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CONTINUITY OF THE MAXIMAL COMMUTATORS IN SOBOLEV SPACES

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Abstract. We prove the Sobolev continuity of maximal commutator and its fractional variant with Lipschitz symbols, both in the global and local cases. The main result in global case answers a question originally posed by Liu and Wang in [29].

1. Introduction

1.1. Background

The regularity theory of maximal operators has been an active topic of current research. The first work was due to Kinnunen [25] who showed that the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy$$

is bounded on the first order Sobolev space $W^{1,p}(\mathbb{R}^n)$ for $1 < p \leq \infty$, where $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at x with radius r and $|B(x,r)|$ denotes its volume. Here

$$W^{1,p}(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{R} \,|\, \|f\|_{W^{1,p}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)} < \infty\},$$

where $\nabla f = (D_1 f, \ldots, D_n f)$ is the weak gradient of f. Since then, more and more scholars were devoted to extending Kinnunen's result to various cases. For examples, see [22,26] for the local case, [27] for the fractional case, [10,30] for the multisublinear case. It is worth pointing out that the $L^p(\mathbb{R}^n)$ boundedness of M plays a key role in deducing the boundedness of $M : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ when $1 < p < \infty$. The endpoint Sobolev regularity of the Hardy–Littlewood maximal operators has been recently studied by many authors, see [2, 28, 42]

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for the case $n = 1$ and [21, 36] for the case $n \geq 2$. Other interesting works related to this topic can be found in [5, 37], among others.

In light of the $W^{1,p}(\mathbb{R}^n)$ boundedness of M for $1 < p < \infty$, it is natural and interesting to ask whether the map $M : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ is continuous where $1 < p < \infty$. This question is attributed to T. Iwaniec and was firstly posed by Hajłasz and Onninen in [22]. It is well known that M is continuous on $L^p(\mathbb{R}^n)$ for all $1 < p \leq \infty$, which follows directly from the well-known L^p bounds and the sublinearity. However, since the maximal operator is not necessarily sublinear at the derivative level, the continuity of $M : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ for $1 < p < \infty$ is certainly a nontrivial issue. This question was first studied by Luiro [34] who established the continuity of $M : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ for $1 < p < \infty$ by establishing some explicit formulas for the derivatives of the maximal function. Later on, Luiro's result was extended to the local case in [35], to the bilinear case in [10]. The endpoint Sobolev continuity was first studied by Carneiro, Madrid and Pierce [9]. More interesting works may be found in [4–8, 18, 19, 38].

1.2. Maximal commutators

In this paper we are concerned with the continuity of maximal commutator and its fractional variant with Lipschitz symbols in Sobolev spaces. Let $0 \leq \alpha < n$ and b be a locally integral function defined on $\mathbb{R}^n$. The fractional maximal commutator with b is defined by

$$M_{b,\alpha}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} |b(x) - b(y)||f(y)|dy, \quad x \in \mathbb{R}^n.$$

Particularly, when $\alpha = 0$, $M_{b,\alpha}$ is the centered maximal commutator M_b.

We point out the following facts, which are useful in our proof of main result.

Remark 1.1. The operator $M_{b,\alpha}$ is positive and sublinear. Let $b \in L^\infty(\mathbb{R}^n)$, $1 < p < \infty$, $1/q = 1/p - \alpha/n$ and $0 \leq \alpha < n/p$. Then the map $M_{b,\alpha} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is bounded and continuous. To see this, observe that

$$M_{b,\alpha}f(x) \leq (|b(x)| + \|b\|_{L^\infty(\mathbb{R}^n)})M_\alpha f(x),$$

where M_α is the usual fractional maximal operator, i.e.,

$$M_\alpha f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} |f(y)|dy, \quad x \in \mathbb{R}^n.$$

It is well known that $M_\alpha : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is bounded and continuous. Applying the above observation and the bounds of M_α, we have

$$\|M_{b,\alpha}f\|_{L^q(\mathbb{R}^n)} \leq C_{\alpha,n,p}\|b\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}. \tag{1.1}$$

Combining (1.1) and the sublinearity of $M_{b,\alpha}$ implies the continuity of $M_{b,\alpha} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$.

The maximal commutator and its fractional variant has been the subject of many recent articles in harmonic analysis. The maximal commutator was first introduced by Garcia-Cuerva et al. [16] who used its L^p bounds to characterize $\mathrm{BMO}(\mathbb{R}^n)$ function. In fact, the maximal commutator plays an important role in the study of commutators of singular integral operators with BMO symbols (see, for instance [39,40]). Recently, Zhang [46,47] used the bounds of maximal commutators on the Lebesgue spaces and variable exponent Lebesgue spaces to characterize the Lipschitz space. The endpoint estimate of maximal commutator can be found in [1,3]. Recently, the investigation on the fractional maximal commutators has attracted the attention of many authors (see [12, 13, 20]). In fact, the investigation on various commutators has always been an active topic in harmonic analysis and PDE (see [11, 14, 15, 41, 43–45] et al.).

Very recently, the regularity of maximal commutators has also attracted a lot of attention. The first work was due to Liu, Xue and Zhang [33] who proved that M_b is bounded on $W^{1,p_1}(\mathbb{R}^n)$, where $1 < p_1, p_2, p < \infty$, $1/p = 1/p_1 + 1/p_2$ and $b \in W^{1,p_2}(\mathbb{R}^n)$. Later on, Liu and Xi [31] extended the above result to the fractional version and proved that $M_{b,\alpha}$ is bounded from $W^{1,p_1}(\mathbb{R}^n)$ to $W^{1,q}(\mathbb{R}^n)$ if $1 < p_1, p_2, p, p_1p_2/(p_1 + p_2) < \infty$, $0 \le \alpha < 1/p_1$, $1/q = 1/p_1 + 1/p_2 - \alpha$ and $b \in W^{1,p_2}(\mathbb{R}^n)$. Meanwhile, the second author and Wang [29] studied the Sobolev boundedness of $M_{b,\alpha}$ with b belonging to the inhomogeneous Lipschitz space $\mathrm{Lip}(\mathbb{R}^n)$. Here

$$\mathrm{Lip}(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{C} \text{ continuous} : \|f\|_{\mathrm{Lip}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\mathrm{Lip}(\mathbb{R}^n)} := \|f\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{Lip(\mathbb{R}^n)}$$

and

$$\|f\|_{Lip(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \sup_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f(x+h) - f(x)|}{|h|}.$$

The following presents the differentiable properties of the Lipschitz function.

Remark 1.2. Let $b \in \mathrm{Lip}(\mathbb{R}^n)$. It was pointed out in [29] that the weak partial derivatives D_ib, $i = 1, \ldots, n$, exist almost everywhere. Moreover, we have that $D_ib(x) = \lim_{h\to 0} \frac{b(x+he_i)-b(x)}{h}$ and $|D_ib(x)| \le \|b\|_{Lip(\mathbb{R}^n)}$ for almost every $x \in \mathbb{R}^n$. Here $e_l = (0, \ldots, 0, i, 0, \ldots, 0)$ is the canonical i-th base vector in $\mathbb{R}^n$ for $i = 1, \ldots, n$.

We now list the main result of [29] as follows.

Theorem A ([29]). *Let $1 < p < \infty$, $0 \le \alpha < n/p$ and $1/q = 1/p - \alpha/n$. If $b \in \mathrm{Lip}(\mathbb{R}^n)$, then the map $M_{b,\alpha} : W^{1,p}(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^n)$ is bounded.*

Based on Theorem A, a natural question was posed by Liu and Wang (see Remark 1.9 in [29]).

Question 1.3. Let $1 < p < \infty$, $0 \le \alpha < n/p$, $1/q = 1/p-\alpha/n$ and $b \in \mathrm{Lip}(\mathbb{R}^n)$. Is the map $M_{b,\alpha} : W^{1,p}(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^n)$ continuous?

This is one of main motivations of this work. We will give a positive answer to Question 1.3.

Theorem 1.4. *Let* $1 < p < \infty$, $0 \le \alpha < n/p$ *and* $1/q = 1/p - \alpha/n$. *Assume* $b \in \mathrm{Lip}(\mathbb{R}^n)$. *Then the map* $M_{b,\alpha} : W^{1,p}(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^n)$ *is continuous.*

1.3. The local case

The second one of main motivations is to establish the Sobolev continuity of maximal commutator and its fractional variant with Lipschitz symbols in the local setting. Let Ω be a subdomain in $\mathbb{R}^n$ and $0 \le \alpha < n$. For a locally integrable function b defined on Ω, we define the local fractional maximal commutator by

$$M_{b,\alpha,\Omega} f(x) = \sup_{0<r<\mathrm{dist}(x,\Omega^c)} \frac{1}{|B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} |b(x) - b(y)||f(y)|dy, \quad x \in \Omega,$$

where $\Omega^c = \mathbb{R}^n \setminus \Omega$. When $\alpha = 0$, $M_{b,\alpha,\Omega}$ reduces to the local maximal commutator $M_{b,\Omega}$.

We point out the following facts, which follow from [29].

Remark 1.5. Let $1 < p < \infty$, $0 \le \alpha < n/p$ and $1/q = 1/p - \alpha/n$. Assume $b \in L^\infty(\Omega)$. The following facts are valid:

(i) Let $M_{\alpha,\Omega}$ be the local fractional maximal operator, i.e.,

$$M_{\alpha,\Omega} f(x) = \sup_{0<r<\mathrm{dist}(x,\Omega^c)} \frac{1}{|B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} |f(y)|dy, \quad x \in \Omega.$$

It is known that the map $M_{\alpha,\Omega} : L^p(\Omega) \to L^q(\Omega)$ is bounded and continuous. Particularly, if $f \in L^p(\Omega)$, then

$$\|M_{\alpha,\Omega} f\|_{L^q(\Omega)} \le C_{\alpha,n,p} \|f\|_{L^p(\Omega)}.$$

(ii) The operator $M_{b,\alpha,\Omega}$ is positive and sublinear. Moreover, the map $M_{b,\alpha,\Omega} : L^p(\Omega) \to L^q(\Omega)$ is bounded and continuous. Particularly, if $f \in L^p(\Omega)$, then

$$\|M_{b,\alpha,\Omega} f\|_{L^q(\Omega)} \le C_{\alpha,n,p} \|b\|_{L^\infty(\Omega)} \|f\|_{L^p(\Omega)}.$$

The Sobolev regularity for maximal operators in local setting was originally studied by Kinnunen and Lindqvist [26] who established the boundedness of the map $M_\Omega : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ for $1 < p \le \infty$ (see also [22]). Here $W^{1,p}(\Omega)$ is given by

$$W^{1,p}(\Omega) := \{f : \Omega \to \mathbb{R} \,|\, \|f\|_{W^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)} < \infty\},$$

where $\nabla f = (D_1 f, \dots, D_n f)$ is the weak gradient of f. Kinnunen and Lindqvist's result was later extended to the fractional case in [24] and to the multilinear case in [23]. Recently, the Sobolev regularity for maximal commutators in local setting has also gotten a lot of attention. In 2021, Liu and Xi [31] first studied the Sobolev boundedness of local maximal commutators in the fractional setting. Very recently, Liu, Xue and Yabuta [32] established the following result.

Theorem B ([32]). *Let* $1 < p_1, p_2, p < \infty$, $1/p = 1/p_1 + 1/p_2$ *and* $b \in W^{1,p_2}(\Omega)$. *If* $|\Omega| < \infty$, *then the map* $M_{b,\Omega} : W^{1,p_1}(\Omega) \to W^{1,p}(\Omega)$ *is bounded and continuous.*

In [29], Liu and Wang introduced the local Lipschitz space $\mathrm{Lip}(\Omega)$ and established the Sobolev boundedness of local maximal commutator and its fractional variant with Lipschitz symbols. Recall that the inhomogeneous Lipschitz space $\mathrm{Lip}(\Omega)$ is given by

$$\mathrm{Lip}(\Omega) := \{f : \Omega \to \mathbb{C} \text{ continuous} : \|f\|_{\mathrm{Lip}(\Omega)} < \infty\},$$

where

$$\|f\|_{\mathrm{Lip}(\Omega)} := \|f\|_{L^\infty(\Omega)} + \|f\|_{Lip(\Omega)}$$

and

$$\|f\|_{Lip(\Omega)} := \sup_{x,\, y \in \Omega} \frac{|f(x) - f(y)|}{|x - y|}.$$

Remark 1.6. It was pointed out in [29] that if $b \in Lip(\Omega)$, then the weak partial derivatives $D_i b$, $i = 1, \ldots, n$, exist almost everywhere. Moreover, for almost every $x \in \Omega$ we have that $D_i b(x) = \lim_{h \to 0} \frac{b(x+he_i) - b(x)}{h}$ and $|D_i b(x)| \le \|b\|_{Lip(\Omega)}$.

We now present partial results of [29] as follows.

Theorem C ([29]). *Let* $b \in \mathrm{Lip}(\Omega)$.

- (i) *Let* $1 < p < \infty$. *Then the map* $M_{b,\Omega} : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ *is bounded.*
- (ii) *Let* $p \in (1, n)$, $\alpha \in [1, n/p)$ *and* $q = np/(n - (\alpha - 1)p)$. *Assume* $|\Omega| < \infty$. *Then the map* $M_{b,\alpha,\Omega} : W^{1,p}(\Omega) \to W^{1,q}(\Omega)$ *is bounded.*

Based on Theorems B and C, it is natural and interesting to ask the following.

Question 1.7. Let $b \in \mathrm{Lip}(\Omega)$, $1 < p < \infty$ and $|\Omega| < \infty$. Is the map $M_{b,\Omega} : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ continuous? What about $M_{b,\alpha,\Omega}$?

