^S|_L\| < 0$$

for any $L \in \tilde{\Lambda}_\sigma$. If $\tilde{\mu}(\tilde{\Lambda}_\sigma) \neq 0$, then we have $\tilde{\mu}(\tilde{\Lambda}_\sigma) = 1$ by the invariant of $\tilde{\Lambda}_\sigma$ and the ergodicity of $\tilde{\mu}$, thus we have

$$\int (\log \|\tilde{P}_S^X|_{\Delta^1(L)}\| - \log \|DX^S|_L\|) d\tilde{\mu}(L) < 0.$$

This is a contradiction. This ends the proof of claim. □

In the following, we will take S is a multiple of T_0 which is big enough such that the above claim is satisfied. One can see S have also the properties of T_0.

For any Borel set $A \subset \Lambda$, we denote by $\tilde{A} = \{L : L = \langle X(x) \rangle$ for some $x \in A\}$. Then we define $\mu(A) = \tilde{\mu}(\tilde{A})$. By the fact that $\tilde{\mu}(\tilde{\Lambda}_\sigma) = 0$ for any $\sigma \in \Lambda \cap Sing(X)$, we know that μ is an ergodic measure support in Λ with $\mu(\Lambda \setminus Sing(X)) = 1$. From the inequality

$$\int (\log \|\tilde{P}_S^X|_{\Delta^1(L)}\| - \log \|DX^S|_L\|) d\tilde{\mu}(L) \geq 0,$$

we have

$$\int_{\Lambda \setminus Sing(X)} (\log \|P_S^X|_{\Delta_x^1}\| - \log \|DX^S|_{\langle X(x) \rangle}\|) d\mu(x) \geq 0.$$

By Lemma 4.4,

$$\int_{\Lambda\cap\Sigma(X)} (\log \|P_S^X|_{\Delta^1(x)}\| - \log \|DX^S|_{\langle X(x)\rangle}\|) d\mu(x) \geq 0.$$

By the ergodic theorem of Birkhoff, there is a point $y \in \Lambda \cap \Sigma(X)$ such that

$$(1) \qquad \lim_{n\to\infty} \frac{1}{nS} \sum_{j=0}^{n-1} (\log \|P_S^X|_{\Delta^1(X^{jS}(y))}\| - \log \|DX^S|_{\langle X(X^{jS}(y))\rangle}\|) \geq 0.$$

Claim. y is not a periodic point of X.

Proof of Claim. By the fact that $\|DX^S|_{\langle X(x)\rangle}\| = \frac{|X(X^S(x))|}{|X(x)|}$, we have

$$\begin{aligned}\sum_{j=0}^{n-1} \log \|DX^s|_{\langle X(X^{jS}(y))\rangle}\| &= \sum_{j=0}^{n-1} \log \frac{|X(X^{j+1}S(y))|}{|X(X^{jS}(y))|} \\ &= \log |X(X^{nS}(y))| - \log |X(y)|.\end{aligned}$$

If $y \in Per(X)$, then by Proposition 4.2, we have

$$\limsup_{n\to\infty} \frac{1}{nS} \sum_{j=0}^{n-1} \log \|P_S^X|_{\Delta^s_{X^{jS}(y)}}\| \leq -\lambda_0.$$

Since $\sup |\log(X(x))|$ is bounded for $x \in Orb(y)$, we have

$$\limsup_{n\to\infty} \frac{1}{nS} \big(\sum_{j=0}^{n-1} \log \|P_S^X|_{\Delta^s_{X^{jS}(y)}}\| - \log |X(X^{nS}(y))| - \log |X(y)|\big) \leq -\lambda.$$

This is contradiction by (1). Thus y is not periodic. □

Since y is a well closable point, for any $n > 0$, there are $X_n \in \mathfrak{X}^1(M)$, $z_n \in M$, and $\tau_n > 0$ such that

(i) $Y_n^{\tau_n}(z_n) = z_n$ and τ_n is the prime period of z_n,
(ii) $d(X^t(y), Y_n^t(z_n)) \leq 1/n$ for any $0 \leq t \leq \tau_n$, and
(iii) $\|Y_n - X\| \leq 1/n$.

Since y is not a periodic point, we have $\tau_n \to +\infty$ as $n \to \infty$. We also have the following uniformly continuity for $P_t^Y|_{\Delta^1}$.