The above question can be addressed by the following theorem.

Theorem 1.8. *Let* $b \in \mathrm{Lip}(\Omega)$ *and* $|\Omega| < \infty$.

- (i) *Let* $1 < p < \infty$. *Then the map* $M_{b,\Omega} : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ *is continuous.*
- (ii) *Let* $p \in (1, n)$, $\alpha \in [1, n/p)$ *and* $q = np/(n - (\alpha - 1)p)$. *Then the map* $M_{b,\alpha,\Omega} : W^{1,p}(\Omega) \to W^{1,q}(\Omega)$ *is continuous.*

The rest of this paper is organized as follows. Section 2 will be devoted to presenting the proof of Theorem 1.4. The proof of Theorem 1.8 will be given in Section 3. It should be pointed out that the main ideas employed in the proofs of Theorems 1.4 and 1.8 are motivated by [29, 32, 34]. Due of the presence of $|b(x) - b(y)|$ in the integral average, the derivative formulas of maximal commutators is more complex. In fact, we have to make use of the global differentiable property of $|b(x) - b(y)|$ when we consider the partial derivatives of maximal commutators.

Throughout this paper, the letter C will stand for positive constants not necessarily the same one at each occurrence but is independent of the essential variables. Especially, the letter $C_{\alpha,\beta}$ denote the positive constants that depend on the parameters α, β. For a set $A \subset \mathbb{R}^n$, we denote $A^c = \mathbb{R}^n \setminus A$. For any arbitrary functions $F(x, y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$, we set

$$\nabla_x F = (D_{1,x}F, \ldots, D_{n,x}F), \quad \nabla_y F = (D_{1,y}F, \ldots, D_{n,y}F),$$

were $D_{i,x}F$ (resp., $D_{i,y}F$) is the i-th weak parital derivative of F in x (resp., y). Throughout this paper, let b be a local integrable function, we denote $F_b(x, y) = |b(x) - b(y)|$. For $l \in \{1, 2, \ldots, n\}$ and $h \in \mathbb{R} \setminus \{0\}$, we set

$$(F_{x,b})^l_h(y) = \frac{1}{h}(F_b(x, y+he_l) - F_b(x, y)), \ (F_{y,b})^l_h(x) = \frac{1}{h}(F_b(x+he_l, y) - F_b(x, y)).$$

2. Proof of Theorem 1.4

In this section we prove Theorem 1.4. The main ingredient of proving Theorem 1.4 is the derivative formulas of maximal commutators (see Lemma 2.3). Let us begin with some necessary notation and lemmas.

2.1. Preliminary notation and lemmas

Let $b \in L^\infty(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$ and $0 \le \alpha < n/p$. For a fixed point $x \in \mathbb{R}^n$, we define the function $A_{b,\alpha,x,f} : [0, \infty) \to \mathbb{R}$ by

$$A_{b,\alpha,x,f}(r) = \begin{cases} 0, & \text{if } r = 0; \\ \dfrac{1}{|B(x,r)|^{1-\alpha/n}} \displaystyle\int_{B(x,r)} |b(x) - b(y)||f(y)|dy, & \text{if } r \in (0, \infty). \end{cases}$$

We define the set $\mathcal{R}_{b,\alpha,f}(x)$ by

$$\mathcal{R}_{b,\alpha,f}(x) := \Big\{ r \ge 0 : M_{b,\alpha}f(x) = \limsup_{r_k \to r} A_{b,\alpha,x,f}(r_k) \text{ for some } r_k > 0 \Big\}.$$

It should be pointed out that the following facts are valid:

(i) $A_{b,\alpha,x,f}$ is continuous on $(0, \infty)$ for all $x \in \mathbb{R}^n$ and at $r = 0$ for almost every $x \in \mathbb{R}^n$;

(ii) $\lim_{r\to\infty} A_{b,\alpha,x,f}(r) = 0$ since

$$A_{b,\alpha,x,f}(r) \le (|b(x)| + \|b\|_{L^\infty(\mathbb{R}^n)})\|f\|_{L^p(\mathbb{R}^n)}|B(x, r)|^{\alpha/n-1/p}, \quad r > 0.$$

(iii) $\mathcal{R}_{b,\alpha,f}(x)$ is nonempty and closed for all $x \in \mathbb{R}^n$ and

$$M_{b,\alpha}f(x) = A_{b,\alpha,x,f}(r) \text{ if } 0 < r \in \mathcal{R}_{b,\alpha,f}(x), \ \forall x \in \mathbb{R}^n,$$

$$M_{b,\alpha}f(x) = A_{b,\alpha,x,f}(0) \text{ for almost every } x \in \mathbb{R}^n \text{ such that } 0 \in \mathcal{R}_{b,\alpha,f}(x).$$

Let u be a function defined on $\mathbb{R}^n$. For all $h \in \mathbb{R}$, $|h| > 0$, $y \in \mathbb{R}^n$ and $i = 1, 2, \ldots, n$, we define the functions u^i_h and $u_{h,i}$ by setting

$$u^i_h(x) = \frac{u(x + he_i) - u(x)}{h} \ \text{ and } \ u_{h,i}(x) = u(x + he_i).$$

It is well known that $u_{h,i} \to u$ in $L^p(\mathbb{R}^n)$ when $h \to 0$, and if $u \in W^{1,p}(\mathbb{R}^n)$, then $u_h^i \to D_i u$ in $L^p(\mathbb{R}^n)$ when $h \to 0$ (see [17, Section 7.11]).

For $R > 0$, we denote by B_R the ball of radius R centered at the origin. For $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we define

$$d(x, A) := \inf_{a \in A} |x - a| \quad \text{and } A_{(\lambda)} := \{x \in \mathbb{R}^n : d(x, A) \leq \lambda\} \quad \text{for } \lambda \geq 0.$$

We now prove the following lemma, which tells us how the sets $\mathcal{R}_{b,\alpha,f}(x)$ and $\mathcal{R}_{b,\alpha,g}(x)$ are related to each other when $\|f - g\|_{L^p(\mathbb{R}^n)}$ is small.

Lemma 2.1. *Let $b \in L^\infty(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$ and $0 \leq \alpha < n/p$. Assume that $f_j \to f$ in $L^p(\mathbb{R}^n)$ when $j \to \infty$. Then for all $R > 0$ and $\lambda > 0$, we have*

$$\lim_{j \to \infty} |\{x \in B(0, R) : \mathcal{R}_{b,\alpha,f_j}(x) \nsubseteq \mathcal{R}_{b,\alpha,f}(x)_{(\lambda)}\}| = 0. \tag{2.1}$$

Proof. We adopt the method of proving [34, Lemma 2.2] to prove this lemma. Let $\lambda > 0$, $R > 0$ and $\epsilon \in (0, 1)$. Applying the argument similar to those used to derive [34, Lemma 2.2], one can conclude that for any $j \in \mathbb{Z}$, the set $\{x \in \mathbb{R}^n : \mathcal{R}_{b,\alpha,f_j}(x) \nsubseteq \mathcal{R}_{b,\alpha,f}(x)_{(\lambda)}\}$ is measurable. Moreover, for almost every $x \in B(0, R)$, there exists $\gamma(x) \in \mathbb{N} \setminus \{0\}$ such that

$$A_{b,\alpha,x,f}(r) < M_{b,\alpha} f(x) - \frac{1}{\gamma(x)}, \quad \text{when } d(r, \mathcal{R}_{b,\alpha,f}(x)) > \lambda.$$

So we can find $\gamma = \gamma(R, \lambda, \epsilon) \in \mathbb{N} \setminus \{0\}$ and a measurable set E with $|E| < \epsilon$ such that

$$\begin{aligned} B(0, R) \subset \{x \in \mathbb{R}^n : &A_{b,\alpha,x,f}(r) < M_{b,\alpha} f(x) - \gamma^{-1} \\ &\text{if } d(r, \mathcal{R}_{b,\alpha,f}(x)) > \lambda\} \cup E. \end{aligned} \tag{2.2}$$

Observe that

$$\begin{aligned} &\{x \in \mathbb{R}^n : A_{b,\alpha,x,f}(r) < M_{b,\alpha} f(x) - \gamma^{-1} \ \text{ if } d(r, \mathcal{R}_{b,\alpha,f}(x)) > \lambda\} \\ &\subset \bigcup_{i=1}^{3} B_{i,j}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} B_{1,j} &:= \{x \in \mathbb{R}^n : |M_{b,\alpha} f_j(x) - M_{b,\alpha} f(x)| \geq (4\gamma)^{-1}\}, \\ B_{2,j} &:= \{x \in \mathbb{R}^n : |A_{b,\alpha,x,f_j}(r) - A_{b,\alpha,x,f}(r)| \geq (2\gamma)^{-1} \\ &\qquad \text{for some } r \text{ such that } d(r, \mathcal{R}_{b,\alpha,f}(x)) > \lambda\}, \\ B_{3,j} &:= \{x \in \mathbb{R}^n : A_{b,\alpha,x,f_j}(r) < M_{b,\alpha} f_j(x) - (4\gamma)^{-1} \ \text{ if } d(r, \mathcal{R}_{b,\alpha,f}(x)) > \lambda\}. \end{aligned}$$

Since $B_{3,j} \subset \{x \in \mathbb{R}^n : \mathcal{R}_{b,\alpha,f_j}(x) \subset \mathcal{R}_{b,\alpha,f}(x)_{(\lambda)}\}$, then we get from (2.2) and (2.3) that

$$\{x \in B_R : \mathcal{R}_{b,\alpha,f_j}(x) \nsubseteq \mathcal{R}_{b,\alpha,f}(x)_{(\lambda)}\} \subset B_{1,j} \cup B_{2,j} \cup E. \tag{2.4}$$

By our assumption, there exist $N_1 = N_1(\epsilon) \in \mathbb{N} \setminus \{0\}$ and $C > 0$ such that $\|f_j - f\|_{L^p(\mathbb{R}^n)} < \frac{\epsilon}{\gamma}$ for any $j \geq N_1$. By the sublinearity of $M_{b,\alpha}$,

$$|M_{b,\alpha} f_j(x) - M_{b,\alpha} f(x)| \leq M_{b,\alpha}(f_j - f)(x).$$

We also note that

$$|A_{b,\alpha,x,f_j}(r) - A_{b,\alpha,x,f}(r)| \leq M_{b,\alpha}(f_j - f)(x).$$

The above facts together with Remark 1.1 imply that for any $j \geq N_1$,

$$|B_{1,j}| + |B_{2,j}| \leq 2 \int_{\mathbb{R}^n} (4\gamma M_{b,\alpha}(f_j - f)(x))^q dx \leq C(\gamma \|f_j - f\|_{L^p(\mathbb{R}^n)})^q \leq C\epsilon^q,$$

where $1/q = 1/p - \alpha/n$ and $C > 0$ is independent of γ. In view of (2.4), we have

$$|\{x \in B(0,R) : \mathcal{R}_{b,\alpha,f_j}(x) \not\subseteq \mathcal{R}_{b,\alpha,f}(x)_{(\lambda)}\}| \leq C\epsilon$$

for any $j \geq N_1$. This proves (2.1). $\square$

Let A, B be two subsets of $\mathbb{R}^n$. The Hausdorff distance of A and B is defined by

$$\pi(A,B) := \inf\{\delta > 0 : A \subset B_{(\delta)} \text{ and } B \subset A_{(\delta)}\}.$$

The following lemma tells us how close the sets $\mathcal{R}_{b,\alpha,f}(x)$ and $\mathcal{R}_{b,\alpha,f}(x+he_l)$ are when h is small enough. This plays a key role in establishing the derivative formulas of maximal commutators.

Lemma 2.2. *Let $b \in \mathrm{Lip}(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$ with $1 < p < \infty$ and $0 \leq \alpha < n/p$. Then for all $R > 0$, $\lambda > 0$ and $l \in \{1, 2, \ldots, n\}$, we have*

$$\lim_{h \to 0} |\{x \in B(0,R) : \pi(\mathcal{R}_{b,\alpha,f}(x), \mathcal{R}_{b,\alpha,f}(x + he_l)) > \lambda\}| = 0. \tag{2.5}$$

Proof. Let us fix $l \in \{1, 2, \ldots, n\}$, $\lambda > 0$ and $R > 0$. For (2.5) it suffices to show that

$$\lim_{h \to 0} |\{x \in B(0,R) : \mathcal{R}_{b,\alpha,f}(x + he_l) \not\subseteq \mathcal{R}_{b,\alpha,f}(x)_{(\lambda)}\}| = 0 \tag{2.6}$$

and

$$\lim_{h \to 0} |\{x \in B(0,R) : \mathcal{R}_{b,\alpha,f}(x) \not\subseteq \mathcal{R}_{b,\alpha,f}(x + he_l)_{(\lambda)}\}| = 0. \tag{2.7}$$

We now prove (2.6). This is similar to the proof of Lemma 2.1, some modifications are made. By a change of variable, it is not difficult to see that $\mathcal{R}_{b,\alpha,f}(x + he_l) = \mathcal{R}_{b_{h,l},\alpha,f_{h,l}}(x)$. Hence, for (2.6) it is enough to show that

$$\lim_{h \to 0} |\{x \in B(0,R) : \mathcal{R}_{b_{h,l},\alpha,f_{h,l}}(x) \not\subseteq \mathcal{R}_{b,\alpha,f}(x)_{(\lambda)}\}| = 0. \tag{2.8}$$

By the proof of Lemma 2.1, we know that the set $\{x \in \mathbb{R}^n : \mathcal{R}_{b_{h,l},\alpha,f_{h,l}}(x) \not\subseteq \mathcal{R}_{b,\alpha,f}(x)_{(\lambda)}\}$ is measurable for any $h \in \mathbb{R}$. Moreover, there exist $\gamma = \gamma(R, \lambda, \epsilon) \in \mathbb{N} \setminus \{0\}$ and a measurable set E with $|E| < \epsilon$ such that

$$\begin{aligned} &B(0,R) \\ &\subset \{x \in \mathbb{R}^n : A_{b,\alpha,x,f}(r) < M_{b,\alpha} f(x) - \gamma^{-1} \ \text{ if } d(r, \mathcal{R}_{b,\alpha,f}(x)) > \lambda\} \cup E. \end{aligned} \tag{2.9}$$

Let $h \in \mathbb{R}$. Observe that

$$\begin{aligned}&\{x \in \mathbb{R}^n : A_{b,\alpha,x,f}(r) < M_{b,\alpha}f(x) - \gamma^{-1} \text{ if } d(r, \mathcal{R}_{b,\alpha,f}(x)) > \lambda\}\\ &\subset \bigcup_{i=1}^{3} B_{i,h},\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}B_{1,h} &:= \{x \in \mathbb{R}^n : |M_{b_{h,l},\alpha} f_{h,l}(x) - M_{b,\alpha} f(x)| \geq (4\gamma)^{-1}\},\\ B_{2,h} &:= \{x \in \mathbb{R}^n : |A_{b_{h,l},\alpha,x,f_{h,l}}(r) - A_{b,\alpha,x,f}(r)| \geq (2\gamma)^{-1}\\ &\qquad\qquad \text{for some } r \text{ such that } d(r, \mathcal{R}_{b,\alpha,f}(x)) > \lambda\},\\ B_{3,h} &:= \{x \in \mathbb{R}^n : A_{b_{h,l},\alpha,x,f_{h,l}}(r) < M_{b_{h,l},\alpha} f_{h,l}(x) - (4\gamma)^{-1}\\ &\qquad\qquad \text{if } d(r, \mathcal{R}_{b,\alpha,f}(x)) > \lambda\}.\end{aligned}$$

Note that $B_{3,h} \subset \{x \in \mathbb{R}^n : \mathcal{R}_{b_{h,l},\alpha,f_{h,l}}(x) \subset \mathcal{R}_{b,\alpha,f}(x)_{(\lambda)}\}$. Thus, we get from (2.9) and (2.10) that

$$\{x \in B(0,R) : \mathcal{R}_{b_{h,l},\alpha,f_{h,l}}(x) \nsubseteq \mathcal{R}_{b,\alpha,f}(x)_{(\lambda)}\} \subset B_{1,h} \cup B_{2,h} \cup E. \tag{2.11}$$

On the other hand, we have

$$\begin{aligned}&|M_{b_{h,l},\alpha} f_{h,l}(x) - M_{b,\alpha} f(x)|\\ &\leq \sup_{r>0} \frac{1}{|B(x,r)|^{1-\alpha/n}}\\ &\quad \times \int_{B(x,r)} ||b_{h,l}(x) - b_{h,l}(y)||f_{h,l}(y)| - |b(x) - b(y)||f(y)||dy\\ &\leq \sup_{r>0} \frac{1}{|B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} |b_{h,l}(x) - b_{h,l}(y)||f_{h,l}(y) - f(y)|dy\\ &\quad + \sup_{r>0} \frac{1}{|B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} ||b_{h,l}(x) - b_{h,l}(y)| - |b(x) - b(y)|||f(y)|dy\\ &\leq M_{b_{h,l},\alpha}(f_{h,l} - f)(x) + |b_{h,l}(x) - b(x)|M_\alpha f(x) + M_\alpha((b_{h,l} - b)f)(x)\\ &\leq M_{b_{h,l},\alpha}(f_{h,l} - f)(x) + 2\|b\|_{Lip(\mathbb{R}^n)}|h|M_\alpha f(x).\end{aligned}$$

Similarly we have

$$|A_{b_{h,l},\alpha,x,f_{h,l}}(r) - A_{b,\alpha,x,f}(r)| \leq M_{b_{h,l},\alpha}(f_{h,l} - f)(x) + 2\|b\|_{Lip(\mathbb{R}^n)}|h|M_\alpha f(x).$$

Since $f \in W^{1,p}(\mathbb{R}^n)$, there exists $\delta > 0$ such that $\|f_{h,l} - f\|_{L^p(\mathbb{R}^n)} < \frac{\epsilon}{\gamma}$ whenever $|h| < \delta$. The above facts together with (1.1) imply that when $|h| < \min\{\delta, (8\gamma)^{-1}\epsilon\}$,

$$\begin{aligned}&|B_{1,h}| + |B_{2,h}|\\ &\leq 2\int_{\mathbb{R}^n} (4\gamma(M_{b_{h,l},\alpha}(f_{h,l} - f)(x) + 2\|b\|_{Lip(\mathbb{R}^n)}|h|M_\alpha f(x)))^q dx\\ &\leq 2(4\gamma)^q \|M_{b_{h,l},\alpha}(f_{h,l} - f)\|^q_{L^q(\mathbb{R}^n)} + 2\|b\|^q_{Lip(\mathbb{R}^n)}\epsilon^q \|M_\alpha f\|^q_{L^q(\mathbb{R}^n)}\end{aligned}$$

$$\leq C_{\alpha,n,p}(2(4\gamma)^q\|b_{h,l}\|_{L^\infty(\mathbb{R}^n)}\|f_{h,l}-f\|^q_{L^p(\mathbb{R}^n)}+2\|b\|^q_{Lip(\mathbb{R}^n)}\|f\|^q_{L^p(\mathbb{R}^n)}\epsilon^q)$$
$$\leq C_{\alpha,n,p}\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\epsilon^q,$$

where $1/q = 1/p - \alpha/n$ and $C_{\alpha,n,p} > 0$ is independent of γ and ϵ. Hence, we get from (2.11) that

$$\{x \in B(0,R) : \mathcal{R}_{b_{h,l},\alpha,f_{h,l}}(x) \nsubseteq \mathcal{R}_{b,\alpha,f}(x)_{(\lambda)}\} \leq C\epsilon$$

when $|h| < \min\{\delta, (8\gamma)^{-1}\epsilon\}$. Here $C > 0$ is independent of γ and ϵ. This proves (2.8).

It remains to prove (2.7). Note that $\mathcal{R}_{b,\alpha,f}(x) = \mathcal{R}_{b_{-h,l},\alpha,f_{-h,l}}(x + he_l)$. It follows that

$$\begin{aligned}&\{x \in B(0,R) : \mathcal{R}_{b,\alpha,f}(x) \nsubseteq \mathcal{R}_{b,\alpha,f}(x+he_l)_{(\lambda)}\}\\ \subsetneq\ &\{x \in B(0,R) : \mathcal{R}_{b_{-h,l},\alpha,f_{-h,l}}(x+he_l) \nsubseteq \mathcal{R}_{b,\alpha,f}(x+he_l)_{(\lambda)}\}\\ \subsetneq\ &\{x \in B(0,R+1) : \mathcal{R}_{b_{-h,l},\alpha,f_{-h,l}}(x) \nsubseteq \mathcal{R}_{b,\alpha,f}(x)_{(\lambda)}\} - he_l.\end{aligned}$$

This together with (2.8) gives (2.7). □

We now establish the derivative formulas of maximal commutators, which is the main ingredient of the proof of Theorem 1.4.

Lemma 2.3. *Let $b \in \mathrm{Lip}(\mathbb{R}^n)$ and $f \in W^{1,p}(\mathbb{R}^n)$ with $1 < p < \infty$ and $0 \leq \alpha < n/p$. Let $l \in \{1, 2, \ldots, n\}$. Then*

(i) *For almost every $x \in \mathbb{R}^n$ and $r \in \mathcal{R}_{b,\alpha,f}(x)$ with $0 < r < \infty$, we have*

$$\begin{aligned}D_lM_{b,\alpha}f(x) = &\frac{1}{|B(x,r)|^{1-\alpha/n}}\int_{B(x,r)}(D_{l,y}|b(x)-b(y)|\\ &+ D_{l,x}|b(x)-b(y)|)|f(y)|dy\\ &+\frac{1}{|B(x,r)|^{1-\alpha/n}}\int_{B(x,r)}|b(x)-b(y)|D_l|f|(y)dy.\end{aligned} \tag{2.12}$$

(ii) *For almost every $x \in \mathbb{R}^n$ and $0 \in \mathcal{R}_{b,\alpha,f}(x)$, we have*

$$D_lM_{b,\alpha}f(x) = 0. \tag{2.13}$$

Proof. Without loss of generality we may assume that $f \geq 0$, since $M_{b,\alpha}f = M_{b,\alpha}|f|$ and $|f| \in W^{1,p}(\mathbb{R}^n)$ if $f \in W^{1,p}(\mathbb{R}^n)$. Let A_1 be the set of all $x \in \mathbb{R}^n$ for which b is differentiable at x. By Remark 1.2 we see that $|A_1^c| = 0$. Let $q = pn/(n-\alpha p)$ and A_2 be the set of all $x \in \mathbb{R}^n$ for which $M_\alpha f(x) < \infty$ and $M_\alpha D_lf(x) < \infty$. Clearly, it follows from the boundedness of M_α that $|A_2^c| = 0$. Let $R > 0$. Invoking Lemma 2.2, there exists a sequence $\{s_k\}_{k=1}^\infty$, $s_k > 0$ and $s_k \to 0$ such that $\lim_{k\to\infty}\pi(\mathcal{R}_{b,\alpha,f}(x), \mathcal{R}_{b,\alpha,f}(x+s_ke_l)) = 0$ for almost every $x \in B(0,R)$. Since $f \in W^{1,p}(\mathbb{R}^n)$, then $\|f_{s_k,l} - f\|_{L^p(\mathbb{R}^n)} \to 0$ and $\|f^l_{s_k} - D_lf\|_{L^p(\mathbb{R}^n)} \to 0$ as $k \to \infty$. It follows that $\|M_\alpha(f_{s_k,l}-f)\|_{L^q(\mathbb{R}^n)} \to 0$ and $\|M_\alpha(f^l_{s_k} - D_lf)\|_{L^q(\mathbb{R}^n)} \to 0$ as $k \to \infty$. We get by Theorem A that $M_{b,\alpha}f \in W^{1,q}(\mathbb{R}^n)$. It follows that $\|(M_{b,\alpha}f)^l_{s_k} - D_lM_{b,\alpha}f\|_{L^q(\mathbb{R}^n)} \to 0$ as $k \to \infty$. Note that $|b^l_{s_k}(y) - D_lb(y)|f(y) \leq 2\|b\|_{Lip(\mathbb{R}^n)}f(y)$ and $\lim_{k\to\infty} b^l_{s_k}(y) = D_lb(y)$

for almost every $y \in \mathbb{R}^n$. By the dominated convergence theorem, we have $\|(b^l_{s_k} - D_l b)f\|_{L^p(\mathbb{R}^n)} \to 0$ as $k \to \infty$. It follows that

$$\|M_\alpha((b^l_{s_k} - D_l b)f)\|_{L^q(\mathbb{R}^n)} \to 0 \quad \text{as} \quad k \to \infty.$$

It is clear that $F_b(x, \cdot) \in Lip(\mathbb{R}^n)$ and $\|F_b(x, \cdot)\|_{Lip(\mathbb{R}^n)} \leq \|b\|_{Lip(\mathbb{R}^n)}$ for all $x \in \mathbb{R}^n$. By Remark 1.2, for a fixed $x \in \mathbb{R}^n$ the function $F_b(x, \cdot)$ is differentiable almost every $y \in \mathbb{R}^n$. Moreover, for almost every $y \in \mathbb{R}^n$, we have $|D_{l,y}F_b(x, y)| \leq \|b\|_{Lip(\mathbb{R}^n)}$. Similarly we see that $F_b(\cdot, y) \in Lip(\mathbb{R}^n)$ and $\|F_b(\cdot, y)\|_{Lip(\mathbb{R}^n)} \leq \|b\|_{Lip(\mathbb{R}^n)}$ for all $y \in \mathbb{R}^n$. Moreover, for a fixed $y \in \mathbb{R}^n$ the function $F_b(\cdot, y)$ is differentiable almost everywhere. Moreover, for almost every $x \in \mathbb{R}^n$, we have $|D_{l,x}F_b(x, y)| \leq \|b\|_{Lip(\mathbb{R}^n)}$. By the above facts, there exist a subsequence $\{h_k\}_{k=1}^\infty$ of $\{s_k\}_{k=1}^\infty$ and a measurable set $A_3 \subset B(0, R)$ with $|B(0, R) \setminus A_3| = 0$ such that for any $x \in A_1$, we have that $\lim_{k\to\infty} M_\alpha(f_{h_k,l} - f)(x) = 0$, $\lim_{k\to\infty} M_\alpha(f^l_{h_k} - D_l f)(x) = 0$, $\lim_{k\to\infty} M_\alpha((b^l_{h_k} - D_l b)f)(x) = 0$, $\lim_{k\to\infty}(M_{b,\alpha}f)^l_{h_k}(x) = D_l M_{b,\alpha}(f)(x)$, $\lim_{k\to\infty} \pi(\mathcal{R}_{b,\alpha,f}(x), \mathcal{R}_{b,\alpha,f}(x + h_k e_l)) = 0$, $\lim_{k\to\infty}(F_{y,b})^l_{h_k}(x) = D_{l,x}F_b(x, y)$ for any $y \in \mathbb{R}^n$. Let A_4 be the set of all Lebesgue points of f and $D_l f$. We set

$$A_5 := \{x \in \mathbb{R}^n : M_{b,\alpha}f(x) = A_{b,\alpha,x,f}(0) \text{ if } 0 \in \mathcal{R}_{b,\alpha,f}(x)\},$$

$$A_6 := \bigcap_{k=1}^{\infty} \{x \in \mathbb{R}^n : M_{b,\alpha}f(x+h_k e_l) = A_{b,\alpha,x+h_k e_l,f}(0) \text{ if } 0 \in \mathcal{R}_{b,\alpha,f}(x+h_k e_l)\},$$

$$A_7 := \Big\{x \in \mathbb{R}^n : \lim_{k\to\infty} (F_{y,b})^l_{h_k}(x) = D_{l,x}F_b(x, y), \quad \text{a.e. } y \in \mathbb{R}^n\Big\},$$

$$A_8 := \Big\{x \in A_1 : \lim_{r\to 0^+} \frac{1}{|B(x, r)|} \int_{B(x,r)} |D_l b(x) - D_l b(y)|f(y)dy = 0\Big\}.$$

One can easily check that $|A^c_i| = 0$ for any $i = 4, 5, 6, 7$. Note that $|D_l b(x) - D_l b(y)|f(y) \leq 2\|b\|_{Lip(\mathbb{R}^n)}f$ for any $x \in A_1$ and almost every $y \in \mathbb{R}^n$. Applying the Lebesgue differentiation theorem, we see that $|A^c_8| = 0$. Hence, we have $|(\bigcap_{j=1}^8 A_j)^c| = 0$.