Claim. For any $\epsilon > 0$ there is $\delta > 0$ and a C^1 neighborhood $\mathcal{U}(X)$ of X such that for any $x, y \in M$, if (i) $x \in \Lambda \setminus Sing(X)$, (ii) there is $Y \in \mathcal{U}(X)$ such that $y \in Per(Y)$, $Orb(y) \subset U$, and $d(x, y) < \delta$, then

$$(2) \qquad |\log \|P_t^X|_{\Delta^1(x)}\| - \log \|P_t^Y|_{\Delta^s(y)}\|| < \epsilon$$

for any $t \in [0, 2S]$. Here $\Delta^s(y)$ denotes the stable subspace of y with respect to the vector field Y.

Proof of Claim. We prove this by deriving a contradiction. Assume the contrary. Then there is $\eta > 0$ such that for any $n > 0$ there exists $t_n \in [0, 2S]$, $X_n \to X$ and two sequences $\{x_n\}$, $\{y_n\}$ such that (i) $x_n \in \Lambda \setminus Sing(X)$, (ii) $y_n \in Per(X_n)$ and $Orb(y_n) \subset U$, (iii) $d(x_n, y_n) < 1/n$, and

$$|\log \|P_{t_n}^X|_{\Delta^1(x_n)}\| - \log \|P_{t_n}^{X_n}|_{\Delta^s(y_n)}\|| \geq \eta.$$

Since $[0, 2S]$ and Λ are compact, we can take sequences $\{t_n\} \subset [0, 2S]$ and $\{x_n\} \subset \Lambda$ (take subsequences if necessary) such that $t_n \to t_0$ and $x_n \to x_0$. Then we have $y_n \to x_0$ by the above item (iii).

If $x_0 \notin Sing(X)$, then by the continuity of dominated splitting, we know $\Delta^1(x_n) \to \Delta^1(x_0)$ and $\Delta^s(y_n) \to \Delta^1(x_0)$ as $n \to \infty$, then we have

$$|\log \|P_{t_0}^X|_{\Delta^1(x_0)}\| - \log \|P_{t_0}^X|_{\Delta^1(x_0)}\|| \geq \eta.$$

This is a contradiction.

If $x_0 \in Sing(X)$, then we can take sequences $\{\langle X(x_n)\rangle\}$, $\{\langle X_n(y_n)\rangle\}$ (take subsequences if necessary) such that $\langle X(x_n)\rangle \to L \in \tilde{\Lambda}_{x_0}$ and $\langle X_n(y_n)\rangle \to L_1 \in \tilde{\Lambda}_{x_0}$. Since both $L, L_1 \in \tilde{\Lambda}_{x_0}$, we have $\tilde{P}_t^X|_{\Delta^1(L)} = \tilde{P}_t^X|_{\Delta^1(L_1)} = DX^t|_{E_{x_0}^{ss}}$ by Lemma 4.3. But on the other hand, we have

$$|\log \|\tilde{P}_t^X|_{\Delta^1(L)}\| - \log \|\tilde{P}_t^X|_{\Delta^1(L_1)}\|| \geq \eta.$$

This is also a contradiction. This ends the proof of Claim. □

By (2), there is n_0 such that for any $k > n_0$, $t \in [0, 2S]$ and $t_0 \in [0, \tau_n]$, one has

$$|\log \|P_t^X|_{\Delta^1_{X^{t_0}(y)}}\| - \log \|P_t^{X_n}|_{\Delta^s(X_n^{t_0}(z_n))}\|| < S\lambda_0/3, \tag{3}$$

where λ_0 as in Proposition 4.2. Let $\tau_n = m_n S + s_n$ ($m_n \in \mathbb{Z}$ and $s_n \in [0, S)$). Then we consider the partition

$$0 = t_0 < t_1 = S < \cdots < t_{m_n - 1} = (m_n - 1)S < t_{m_n} = \tau_n.$$

According to Proposition 4.2, we know

$$\sum_{j=0}^{m_n-2} \log \|P_S^{X_n}|_{\Delta^s(X_n^{jS}(z_n))}\| + \log \|P_{S+s_n}^{X_n}|_{\Delta^s(X_n^{(m_n-1)S}(z_n))}\| \leq -\tau_n \lambda_0.$$

Then by (3) we have

$$\begin{aligned} &\sum_{j=0}^{m_n-2} \log \|P_S^X|_{\Delta^1(X^{jS}(y))}\| + \log \|P_{S+s_n}^X|_{\Delta^1(X^{(m_n-1)S}(y))}\| \\ &\leq m_n S\lambda_0/3 - \tau_n \lambda_0 = -2m_n S\lambda_0/3 - s_n\lambda_0 \leq -2m_n S\lambda_0/3. \end{aligned}$$

For sufficiently small $r > 0$, let $B_r(y)$ be a neighborhood of $X^{[-2S,0]}(y)$ such that $B_r(y) \cap Sing(X) = \emptyset$. Denote by $C = \sup\{|\log |X(x)|| : x \in B_r(y)\} + \sup\{|\log ||P_t^X|_{\Delta^s(x)}||| : x \in B_r(y), t \in [0, 2S]\} < \infty$. Since $d(y, z_n) < 1/n$ and $d(X^{\tau_n}(y), z_n) = d(X^{\tau_n}(y), X^{\tau_n}(z_n)) < 1/n$, we know $d(X^{\tau_n}(y), y) < 2/n$.

Thus there is $n_1 > n_0$ such that for any $n > n_1$ and $t \in [0, 2S]$ we have $X^{\tau_n - t}(y) \in B_r(y)$. Since $\tau_n - (m_n - 1)S = S + s_n < 2S$, we know

$$|\log |X(X^{(m_n-1)S}(y))|| + |\log \|P^X_{S+s_n}|_{\Delta^s(P^X_{(m_n-1)S}(y))}\|| \le C. \tag{4}$$

By (1) and $m_n \to +\infty$ as $n \to +\infty$, there is $n_2 \ge n_1$ such that for any $n > n_2$

$$\sum_{j=0}^{m_n-2} \log \|P^X_S|_{\Delta^1(X^{jS}(y))}\| - (\log |X(X^{(m_n-1)S}(y))| - \log |X(y)|)$$
$$\ge -(m_n - 1)S\lambda_0/3.$$

Then by

$$\sum_{j=0}^{m_n-2} \log \|P^X_S|_{\Delta^s(X^{jS}(y))}\| + \log \|P^X_{S+s_n}|_{\Delta^s(X^{(m_n-1)S}(y))}\| \le -2m_n S\lambda_0/3,$$

and (4), we have

$$-(m_n - 1)S\lambda_0/3 \le -2m_n S\lambda_0/3 + C + \log |X(y)|.$$

If n is big enough, then it does not happen, and so, it is a contradiction. This proves that for any $L \in \tilde{\Lambda}$, there is a $T > 0$ such that

$$\log \|\tilde{P}^X_T|_{\Delta^1(L)}\| - \log \|DX^T|_L\| < 0.$$

And then by the compactness of $\tilde{\Lambda}$, we can find $C > 1$ and $\lambda > 0$ such that for any $L \in \tilde{\Lambda}$ and any $t > 0$, we have

$$\|DX^t|_L\|^{-1} \cdot \|\tilde{P}^X_t|_{\Delta^1(L)}\| < C\mathrm{e}^{-\lambda t}.$$

By a similar argument we can prove that for any $L \in \tilde{\Lambda}$, there is a $T > 0$ such that

$$\log \|\tilde{P}^X_{-T}|_{\Delta^2(L)}\| + \log \|DX^{-T}|_L\| < 0,$$

and then there exist $C > 1$ and $\lambda > 0$ such that for any $L \in \tilde{\Lambda}$ and any $t > 0$, we have

$$\|DX^{-t}|_L\| \cdot \|\tilde{P}^X_{-t}|_{\Delta^2(L)}\| < C\mathrm{e}^{-\lambda t}.$$

This ends the proof of the lemma. □

Theorem A is a direct corollary of Lemma 4.5 and the following lemma in [19].

Lemma 4.6 ([19, Theorem A]). *Assume Λ is a non-trivial transitive set such that all singularity in Λ is hyperbolic. If there is a dominated splitting $N_{\Lambda \setminus Sing(X)} = \Delta^1 \oplus \Delta^2$ on $\Lambda \setminus Sing(X)$ with respect to P^X_t and there are constants $C > 1$ and $\lambda > 0$ such that*

$$\|DX^t|_{\langle X(x)\rangle}\|^{-1} \cdot \|P^X_t|_{\Delta^1(x)}\| < C\mathrm{e}^{-\lambda t},$$
$$\|DX^{-t}|_{\langle X(x)\rangle}\| \cdot \|P^X_{-t}|_{\Delta^2(x)}\| < C\mathrm{e}^{-\lambda t}$$

for all $x \in \Lambda \setminus Sing(X)$ and $t \ge 0$, then Λ is positively singular hyperbolic.

Proof of Theorem A. Let $X \in \mathcal{T}$ and Λ be an isolated transitive set of X. If there is a singularity $\sigma \in \Lambda$ with index 2, then Λ is positively singular hyperbolic by Lemma 4.5 and Lemma 4.6. If there is a singularity $\sigma \in \Lambda$ with index 1, then by reversing the vector fields, we know that Λ is negatively singular hyperbolic. This ends of the proof of Theorem A. □

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