Let $x \in \bigcap_{j=1}^8 A_j$ and $r \in \mathcal{R}_{b,\alpha,f}(x)$. Then there exists $r_k \in \mathcal{R}_{b,\alpha,f}(x + h_k e_l)$ such that $\lim_{k\to\infty} r_k = r$. It is easy to see that

$$D_l M_{b,\alpha}f(x) = \lim_{k\to\infty} \frac{1}{h_k}(M_{b,\alpha}f(x + h_k e_l) - M_{b,\alpha}f(x)). \tag{2.14}$$

We consider two cases:

Case A ($r > 0$). Due to the fact that $\lim_{k\to\infty} r_k = r$, without loss of generality we may assume that $r_k \in (0, 2r)$ for all $k \geq 1$. By a change of variable, we have

$$A_{b,\alpha,x+h_k e_l,f}(r_k) = \frac{1}{|B(x, r_k)|^{1-\alpha/n}} \int_{B(x,r_k)} |b(x+h_k e_l) - b(y+h_k e_l)||f_{h_k,l}(y)|dy.$$

It follows that

$$\begin{aligned}
&\frac{1}{h_k}(M_{b,\alpha}f(x+h_ke_l)-M_{b,\alpha}f(x))\\
&\le \frac{1}{h_k}(A_{b,\alpha,x+h_ke_l,f}(r_k)-A_{b,\alpha,x,f}(r_k))\\
&=\frac{1}{h_k}\frac{1}{|B(x,r_k)|^{1-\alpha/n}}\\
&\quad\times\int_{B(x,r_k)}(F_b(x+h_ke_l,y+h_ke_l)f_{h_k,l}(y)-F_b(x,y)f(y))dy\\
&=\frac{1}{|B(x,r_k)|^{1-\alpha/n}}\int_{B(x,r_k)}F_b(x,y)f_{h_k}^l(y)dy+\frac{1}{|B(x,r_k)|^{1-\alpha/n}}\\
&\quad\times\int_{B(x,r_k)}\frac{F_b(x+h_ke_l,y+h_ke_l)-F_b(x,y)}{h_k}f_{h_k,l}(y)dy.
\end{aligned}\tag{2.15}$$

Note that

$$\begin{aligned}
&\Big|\int_{B(x,r_k)}F_b(x,y)f_{h_k}^l(y)dy-\int_{B(x,r)}F_b(x,y)D_lf(y)dy\Big|\\
&\le \int_{B(x,r_k)}F_b(x,y)|f_{h_k}^l(y)-D_lf(y)|dy\\
&\quad+\Big|\int_{B(x,r_k)}F_b(x,y)D_lf(y)dy-\int_{B(x,r)}F_b(x,y)D_lf(y)dy\Big|\\
&\le \|b\|_{Lip(\mathbb{R}^n)}r_k|B(x,r_k)|^{1-\alpha/n}M_\alpha(f_{h_k}^l-D_lf)(x)\\
&\quad+\|b\|_{Lip(\mathbb{R}^n)}r_k\int_{B(x,2r)}|D_lf(y)||(\chi_{B(x,r_k)}(y)-\chi_{B(x,r)}(y))|dy.
\end{aligned}$$

By the Hölder's inequality, we see that $D_lf\in L^1(B(x,2r))$. Applying the dominated convergence theorem,

$$\lim_{k\to\infty}\int_{B(x,r_k)}F_b(x,y)f_{h_k}^l(y)dy=\int_{B(x,r)}F_b(x,y)D_lf(y)dy.\tag{2.16}$$

On the other hand, we have

$$\begin{aligned}
&\int_{B(x,r_k)}\frac{F_b(x+h_ke_l,y+h_ke_l)-F_b(x,y)}{h_k}f_{h_k,l}(y)dy\\
&=\int_{B(x,r_k)}(F_{x,b})_{h_k}^l(y)f_{h_k,l}(y)dy\\
&\quad+\int_{B(x,r_k)}(F_{y+h_ke_l,b})_{h_k}^l(x)f_{h_k,l}(y)dy.
\end{aligned}\tag{2.17}$$

Note that

$$\begin{aligned}
&\Big|\int_{B(x,r_k)}(F_{x,b})^l_{h_k}(y)f_{h_k,l}(y)dy-\int_{B(x,r)}D_{l,y}F_b(x,y)f(y)dy\Big|\\
&\leq \int_{B(x,r_k)}|(F_{x,b})^l_{h_k}(y)f_{h_k,l}(y)-D_{l,y}F_b(x,y)f(y)|dy\\
&\leq \int_{B(x,r_k)}|(F_{x,b})^l_{h_k}(y)||f_{h_k,l}(y)-f(y)|dy\\
&\quad+\int_{B(x,r_k)}|(F_{x,b})^l_{h_k}(y)-D_{l,y}F_b(x,y)||f(y)|dy.
\end{aligned}\tag{2.18}$$

Observe that

$$\begin{aligned}
&\int_{B(x,r_k)}|(F_{x,b})^l_{h_k}(y)||f_{h_k,l}(y)-f(y)|dy\\
&\leq \|b\|_{Lip(\mathbb{R}^n)}|B(x,r_k)|^{1-\alpha/n}M_\alpha(f_{h_k,l}-f)(x)\to 0 \quad\text{as}\quad k\to\infty.
\end{aligned}$$

Note that $((F_{x,b})^l_{h_k}(y)-D_{l,y}F_b(x,y))\chi_{B(x,r_k)}(y)\to 0$ as $k\to\infty$ for almost every $y\in\mathbb{R}^n$. Moreover, $|(F_{x,b})^l_{h_k}(y)-D_{l,y}F_b(x,y)||f(y)|\leq 2\|b\|_{Lip(\mathbb{R}^n)}|f(y)|$ for almost every $y\in\mathbb{R}^n$. These facts together with the fact that $f\in L^1(B(x,2r))$ and the dominated convergence theorem imply

$$\int_{B(x,r)}|(F_{x,b})^l_{h_k}(y)-D_{l,y}F_b(x,y)||f(y)|dy\to 0\quad\text{as}\quad k\to\infty.$$

Hence, we get from (2.18) that

$$\lim_{k\to\infty}\int_{B(x,r_k)}(F_{x,b})^l_{h_k}(y)f_{h_k,l}(y)dy=\int_{B(x,r)}D_{l,y}F_b(x,y)f(y)dy.\tag{2.19}$$

We now prove that

$$\lim_{k\to\infty}\int_{B(x,r_k)}(F_{y+h_ke_l,b})^l_{h_k}(x)f_{h_k,l}(y)dy=\int_{B(x,r)}D_{l,x}F_b(x,y)f(y)dy.\tag{2.20}$$

By a change of variable,

$$\int_{B(x,r_k)}(F_{y+h_ke_l,b})^l_{h_k}(x)f_{h_k,l}(y)dy=\int_{B(x+h_ke_l,r_k)}(F_{y,b})^l_{h_k}(x)f(y)dy.$$

Hence, for (2.20), it suffices to show that

$$\lim_{k\to\infty}\int_{B(x+h_ke_l,r_k)}(F_{y,b})^l_{h_k}(x)f(y)dy=\int_{B(x,r)}D_{l,x}F_b(x,y)f(y)dy.\tag{2.21}$$

Due to $\lim_{k\to\infty}h_k=0$, without loss of generality we may assume that $h_k\leq r$ for all $k\geq 1$. Note that $B(x+h_ke_l,r_k)\subset B(x,3r)$, $|(F_{y,b})^l_{h_k}(x)|\leq\|b\|_{Lip(\mathbb{R}^n)}$ and $f\in L^1(B(x,3r))$. An application of the dominated convergence theorem gives

$$\Big|\int_{B(x+h_ke_l,r_k)}(F_{y,b})^l_{h_k}(x)f(y)dy-\int_{B(x,r)}(F_{y,b})^l_{h_k}(x)f(y)dy\Big|$$

$$\leq \|b\|_{Lip(\mathbb{R}^n)} \int_{B(x,3r)} |f(y)(\chi_{B(x+h_k e_l, r_k)}(y) - \chi_{B(x,r)}(y))| dy$$
$$\to 0 \quad \text{as} \quad k \to \infty.$$

Thus, to prove (2.20), it is enough to show that

$$\lim_{k\to\infty} \int_{B(x,r)} (F_{y,b})^l_{h_k}(x) f(y) dy = \int_{B(x,r)} D_{l,x} F_b(x,y) f(y) dy. \tag{2.22}$$

Note that $\lim_{k\to\infty} (F_{y,b})^l_{h_k}(x) = D_{l,x} F_b(x,y)$ and $|(F_{y,b})^l_{h_k}(x) - D_{l,x}F_b(x,y)| \leq 2\|b\|_{Lip(\mathbb{R}^n)}$. These facts together with the fact that $f \in L^1(B(x,r))$ and the dominated convergence theorem imply (2.22). It follows from (2.17), (2.19) and (2.20) that

$$\begin{aligned} &\lim_{k\to\infty} \int_{B(x,r_k)} \frac{F_b(x+h_k e_l, y+h_k e_l) - F_b(x,y)}{h_k} f_{h_k,l}(y) dy \\ &= \int_{B(x,r)} (D_{l,y} F_b(x,y) + D_{l,x} F_b(x,y)) f(y) dy. \end{aligned} \tag{2.23}$$

Then we get from (2.14)-(2.16) and (2.23) that

$$\begin{aligned} D_l M_{b,\alpha} f(x) \leq &\frac{1}{|B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} (D_{l,y}|b(x) - b(y)| \\ &+ D_{l,x}|b(x) - b(y)|) f(y) dy \\ &+ \frac{1}{|B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} |b(x) - b(y)| D_l f(y) dy. \end{aligned} \tag{2.24}$$

On the other hand, we have

$$\begin{aligned} &\frac{1}{h_k}(M_{b,\alpha} f(x+h_k e_l) - M_{b,\alpha} f(x)) \\ &\geq \frac{1}{h_k}(A_{b,\alpha,x+h_k e_l,f}(r) - A_{b,\alpha,x,f}(r)) \\ &= \frac{1}{h_k} \frac{1}{|B(x,r)|^{1-\alpha/n}} \\ &\quad \times \int_{B(x,r)} (F_b(x+h_k e_l, y+h_k e_l) f_{h_k,l}(y) - F_b(x,y) f(y)) dy \\ &= \frac{1}{|B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} F_b(x,y) f^l_{h_k}(y) dy \\ &\quad + \frac{1}{|B(x,r)|^{1-\alpha/n}} \\ &\quad \times \int_{B(x,r)} \frac{F_b(x+h_k e_l, y+h_k e_l) - F_b(x,y)}{h_k} f_{h_k,l}(y) dy. \end{aligned} \tag{2.25}$$

By (2.14), (2.25) and the arguments similar to those used to derive (2.24),

$$\begin{aligned}&D_l M_{b,\alpha}(f)(x)\\ &\geq \frac{1}{|B(x,r)|^{1-\alpha/n}}\int_{B(x,r)}(D_{l,y}(|b(x)-b(y)|)+D_{l,x}(|b(x)-b(y)|)f(y)dy\\ &\quad+\frac{1}{|B(x,r)|^{1-\alpha/n}}\int_{B(x,r)}|b(x)-b(y)|D_l f(y)dy.\end{aligned}$$

This together with (2.24) implies that (2.12) holds for almost every $x\in B(0,R)$.

Case B ($r=0$). Due to $0\in\mathcal{R}_{b,\alpha,f}(x)$, we have $M_{b,\alpha}f(x)=A_{b,\alpha,x,f}(0)=0$. Then we get from (2.14) that

$$D_l M_{b,\alpha}f(x)=\lim_{k\to\infty}\frac{1}{h_k}M_{b,\alpha}f(x+h_ke_l)=\lim_{k\to\infty}\frac{1}{h_k}A_{b,\alpha,x+h_ke_l,f}(r_k). \tag{2.26}$$

If $r_k=0$ for infinitely many k, then (2.26) gives $D_lM_{b,\alpha}f(x)=0$. In what follows, without loss of generality we may assume that $r_k\in(0,1)$ for all $k\geq 1$. Since $M_{b,\alpha}f(x)=0$, then $|b(x)-b(y)|f(y)=0$ for almost every $y\in\mathbb{R}^n$. It follows that

$$\begin{aligned}&\int_{B(x+h_ke_l,r_k)}|b(x+h_ke_l)-b(y)|f(y)dy\\ &=\int_{B(x,r_k)}|b(x+h_ke_l)-b(y+h_ke_l)|f(y+h_ke_l)dy\\ &\leq\int_{B(x,r_k)}|b(x+h_ke_l)-b(y+h_ke_l)||f(y+h_ke_l)-f(y)|dy\\ &\quad+\int_{B(x,r_k)}|b(x+h_ke_l)-b(y+h_ke_l)-(b(x)-b(y))|f(y)dy.\end{aligned}$$

It follows that

$$\begin{aligned}&\frac{1}{h_k}A_{b,\alpha,x+h_ke_l,f}(r_k)\\ &\leq\frac{1}{|B(x,r_k)|^{1-\alpha/n}}\int_{B(x,r_k)}|b(x+h_ke_l)-b(y+h_ke_l)||f_{h_k}^l(y)|dy\\ &\quad+\frac{1}{|B(x,r_k)|^{1-\alpha/n}}\int_{B(x,r_k)}|b_{h_k}^l(x)-b_{h_k}^l(y)|f(y)dy.\end{aligned} \tag{2.27}$$

Observe that

$$\begin{aligned}&\frac{1}{|B(x,r_k)|^{1-\alpha/n}}\int_{B(x,r_k)}|b(x+h_ke_l)-b(y+h_ke_l)||f_{h_k}^l(y)|dy\\ &\leq\|b\|_{Lip(\mathbb{R}^n)}r_k(M_\alpha(f_{h_k}^l-D_lf)(x)+M_\alpha(D_lf)(x))\\ &\to 0\ \text{ as }\ k\to\infty.\end{aligned} \tag{2.28}$$

We write

$$\frac{1}{|B(x,r_k)|^{1-\alpha/n}}\int_{B(x,r_k)}|b_{h_k}^l(x)-b_{h_k}^l(y)|f(y)dy$$

$$
\begin{aligned}
&\leq \frac{1}{|B(x,r_k)|^{1-\alpha/n}}\int_{B(x,r_k)}|b_{h_k}^l(x)-b_{h_k}^l(y)-(D_lb(x)-D_lb(y))|f(y)dy\\
&\quad+\frac{1}{|B(x,r_k)|^{1-\alpha/n}}\int_{B(x,r_k)}|D_lb(x)-D_lb(y)|f(y)dy\\
&\leq |b_{h_k}^l(x)-D_lb(x)|M_\alpha f(x)+M_\alpha((b_{h_k}^l-D_lb)f)(x)\\
&\quad+\frac{1}{|B(x,r_k)|^{1-\alpha/n}}\int_{B(x,r_k)}|D_lb(x)-D_lb(y)|f(y)dy\\
&\to 0 \quad \text{as} \quad k\to\infty.
\end{aligned}
$$

This together with (2.26)-(2.28) implies $D_lM_{b,\alpha}f(x)=0$. Since R was arbitrary and $|B(0,R)\setminus(\bigcap_{j=1}^8 A_j)|=0$. This proves Lemma 2.3. □

Finally, we present an important property of $W^{1,p}(\mathbb{R}^n)$ with $p>1$. This play a key role in the proof of Theorem 1.4.

Lemma 2.4. *Let $p>1$, $f\in W^{1,p}(\mathbb{R}^n)$ and $\{f_j\}_{j\geq1}\subset W^{1,p}(\mathbb{R}^n)$. Assume that $f_j\to f$ in $W^{1,p}(\mathbb{R}^n)$ as $j\to\infty$. Then $\nabla|f_j|\to\nabla|f|$ in $L^p(\mathbb{R}^n)$ as $j\to\infty$.*

Proof. It was shown in [37, Theorem 6.17] that if $u\in W^{1,p}(\mathbb{R}^n)$ for $p>1$, then $|u|\in W^{1,p}(\mathbb{R}^n)$. Moreover, we have

$$
\nabla|g|=\begin{cases}\nabla g, & \text{in } \{x\in\mathbb{R}^n: g(x)>0\} \ \text{ a.e.;}\\ 0, & \text{in } \{x\in\mathbb{R}^n: g(x)=0\} \ \text{ a.e.;}\\ -\nabla g, & \text{in } \{x\in\mathbb{R}^n: g(x)<0\} \ \text{ a.e.}\end{cases}
$$

Define the sets

$$
\begin{gathered}
X=\{x\in\mathbb{R}^n: f(x)>0\},\ Y=\{x\in\mathbb{R}^n: f(x)=0\},\ Z=\{x\in\mathbb{R}^n: f(x)<0\},\\
X_j=\{x\in\mathbb{R}^n: f_j(x)>0\},\quad Y_j=\{x\in\mathbb{R}^n: f_j(x)=0\},\quad \forall j\geq1,\\
Z_j=\{x\in\mathbb{R}^n: f_j(x)<0\},\quad \forall j\geq1.
\end{gathered}
$$

Observe that

$$
\begin{aligned}
&\int_X|\nabla|f_j|(y)-\nabla|f|(y)|^pdy\\
&=\int_X|\nabla|f_j|(y)-\nabla f(y)|^pdy\\
&=\int_{X\cap(X_j\cup Y_j)}|\nabla|f_j|(y)-\nabla f(y)|^pdy+\int_{X\cap Z_j}|\nabla|f_j|(y)-\nabla f(y)|^pdy\\
&=\int_{X\cap(X_j\cup Y_j)}|\nabla f_j(y)-\nabla f(y)|^pdy+\int_{X\cap Z_j}|-\nabla f_j(y)-\nabla f(y)|^pdy\\
&\leq\int_X|\nabla f_j(y)-\nabla f(y)|^pdy\\
&\quad+C_p\Big(\int_{X\cap Z_j}|\nabla f_j(y)-\nabla f(y)|^pdy+\int_{X\cap Z_j}|\nabla f(y)|^pdy\Big)
\end{aligned}
$$

$$\leq C_p\Big(\|\nabla f_j-\nabla f\|^p_{L^p(\mathbb{R}^n)}+\int_{X\cap Z_j}|\nabla f(y)|^p dy\Big).$$

The assumption implies that $\|\nabla f_j-\nabla f\|^p_{L^p(\mathbb{R}^n)}\to 0$ as $j\to\infty$. We now prove that

$$\int_{X\cap Z_j}|\nabla f(y)|^p dy\to 0 \ \text{ as } \ j\to\infty. \tag{2.29}$$

Assume that (2.29) is not true, there exist $\epsilon_0>0$ and a subsequence $\{j_k\}_{k\geq 1}$ such that

$$\int_{X\cap Z_{j_k}}|\nabla f(y)|^p dy\geq \epsilon_0.$$

Passing to a further subsequence, if necessary, we may assume that $f_{j_k}\to f$ pointwise a.e. Note that

$$\int_{X\cap Z_{j_k}}|\nabla f(y)|^p dy=\int_{\mathbb{R}^n}|\nabla f(y)|^p\chi_{X\cap Z_{j_k}}(y)dy.$$

We also note that $\chi_{X\cap Z_{j_k}}(y)\to 0$ as $k\to\infty$ for almost every $y\in\mathbb{R}^n$. In fact, for almost every $y\in Z_{j_k}$ we see that $f_{j_k}(y)<0$. It follows that $f(y)\leq 0$ by our assumption. So $y\notin X$. By using dominated convergence one sees that

$$\int_{X\cap Z_{j_k}}|\nabla f(y)|^p dy=\int_{\mathbb{R}^n}|\nabla f(y)|^p\chi_{X\cap Z_{j_k}}(y)dy\to 0 \ \text{ as } \ k\to\infty,$$

which is a contradiction. Hence, (2.29) holds. So we have

$$\int_X|\nabla|f_j|(y)-\nabla|f|(y)|^p dy\to 0 \ \text{ as } \ j\to\infty.$$

Similarly we can prove that

$$\int_Y|\nabla|f_j|(y)-\nabla|f|(y)|^p dy\to 0 \ \text{ as } \ j\to\infty,$$

$$\int_Z|\nabla|f_j|(y)-\nabla|f|(y)|^p dy\to 0 \ \text{ as } \ j\to\infty.$$

This concludes the proof. □

2.2. Proof of Theorem 1.4

Let $b\in\mathrm{Lip}(\mathbb{R}^n)$ and $f\in W^{1,p}(\mathbb{R}^n)$ for $1<p<\infty$. Let $0\leq\alpha<n/p$ and $1/q=1/p-\alpha/n$. Let $\{f_j\}_{j\geq 1}\subset W^{1,p}(\mathbb{R}^n)$ and $f_j\to f$ in $W^{1,p}(\mathbb{R}^n)$ as $j\to\infty$. We want to show that

$$\|M_{b,\alpha}f_j-M_{b,\alpha}f\|_{W^{1,q}(\mathbb{R}^n)}\to 0 \ \text{ as } \ j\to\infty. \tag{2.30}$$

By Remark 1.1 we have that $\|M_{b,\alpha}f_j-M_{b,\alpha}f\|_{L^q(\mathbb{R}^n)}\to 0$ as $k\to\infty$. Hence, for (2.30) it is enough to prove that

$$\|D_lM_{b,\alpha}f_j-D_lM_{b,\alpha}f\|_{L^q(\mathbb{R}^n)}\to 0 \ \text{ when } j\to\infty \tag{2.31}$$

for any $l = 1, 2, \dots, n$. We only work with (2.31) for $l = 1$ and other cases are analogous. In view of Lemma 2.4, we may assume that all $f_j \geq 0$ and $f \geq 0$.

For convenience, for a fixed $x \in \mathbb{R}^n$ and arbitrary function $g \in W^{1,p}(\mathbb{R}^n)$, we define the function $u_{b,\alpha,x,g} : [0, \infty) \to \mathbb{R}$ by

$$u_{b,\alpha,x,g}(0) = 0,$$

$$\begin{aligned} &u_{b,\alpha,x,g}(r) \\ &= \frac{1}{|B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} (D_{1,y}|b(x) - b(y)| + D_{1,x}|b(x) - b(y)|)g(y)dy \\ &\quad + \frac{1}{|B(x,r)|^{1-\alpha/n}} \int_{B(x,r)} |b(x) - b(y)| D_1 g(y)dy, \quad 0 < r < \infty. \end{aligned}$$

We define the operator T_α by

$$T_\alpha(g)(x) = 2\|b\|_{Lip(\mathbb{R}^n)} M_\alpha g(x) + 2\|b\|_{L^\infty(\mathbb{R}^n)} M_\alpha(D_1 g)(x).$$

It is clear that T_α is sublinear and

$$\|T_\alpha(g)\|_{L^q(\mathbb{R}^n)} \leq C\|b\|_{\mathrm{Lip}(\mathbb{R}^n)} \|g\|_{W^{1,p}(\mathbb{R}^n)}, \quad \forall g \in W^{1,p}(\mathbb{R}^n). \tag{2.32}$$

We set $F_b(x, y) = |b(x) - b(y)|$. It is clear that $F_b(x, \cdot) \in Lip(\mathbb{R}^n)$ and $\|F_b(x, \cdot)\|_{Lip(\mathbb{R}^n)} \leq \|b\|_{Lip(\mathbb{R}^n)}$ for all $x \in \mathbb{R}^n$. By Remark 1.2, for a fixed $x \in \mathbb{R}^n$, the function $F_b(x, \cdot)$ is differential almost every $y \in \mathbb{R}^n$. Moreover, for almost every $y \in \mathbb{R}^n$, we have that $|D_{1,y} F_b(x, y)| \leq \|b\|_{Lip(\mathbb{R}^n)}$. Similarly we see that $F_b(\cdot, y) \in Lip(\mathbb{R}^n)$ and $\|F_b(\cdot, y)\|_{Lip(\mathbb{R}^n)} \leq \|b\|_{Lip(\mathbb{R}^n)}$ for all $y \in \mathbb{R}^n$. Moreover, for a fixed $y \in \mathbb{R}^n$ the function $F_b(\cdot, y)$ is differential almost everywhere. Moreover, for almost every $x \in \mathbb{R}^n$, we have that $|D_{1,x} F_b(x, y)| \leq \|b\|_{Lip(\mathbb{R}^n)}$. Hence, we have that for any $g \in W^{1,p}(\mathbb{R}^n)$ and almost every $x \in \mathbb{R}^n$,

$$|u_{b,\alpha,x,g}(r)| \leq T_\alpha(g)(x). \tag{2.33}$$

Let $\epsilon > 0$. There exists $R > 0$ such that $\|T_\alpha(f)\|_{L^q((B(0,R))^c)} < \epsilon$. By the absolute continuity of integration, we can find $\eta > 0$ such that $\|T_\alpha(f)\|_{L^q(A)} < \epsilon$ whenever $|A| < \eta$ and A is a measurable subset of $B(0, R)$. By the proof of Lemma 2.3, one sees that for almost every $x \in \mathbb{R}^n$, the function $u_{b,\alpha,x,f}$ is continuous at $r = 0$. Moreover, in view of the properties of b, one can easily check that for almost every $x \in \mathbb{R}^n$, the function $u_{b,\alpha,x,f}$ is continuous in $(0, \infty)$. Hence, for almost every $x \in \mathbb{R}^n$, the function $u_{b,\alpha,x,f}$ is continuous in $[0, \infty)$. On the other hand, by the proof of Lemma 2.3, we see that for almost every $x \in \mathbb{R}^n$,

$$|u_{b,\alpha,x,f}(x)| \leq (2\|b\|_{Lip(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} + 2\|b\|_{L^\infty(\mathbb{R}^n)} \|D_1 f\|_{L^p(\mathbb{R}^n)}) |B(x, r)|^{-1/q}.$$

This yields that for almost every $x \in \mathbb{R}^n$, we have $\lim_{r\to\infty} u_{b,\alpha,x,f}(x) = 0$. Therefore, we have that for almost every $x \in \mathbb{R}^n$, the function $u_{b,\alpha,x,f}$ is uniformly continuous on $[0, \infty)$. It follows that there exists $\delta_x > 0$ such that

$$|u_{b,\alpha,x,f}(r_1) - u_{b,\alpha,x,f}(r_2)| < |B(0, R)|^{-1/q} \epsilon \ \text{ whenever } |r_1 - r_2| < \delta_x.$$

So we can write

$$B(0,R):=\Big(\bigcup_{j=1}^{\infty}\Big\{x\in B(0,R):\delta_x>\frac{1}{j}\Big\}\Big)\cup E,$$

where $|E|=0$. So there exists $\delta>0$ such that

$$\begin{aligned}&|\{x\in B(0,R):|u_{b,\alpha,x,f}(r_1)-u_{b,\alpha,x,f}(r_2)|\geq|B(0,R)|^{-1/q}\epsilon\\&\qquad\text{for some } r_1,r_2 \text{ with } |r_1-r_2|<\delta\}|\\&=:|G_1|<\frac{\eta}{2}.\end{aligned}$$

Invoking Lemma 2.1, there exists $N_1\in\mathbb{N}\setminus\{0\}$ such that

$$|\{x\in B(0,R):\mathcal{R}_{b,\alpha,f_j}(x)\nsubseteq\mathcal{R}_{b,\alpha,f}(x)_{(\delta)}\}|=:|B^j|<\frac{\eta}{2},\quad\forall j\geq N_1.$$

Let us fix $j\geq 1$. By (2.33) we see that for any $r\in[0,\infty)$ and almost every $x\in\mathbb{R}^n$,

$$|u_{b,\alpha,x,f_j}(r)-u_{b,\alpha,x,f}(r)|=|u_{b,\alpha,x,f_j-f}(r)|\leq T_\alpha(f_j-f)(x). \tag{2.34}$$

In view of Lemma 2.2, we get from (2.34) that for almost every $x\in\mathbb{R}^n$,

$$\begin{aligned}&|D_1M_{b,\alpha}f_j(x)-D_1M_{b,\alpha}f(x)|\\&=|u_{b,\alpha,x,f_j}(r_1)-u_{b,\alpha,x,f}(r_2)|\\&\leq|u_{b,\alpha,x,f_j}(r_1)-u_{b,\alpha,x,f}(r_1)|+|u_{b,\alpha,x,f}(r_1)-u_{b,\alpha,x,f}(r_2)|\\&\leq T_\alpha(f_j-f)(x)+|u_{b,\alpha,x,f}(r_1)-u_{b,\alpha,x,f}(r_2)|\end{aligned} \tag{2.35}$$

for any $r_1\in\mathcal{R}_{b,\alpha,f_j}(x)$ and $r_2\in\mathcal{R}_{b,\alpha,f}(x)$. By our assumption and (2.32), there exists $N_2\in\mathbb{N}\setminus\{0\}$ such that

$$\|T_\alpha(f_j-f)\|_{L^q(\mathbb{R}^n)}<\epsilon,\quad\forall j\geq N_2. \tag{2.36}$$

If $x\in B(0,R)\setminus(G_1\cap B_j)$, then we can choose $r_1\in\mathcal{R}_{b,\alpha,f_j}(x)$ and $r_2\in\mathcal{R}_{b,\alpha,f}(x)$ such that $|r_1-r_2|<\delta$ and

$$|u_{b,\alpha,x,f}(r_1)-u_{b,\alpha,x,f}(r_2)|<|B(0,R)|^{-1/q}\epsilon.$$

On the other hand, we get by (2.32) that for almost every $x\in\mathbb{R}^n$,

$$|u_{b,\alpha,x,f}(r_1)-u_{b,\alpha,x,f}(r_2)|\leq 2T_\alpha(f)(x)$$

for any $r_1\in\mathcal{R}_{b,\alpha,f_j}(x)$ and $r_2\in\mathcal{R}_{b,\alpha,f}(x)$. Hence, we get from (2.36) that for almost every $x\in\mathbb{R}^n$,

$$\begin{aligned}&|D_1M_{b,\alpha}f_j(x)-D_1M_{b,\alpha}f(x)|\\&\leq T_\alpha(f_j-f)(x)+|B(0,R)|^{-1/q}\epsilon\chi_{B(0,R)\setminus(G_1\cup B_j)}(x)\\&\quad+2T_\alpha(f)(x)\chi_{G_1\cup B_j\cup(B(0,R))^c}(x).\end{aligned} \tag{2.37}$$

Observe that $|G_1\cup B^j|<\eta$ for $j\geq B_1$. By (2.37) and Minkowski's inequality, we have

$$\|D_1M_{b,\alpha}f_j-D_1M_{b,\alpha}f\|_{L^q(\mathbb{R}^n)}$$

$$\leq \|T_\alpha(f_j - f)\|_{L^q(\mathbb{R}^n)} + \||B_R|^{-1/q}\epsilon\|_{L^q(B_R)} + 2\|T_\alpha(f)\|_{L^q(G_1\cup B^j\cup B_R^c)} \leq 6\epsilon$$

for any $j \geq \max\{N_1, N_2\}$. This gives that

$$\lim_{j\to\infty} \|D_1 M_{b,\alpha} f_j - D_1 M_{b,\alpha} f\|_{L^q(\mathbb{R}^n)} = 0.$$

Then Theorem 1.4 is proved. □

3. Proof of Theorem 1.8

In this section we prove Theorem 1.8. Let us give some notation and lemmas.

3.1. Preliminary notation and lemmas

Denote $\delta(x) = \text{dist}(x, \Omega^c)$. According to Rademacher's theorem, as a Lipschitz function δ is differentiable almost everywhere in Ω. Moreover, $|\nabla\delta(x)| = 1$ for almost every $x \in \Omega$. The notation $K \subset\subset \Omega$ means that K is open, bounded and $\overline{K} \subset \Omega$. It is well known that $u_{h,i} \to u$ in $L^p(K)$ for all $K \subset\subset \Omega$ when $h \to 0$, and if $u \in W^{1,p}(\Omega)$, then $u_h^i \to D_i u$ in $L^p(k)$ when $h \to 0$ (see [17, 7.11]).

Let $b \in L^\infty(\Omega)$ and $f \in L^p(\Omega)$ with $p \in (1, n)$ and $\alpha \in [1, n/p)$. For every $x \in \Omega$, we define the function $A_{b,\alpha,x,f}(r) : [0, \delta(x)] \to [-\infty, \infty]$ by

$$A_{b,\alpha,x,f}(r) = \begin{cases} 0, & \text{if } r = 0; \\ \dfrac{1}{|B(x,r)|^{1-\alpha/n}} \displaystyle\int_{B(x,r)} |b(x) - b(y)||f(y)|dy, & \text{if } r \in (0, \delta(x)]. \end{cases}$$

Define the set $I_{b,\alpha,f}(x)$ by

$$I_{b,\alpha,f}(x) = \{r \in [0, \delta(x)] : M_{b,\alpha,\Omega} f(x) = A_{b,\alpha,x,f}(r)\}.$$

When $\alpha = 0$, we denote $I_{b,\alpha,f}(x) = I_{b,f}(x)$. By the Lebesgue differentiation theorem, we see that $\lim_{r\to 0^+} A_{b,\alpha,x,f}(r) = 0$ for almost everywhere $x \in \Omega$. It follows that the functions $A_{b,\alpha,x,f}$ are continuous on $(0, \delta(x)]$ for all $x \in \Omega$ and at $r = 0$ for almost every $x \in \Omega$.

By the arguments similar to those used to derive [32, Lemma 3.2] and Lemma 2.1, we can get the following result. The details are omitted.

Lemma 3.1. *Let $b \in L^\infty(\Omega)$, $p \in (1, \infty)$ and $f_j \to f$ in $L^p(\Omega)$ as $j \to \infty$. Assume that $\alpha = 0$ or $\alpha \in [1, n/p)$. Then for all $R > 0$ and $\lambda > 0$, it holds that*

$$\lim_{j\to\infty} |\{x \in \Omega_R : I_{b,\alpha,f_j}(x) \nsubseteq I_{b,\alpha,f}(x)_{(\lambda)}\}| = 0.$$

Here $\Omega_R = \Omega \cap B(0, R)$.

We now establish the following lemma, which tells us how close the sets $I_{b,\alpha,f}(x)$ and $I_{b,\alpha,f}(x + he_l)$ are when h is small enough.

Lemma 3.2. *Let $b \in \text{Lip}(\Omega)$ and $f \in L^p(\Omega)$ for some $p \in (1, \infty)$. Assume that $\alpha = 0$ or $\alpha \in [1, n/p)$. Then for $K \subset\subset \Omega$, $\lambda > 0$ and $l = 1, 2, \ldots, n$, it holds that*

$$|\{x \in K : \pi(I_{b,\alpha,f}(x), I_{b,\alpha,f}(x + he_l)) > \lambda\}| \to 0 \quad \text{when } h \to 0. \tag{3.1}$$

Proof. The proof is similar to that of [32, Lemma 3.3]. However, some modifications are needed. For (3.1) it suffices to show that

$$\lim_{h\to 0} |\{x \in K : I_{b,\alpha,f}(x+he_l) \not\subseteq I_{b,\alpha,f}(x)_{(\lambda)}\}| = 0, \tag{3.2}$$

$$\lim_{h\to 0} |\{x \in K : I_{b,\alpha,f}(x) \not\subseteq I_{b,\alpha,f}(x+he_l)_{(\lambda)}\}| = 0. \tag{3.3}$$

We only prove (3.2) since (3.3) can be proved similarly. Let $\epsilon \in (0,1)$ and $\lambda > 0$. An argument similar to (2.9) shows that there exist a positive integer $\gamma = \gamma(\lambda,\epsilon)$ and a measurable set E with $|E| < \epsilon$ such that

$$\begin{aligned} K \subset \{x \in K : A_{b,\alpha,x,f}(r) < M_{b,\alpha,\Omega} f(x) - \gamma^{-1} \text{ if } d(r, I_{b,\alpha,f}(x)) > \lambda\} \cup E \\ =: G \cup E. \end{aligned} \tag{3.4}$$

Fix $h \in \mathbb{R}$, and let

$$B_{1,h} := \{x \in K : |M_{b,\alpha,\Omega} f(x+he_l) - M_{b,\alpha,\Omega} f(x)| > (4\gamma)^{-1}\},$$

$$B_{2,h} := \{x \in K : M_{b_{h,l},\alpha,\Omega}(f_{h,l} - f)(x) + 2\|b\|_{Lip(\Omega)}|h| M_{\alpha,\Omega} f(x) > (2\gamma)^{-1}\},$$

$$\begin{aligned} B_{3,h} := \{x \in \Omega : \exists\, r \in [\delta(x) - 2|h|, \delta(x+he_l)] \ \text{ such that} \\ |A_{b,\alpha,x+he_l,f}(r) - A_{b,\alpha,x+he_l,f}(\delta(x+he_l) - |h|)| > (8\gamma)^{-1}\}. \end{aligned}$$

Firstly we prove that

$$\begin{aligned} \{x \in K : I_{b,\alpha,f}(x+he_l) \not\subseteq I_{b,\alpha,f}(x)_{(2\lambda)}\} \\ \subset B_{1,h} \cup B_{2,h} \cup (B_{3,h} - he_l) \cup E =: B_h \end{aligned} \tag{3.5}$$

when h is small enough. Let $h_0 \in (0,\lambda)$ be such that $K_{(2h_0)} \subset \Omega$. For (3.5) it is enough to prove that for $x \in G \setminus B_h$ with $|h| < \frac{1}{2}\min\{h_0, \delta(x)\}$, there exists $r \in I_{b,\alpha,f}(x+he_l)$ such that $d(r, I_{b,\alpha,f}(x)) \le 2\lambda$. If not, we assume that $d(r, I_{b,\alpha,f}(x)) > 2\lambda$ and consider two cases:

Case (i) ($r < \delta(x) - |h|$). In view of (3.4),

$$\begin{aligned} &M_{b,\alpha,\Omega} f(x+he_l) \\ &= A_{b,\alpha,x+he_l,f}(r) \le A_{b,\alpha,x+he_l,f}(r) - A_{b,\alpha,x,f}(r) + A_{b,\alpha,x,f}(r) \\ &\le |A_{b,\alpha,x+he_l,f}(r) - A_{b,\alpha,x,f}(r)| + M_{b,\alpha,\Omega} f(x) - \gamma^{-1}. \end{aligned} \tag{3.6}$$

By the argument similar to the proof of Lemma 2.2,

$$\begin{aligned} &|A_{b,\alpha,x+he_l,f}(t) - A_{b,\alpha,x,f}(t)| \\ &\le M_{b_{h,l},\alpha,\Omega}(f_{h,l} - f)(x) + 2\|b\|_{Lip(\Omega)}|h| M_{\alpha,\Omega} f(x) \end{aligned} \tag{3.7}$$

for any $t \in (0, \min\{\delta(x), \delta(x+he_l)\})$. From (3.6) and (3.7) we see that

$$\begin{aligned} &M_{b,\alpha,\Omega} f(x+he_l) \\ &\le M_{b_{h,l},\alpha,\Omega}(f_{h,l} - f)(x) + 2\|b\|_{Lip(\Omega)}|h| M_{\alpha,\Omega} f(x) + M_{b,\alpha,\Omega} f(x) - \gamma^{-1} \\ &\le (2\gamma)^{-1} + M_{b,\alpha,\Omega} f(x) - \gamma^{-1} \le M_{b,\alpha,\Omega} f(x) - (2\gamma)^{-1}. \end{aligned}$$

So $|M_{b,\alpha,\Omega} f(x+he_l) - M_{b,\alpha,\Omega} f(x)| \ge (2\gamma)^{-1}$. This gives $x \in B_{1,h}$ and a contradiction.

Case (ii) ($r \in [\delta(x)-|h|, \delta(x+he_l)]$). In this case we note that $d(\delta(x)-|h|, I_{b,\alpha,f}(x)) > \lambda$, $\delta(x+he_l)-|h| < \delta(x)$ and $\delta(x+he_l)-r \in [0, 2|h|]$. Hence, we have $|r-(\delta(x+he_l)-|h|)| = ||h|-(\delta(x+he_l)-r)| \le |h|$ and

$$d(\delta(x+he_l)-|h|, I_{b,\alpha,f}(x)) \ge d(r, I_{b,\alpha,f}(x)) - |r-(\delta(x+he_l)-|h|)| > 2\lambda - |h| > \lambda.$$

Combining (3.7) with (3.4) yields that

$$\begin{aligned}
&M_{b,\alpha,\Omega}f(x+he_l)\\
&= A_{b,\alpha,x+he_l,f}(r)\\
&\le |A_{b,\alpha,x+he_l,f}(r) - A_{b,\alpha,x+he_l,f}(\delta(x+he_l)-|h|)|\\
&\quad + |A_{b,\alpha,x+he_l,f}(\delta(x+he_l)-|h|) - A_{b,\alpha,x,f}(\delta(x+he_l)-|h|)|\\
&\quad + A_{b,\alpha,x,f}(\delta(x+he_l)-|h|)\\
&\le (8\gamma)^{-1} + M_{b_{h,l},\alpha,\Omega}(f_{h,l}-f)(x) + 2\|b\|_{Lip(\Omega)}|h| M_{\alpha,\Omega}f(x)\\
&\quad + M_{b,\alpha,\Omega}f(x) - \gamma^{-1}\\
&\le (8\gamma)^{-1} + (2\gamma)^{-1} + M_{b,\alpha,\Omega}f(x) - \gamma^{-1} \le M_{b,\alpha,\Omega}f(x) - (4\gamma)^{-1},
\end{aligned}$$

which leads to $|M_{b,\alpha,\Omega}f(x) - M_{b,\alpha,\Omega}f(x+he_l)| > (4\gamma)^{-1}$ and $x \in B_{1,h}$. This is a contradiction. Hence (3.5) is proved.

In view of (3.5), for (3.2), it is enough to show that

$$\lim_{h\to 0} |B_h| = 0. \tag{3.8}$$

Clearly, $|B_{3,h} - he_l| \to 0$ when $h \to 0$. Let $q_\alpha = np/(n-\alpha)$. By Remark 1.5 we see that $M_{b,\Omega}f \in L^p(\Omega)$. So $M_{b,\alpha,\Omega}f(\cdot+he_l) \to M_{b,\alpha,\Omega}f$ in $L^{q_\alpha}(K)$ as $h \to 0$. There exists $\delta_1 > 0$ such that $\|M_{b,\alpha,\Omega}f(\cdot+he_l) - M_{b,\alpha,\Omega}f\|_{L^{q_\alpha}(K)} < (4\gamma)^{-1}\epsilon$ when $|h| < \delta_1$. It follows that

$$|B_{1,h}| \le (4\gamma)^{q_\alpha} \|M_{b,\alpha,\Omega}f(\cdot+he_l) - M_{b,\alpha,\Omega}f\|^{q_\alpha}_{L^{q_\alpha}(\Omega)} < \epsilon^{q_\alpha} \tag{3.9}$$

when $|h| < \delta_1$. On the other hand, by the arguments similar to those used to derive Lemma 2.2, there exists $\delta_2 > 0$ such that

$$|B_{2,h}| \le C\epsilon^{q_\alpha} \tag{3.10}$$

for some $C > 0$ when $|h| < \min\{\delta_2, (8\gamma)^{-1}\epsilon\}$. Here $C > 0$ is independent of γ and ϵ. Combining (3.10) with (3.9) implies that

$$|B_{1,h}| + |B_{2,h}| < C\epsilon$$

when $|h| < \min\{\delta_1, \delta_2, (8\gamma)^{-1}\epsilon\}$. Here $C > 0$ is independent of γ and ϵ. This proves (3.8). Lemma 3.2 is now proved. □

The following lemma presents the derivative formulas of local maximal commutator, which plays a pivotal role in the proof of Theorem 1.8.

Lemma 3.3. *Let $f \in W^{1,p}(\Omega)$ for some $p \in (1,\infty)$ and $b \in \mathrm{Lip}(\Omega)$. Assume $|\Omega| < \infty$. Then*

(i) *For any* $l \in \{1, 2, \ldots, n\}$*, almost every* $x \in \Omega$ *and* $r \in I_{b,f}(x)$ *with* $0 < r < \delta(x)$*, it holds that*

$$\begin{aligned} &D_l M_{b,\Omega} f(x) \\ &= \frac{1}{|B(x,r)|} \int_{B(x,r)} (D_{l,y}|b(x)-b(y)| + D_{l,x}|b(x)-b(y)|)|f(y)|dy \\ &\quad + \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(x)-b(y)| D_l|f|(y)dy. \end{aligned} \tag{3.11}$$

(ii) *For any* $l \in \{1, 2, \ldots, n\}$*, almost every* $x \in \Omega$ *and* $0 \in I_{b,f}(x)$*, it holds that*

$$D_l M_{b,\Omega} f(x) = 0. \tag{3.12}$$

Proof. Without loss of generality we may assume $f \geq 0$. Fix $l \in \{1, 2, \ldots, n\}$. Note that $F_b(x, \cdot) \in Lip(\Omega)$ for all $x \in \Omega$ and $F_b(\cdot, y) \in Lip(\Omega)$ for all $y \in \Omega$. It follows that $|(F_{x,b})_h^l(y)| \leq \|b\|_{Lip(\Omega)}$ and $|(F_{y,b})_h^l(x)| \leq \|b\|_{Lip(\Omega)}$ for all x, $y \in \Omega$. Moreover, for a fixed $x \in \Omega$ the function $F_b(x, \cdot)$ is differentiable almost every $y \in \Omega$. For almost every $y \in \Omega$, we have $|D_{l,y} F_b(x, y)| \leq \|b\|_{Lip(\Omega)}$. Similarly we have $F_b(\cdot, y) \in Lip(\Omega)$ and $\|F_b(\cdot, y)\|_{Lip(\Omega)} \leq \|b\|_{Lip(\Omega)}$ for all $y \in \Omega$. Moreover, for a fixed $y \in \Omega$ the function $F_b(\cdot, y)$ is differentiable almost everywhere. Moreover, for almost every $x \in \Omega$, we have that $|D_{l,x} F_b(x, y)| \leq \|b\|_{Lip(\Omega)}$. Let $K \subset\subset \Omega$. In view of Lemma 3.2, there exists $\{s_k\}_{k=1}^{\infty} \subset (0, \infty)$ with $s_k \to 0$ such that

$$\lim_{k\to\infty} \pi(I_{b,f}(x), I_{b,f}(x + s_k e_l)) = 0$$

for almost every $x \in K$. By Remark 1.6 we see that $|b_{s_k}^l(y) - D_l b(y)|f(y) \leq 2\|b\|_{Lip(\Omega)} f(y)$ and $\lim_{k\to\infty} b_{s_k}^l(y) = D_l b(y)$ for almost every $y \in \Omega$. Then $\lim_{k\to\infty} \|f_{s_k,l} - f\|_{L^p(K)} = 0$, $\lim_{k\to\infty} \|f_{s_k}^l - D_l f\|_{L^p(K)} = 0$, $\lim_{k\to\infty} \|M_\Omega(f_{s_k,l} - f)\|_{L^p(K)} = 0$, $\lim_{k\to\infty} \|M_\Omega(f(b_{s_k}^l - D_l b))\|_{L^p(K)} = 0$, $\lim_{k\to\infty} \|M_\Omega(((F_{x,b})_{s_k}^l - D_{l,y} F_b(x, \cdot))f)\|_{L^p(K)} = 0$ and $\lim_{k\to\infty} \|M_{b,\Omega}(f_{s_k}^l - D_l f)\|_{L^p(K)} = 0$. By Theorem B, we have $M_{b,\Omega} f \in W^{1,p}(\Omega)$. It holds that $\lim_{k\to\infty} \|(M_{b,\Omega} f)_{s_k}^l - D_l M_{b,\Omega} f\|_{L^p(K)} = 0$. From the above facts, there exist a subsequence $\{h_k\}_{k=1}^{\infty}$ of $\{s_k\}_{k=1}^{\infty}$ and a measurable set $B_1 \subset K$ such that $|K \setminus B_1| = 0$ and for any $x \in B_1$, we have that $\lim_{k\to\infty} M_\Omega(f(b_{h_k}^l - D_l b))(x) = 0$, $\lim_{k\to\infty} M_\Omega(f_{h_k,l} - f)(x) = 0$, $\lim_{k\to\infty} M_{b,\Omega}(f_{h_k}^l - D_l f)(x) = 0$, $\lim_{k\to\infty} M_\Omega(((F_{x,b})_{h_k}^l - D_{l,y} F_b(x, \cdot))f)(x) = 0$, $\lim_{k\to\infty} (M_{b,\Omega}(f))_{h_k}^l(x) = D_l M_{b,\Omega} f(x)$, $\lim_{k\to\infty} \pi(I_{b,f}(x), I_{b,f}(x + h_k e_l)) = 0$ and $\lim_{k\to\infty} (F_{y,b})_{h_k}^l(x) = D_{l,x} F_b(x, y)$ for all $y \in \Omega$. We set

$$B_2 := \{x \in K : M_{b,\Omega} f(x) = A_{b,x,f}(0) \text{ if } 0 \in I_{b,f}(x)\},$$

$$B_3 := \bigcap_{k=1}^{\infty} \{x \in K : M_{b,\Omega} f(x + h_k e_l) = A_{b,x+h_k e_l,f}(0) \text{ if } 0 \in I_{b,f}(x + h_k e_l)\},$$

$$B_4 := \Big\{x \in K : \lim_{r\to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |D_l b(x) - D_l b(y)| f(y) dy = 0\Big\},$$

$$B_5 := \Big\{x \in \Omega : \lim_{k\to\infty} (F_{y,b})^l_{h_k}(x) = D_{l,x}F_b(x,y), \quad \text{a.e. } y \in \Omega\Big\}.$$

Clearly, $|K \setminus B_i| = 0$ for any $2 \le i \le 5$. Let B_6 be the set of all $x \in \Omega$ for which b is differentiable at x. By Remark 1.2 we see that $|B_6^c| = 0$. Let B_7 be the set of all $x \in \Omega$ for which $M_\Omega f(x) < \infty$ and $M_\Omega D_l f(x) < \infty$. It follows from the boundedness of M_Ω that $|B_7^c| = 0$. Let B_8 be the set of all Lebesgue points of f. Clearly, $|B_8^c| = 0$. Hence, $|(\bigcap_{j=1}^8 B_j)^c| = 0$.

Let us fix $x \in \bigcap_{j=1}^8 B_j$ and $r \in I_{b,f}(x)$ satisfying $r < \delta(x)$. We note that $\lim_{k\to\infty} \pi(I_{b,f}(x), I_{b,f}(x + he_k e_l)) = 0$. There exists radii $r_k \in I_{b,f}(x + h_k e_l)$ such that $\lim_{k\to\infty} r_k = r$. It is easy to see that

$$D_l M_{b,\Omega} f(x) = \lim_{k\to\infty} \frac{1}{h_k}(M_{b,\Omega} f(x + h_k e_l) - M_{b,\Omega} f(x)). \tag{3.13}$$

We consider two cases:

Case A ($r > 0$). We may assume without loss of generality that all $r_k \in (0, \delta(x))$. Note that $\lim_{k\to\infty} h_k = 0$. Hence, we may assume that $r_k + h_k \in (0, \min\{2r, \delta(x)\})$ for all $k \ge 1$. In view of (2.15) and (2.17),

$$\begin{aligned}
&\frac{1}{h_k}(M_{b,\Omega} f(x + h_k e_l) - M_{b,\Omega} f(x)) \\
&\le \frac{1}{|B(x, r_k)|} \int_{B(x,r_k)} F_b(x,y) f^l_{h_k}(y)dy \\
&\quad + \frac{1}{|B(x, r_k)|}\Big(\int_{B(x,r_k)} (F_{x,b})^l_{h_k}(y) f_{h_k,l}(y)dy \\
&\quad + \int_{B(x,r_k)} (F_{y+h_k e_l,b})^l_{h_k}(x) f_{h_k,l}(y)dy\Big).
\end{aligned} \tag{3.14}$$

An argument similar to (2.16) gives that

$$\lim_{k\to\infty} \int_{B(x,r_k)} F_b(x,y) f^l_{h_k}(y)dy = \int_{B(x,r)} F_b(x,y) D_l f(y)dy. \tag{3.15}$$

We now prove that

$$\lim_{k\to\infty} \int_{B(x,r_k)} (F_{x,b})^l_{h_k}(y) f_{h_k,l}(y)dy = \int_{B(x,r)} D_{l,y}F_b(x,y) f(y)dy \tag{3.16}$$

and

$$\lim_{k\to\infty} \int_{B(x,r_k)} (F_{y+h_k e_l,b})^l_{h_k}(x) f_{h_k,l}(y)dy = \int_{B(x,r)} D_{l,x}F_b(x,y) f(y)dy. \tag{3.17}$$

Note that $D_{l,y}F_b(x,\cdot)f(\cdot) \in L^1(\Omega)$. This gives

$$\lim_{k\to\infty} \int_{B(x,r_k)} D_{l,y}F_b(x,y) f(y)dy = \int_{B(x,r)} D_{l,y}F_b(x,y) f(y)dy.$$

It follows that

$$\Big|\int_{B(x,r_k)} (F_{x,b})^l_{h_k}(y) f_{h_k,l}(y)dy - \int_{B(x,r)} D_{l,y}F_b(x,y) f(y)dy\Big|$$

$$\begin{aligned}
&\leq \int_{B(x,r_k)} |(F_{x,b})^l_{h_k}(y) f_{h_k,l}(y) - D_{l,y}F_b(x,y)f(y)|dy \\
&\leq \int_{B(x,r_k)} |(F_{x,b})^l_{h_k}(y)||f_{h_k,l}(y) - f(y)|dy \\
&\quad + \int_{B(x,r_k)} |(F_{x,b})^l_{h_k}(y) - D_{l,y}F_b(x,y)||f(y)|dy \\
&\leq |B(x,r_k)|(\|b\|_{Lip(\Omega)} M_\Omega(f_{h_k,l} - f)(x) \\
&\quad + M_\Omega(((F_{x,b})^l_{h_k} - D_{l,y}F_b(x,\cdot))f)(x)),
\end{aligned}$$

which gives (3.16).

We now prove (3.17). Note that $|(F_{y+h_ke_l,b})^l_{h_k}(x)| \leq \|b\|_{Lip(\Omega)}$. This yields that

$$\begin{aligned}
&\int_{B(x,r_k)} (F_{y+h_ke_l,b})^l_{h_k}(x)(f_{h_k,l}(y) - f(y))dy \\
&\leq 2\|b\|_{Lip(\Omega)}|B(x,r_k)|M_\Omega(f_{h_k,l} - f)(x).
\end{aligned}$$

Thus, for (3.17) it is enough to prove that

$$\lim_{k\to\infty} \int_{B(x,r_k)} (F_{y+h_ke_l,b})^l_{h_k}(x)f(y)dy = \int_{B(x,r)} D_{l,x}F_b(x,y)f(y)dy. \tag{3.18}$$

By a change of variable,

$$\int_{B(x,r_k)} (F_{y+h_ke_l,b})^l_{h_k}(x)f(y)dy = \int_{B(x+h_ke_l,r_k)} (F_{y,b})^l_{h_k}(x)f_{h_k,l}(y)dy.$$

Note that

$$\begin{aligned}
&\Big|\int_{B(x+h_ke_l,r_k)} (F_{y,b})^l_{h_k}(x)f_{h_k,l}(y)dy - \int_{B(x+h_ke_l,r_k)} (F_{y,b})^l_{h_k}(x)f(y)dy\Big| \\
&\leq \|b\|_{Lip(\Omega)} \int_{B(x+h_ke_l,r_k)} |f_{h_k,l}(y) - f(y)|dy \\
&\leq \|b\|_{Lip(\Omega)} \int_{B(x,r_k+h_k)} |f_{h_k,l}(y) - f(y)|dy \\
&\leq \|b\|_{Lip(\Omega)}|B(x,r_k+h_k)|M_\Omega(f_{h_k,l} - f)(x).
\end{aligned}$$

It follows that

$$\lim_{k\to\infty} \int_{B(x+h_ke_l,r_k)} (F_{y,b})^l_{h_k}(x)f_{h_k,l}(y)dy = \lim_{k\to\infty} \int_{B(x+h_ke_l,r_k)} (F_{y,b})^l_{h_k}(x)f(y)dy.$$

Therefore, to prove (3.18), it suffices to show that

$$\lim_{k\to\infty} \int_{B(x+h_ke_l,r_k)} (F_{y,b})^l_{h_k}(x)f(y)dy = \int_{B(x,r)} D_{l,x}F_b(x,y)f(y)dy. \tag{3.19}$$

Note that $(F_{y,b})^l_{h_k}(x)\chi_{B(x+h_ke_l,r_k)}(y) \to D_{l,x}F_b(x,y)\chi_{B(x,r)}(y)$ as $k \to \infty$ and $|(F_{y,b})^l_{h_k}(x)\chi_{B(x+h_ke_l,r_k)}(y) - D_{l,x}F_b(x,y)\chi_{B(x,r)}(y)| \leq 2\|b\|_{Lip(\Omega)}$ for almost every $y \in \Omega$. Moreover, we have that $f \in L^1(\Omega)$ by the fact that $|\Omega| < \infty$ and

Hölder's inequality. By the dominated convergence theorem, we have (3.19). It follows from (3.13)-(3.17) that for almost every $x \in K$,

$$\begin{aligned} &D_l M_{b,\Omega} f(x) \\ \leq &\frac{1}{|B(x,r)|}\int_{B(x,r)} (D_{l,y}|b(x)-b(y)| + D_{l,x}|b(x)-b(y)|)f(y)dy \\ &+ \frac{1}{|B(x,r)|}\int_{B(x,r)} |b(x)-b(y)|D_l f(y)dy. \end{aligned} \tag{3.20}$$

By the arguments similar to those used to derive Lemma 2.3 and derive (3.20), we have that for almost every $x \in K$,

$$\begin{aligned} &D_l M_{b,\Omega} f(x) \\ \geq &\frac{1}{|B(x,r)|}\int_{B(x,r)} (D_{l,y}|b(x)-b(y)| + D_{l,x}|b(x)-b(y)|)f(y)dy \\ &+ \frac{1}{|B(x,r)|}\int_{B(x,r)} |b(x)-b(y)|D_l f(y)dy. \end{aligned} \tag{3.21}$$

Combining (3.21) with (3.20) implies that (3.11) holds for almost every $x \in K$.

Case B ($r = 0$). Because of $0 \in I_{b,f}(x)$, then $M_{b,\Omega}f(x) = A_{b,x,f}(0) = 0$. We write

$$\begin{aligned} D_l M_{b,\Omega} f(x) &= \lim_{k\to\infty} \frac{1}{h_k} M_{b,\Omega} f(x + h_k e_l) \\ &= \lim_{k\to\infty} \frac{1}{h_k} A_{b,x+h_k e_l,f}(r_k). \end{aligned} \tag{3.22}$$

If we have $r_k = 0$ for infinitely many k, then $D_l M_{b,\Omega} f(x) = 0$. Otherwise, there exists $k_0 \in \mathbb{N}$ such that $r_k > 0$ when $k \geq k_0$. Since $M_{b,\Omega}f(x) = 0$, then $|b(x) - b(y)|f(y) = 0$ for almost every $y \in B(x, \delta(x))$. An argument similar to (2.27) gives that

$$\begin{aligned} &\frac{1}{h_k} A_{b,x+h_k e_l,f}(r_k) \\ \leq &\frac{1}{|B(x,r_k)|}\int_{B(x,r_k)} |b(x+h_k e_l) - b(y+h_k e_l)||f^l_{h_k}(y)|dy \\ &+ \frac{1}{|B(x,r_k)|}\int_{B(x,r_k)} |b^l_{h_k}(x) - b^l_{h_k}(y)|f(y)dy. \end{aligned} \tag{3.23}$$

It is easy to see that

$$\begin{aligned} &\frac{1}{|B(x,r_k)|}\int_{B(x,r_k)} |b(x+h_k e_l) - b(y+h_k e_l)||f^l_{h_k}(y)|dy \\ &\leq \|b\|_{Lip(\mathbb{R}^n)} r_k (M_\Omega(f^l_{h_k} - D_l f)(x) + M_\Omega(D_l f)(x)) \to 0 \quad \text{as} \quad k \to \infty \end{aligned} \tag{3.24}$$

and

$$
\begin{aligned}
&\frac{1}{|B(x,r_k)|}\int_{B(x,r_k)}|b^l_{h_k}(x)-b^l_{h_k}(y)|f(y)dy\\
&\le |b^l_{h_k}(x)-D_lb(x)|M_\Omega f(x)+M_\Omega((b^l_{h_k}-D_lb)f)(x)\\
&\quad+\frac{1}{|B(x,r_k)|}\int_{B(x,r_k)}|D_lb(x)-D_lb(y)|f(y)dy\\
&\to 0 \quad \text{as} \quad k\to\infty.
\end{aligned}
\tag{3.25}
$$

It follows from (3.22)-(3.25) that (3.12) holds for almost every $x\in K$. Since $K\subset\subset\Omega$ is arbitrary, this gives the claim in Ω. □

Applying the arguments similar to those used to derive Lemma 3.3, we can get the following result. The details are omitted.

Lemma 3.4. *Let $f\in W^{1,p}(\Omega)$ with $p\in(1,n)$ and $b\in \mathrm{Lip}(\Omega)$. Assume that $\alpha\in[1,n/p)$ and $|\Omega|<\infty$. Then*

(i) *For any $l\in\{1,2,\dots,n\}$, almost every $x\in\Omega$ and $r\in I_{b,\alpha,f}(x)$ with $0<r<\delta(x)$, it holds that*

$$
\begin{aligned}
&D_lM_{b,\alpha,\Omega}f(x)\\
&=\frac{1}{|B(x,r)|^{1-\alpha/n}}\int_{B(x,r)}(D_{l,y}|b(x)-b(y)|+D_{l,x}|b(x)-b(y)|)|f(y)|dy\\
&\quad+\frac{1}{|B(x,r)|^{1-\alpha/n}}\int_{B(x,r)}|b(x)-b(y)|D_l|f|(y)dy.
\end{aligned}
$$

(ii) *For any $l\in\{1,2,\dots,n\}$, almost every $x\in\Omega$ and $0\in I_{b,\alpha,f}(x)$, it holds that*

$$D_lM_{b,\alpha,\Omega}f(x)=0.$$

Lemma 3.5. *Let $f\in W^{1,p}(\Omega)$ with $1<p<\infty$ and $b\in\mathrm{Lip}(\Omega)$. Let $\{h_k\}_{k\ge1}\subset(0,\infty)$ be such that $\lim_{k\to\infty}h_k=0$ and $l\in\{1,2,\dots,n\}$. Assume that $|\Omega|<\infty$ and $\delta(x)\le\delta(x+h_ke_l)$ for almost every $x\in\Omega$ and all $k\ge1$. Then, for almost every $x\in\Omega$,*

$$
\begin{aligned}
&\lim_{k\to\infty}\int_{B(x,\delta(x))}\frac{F_b(x+h_ke_l,y+h_ke_l)f_{h_k,l}(y)-F_b(x,y)f(y)}{h_k}dy\\
&=\int_{B(x,\delta(x))}(D_{l,y}F_b(x,y)+D_{l,x}F_b(x,y))f(y)dy\\
&\quad+\int_{B(x,\delta(x))}F_b(x,y)D_lf(y)dy.
\end{aligned}
\tag{3.26}
$$

Proof. We shall adapt the method of [32] to prove this lemma. It is not difficult to see that

$$\int_{B(x,\delta(x))}\frac{F_b(x+h_ke_l,y+h_ke_l)f_{h_k,l}(y)-F_b(x,y)f(y)}{h_k}dy$$

$$\begin{aligned}(3.27)\qquad &= \int_{B(x,\delta(x))} F_b(x,y) f^l_{h_k}(y)dy + \int_{B(x,\delta(x))} (F_{x,b})^l_{h_k}(y) f_{h_k,l}(y)dy \\ &\quad + \int_{B(x,\delta(x))} (F_{y+h_k e_l,b})^l_{h_k}(x) f_{h_k,l}(y)dy,\end{aligned}$$

where

$$(F_{x,b})^l_{h_k}(y) = \frac{1}{h_k}(F_b(x, y + h_k e_l) - F_b(x,y)),$$
$$(F_{y,b})^l_{h_k}(x) = \frac{1}{h_k}(F_b(x + h_k e_l, y) - F_b(x,y)).$$

In view of (3.27), to prove (3.26) it is enough to show that for almost every $x \in \Omega$,

$$(3.28)\qquad \lim_{k\to\infty} \int_{B(x,\delta(x))} F_b(x,y) f^l_{h_k}(y)dy = \int_{B(x,\delta(x))} F_b(x,y) D_l f(y)dy;$$

$$(3.29)\qquad \lim_{k\to\infty} \int_{B(x,\delta(x))} (F_{x,b})^l_{h_k}(y) f_{h_k,l}(y)dy = \int_{B(x,\delta(x))} D_{l,y} F_b(x,y) f(y)dy;$$

$$\begin{aligned}(3.30)\qquad &\lim_{k\to\infty} \int_{B(x,\delta(x))} (F_{y+h_k e_l,b})^l_{h_k}(x) f_{h_k,l}(y)dy \\ &= \int_{B(x,\delta(x))} D_{l,x} F_b(x,y) f(y)dy.\end{aligned}$$

We first prove (3.28). Note that $f^l_{h_k} \to D_l f$ in $L^p_{\mathrm{loc}}(\Omega)$. Let $x \in \Omega$. By [35, Lemma 2.8], we can get

$$(3.31)\qquad |f^l_{h_k}(y)| \le C(M(|\nabla f|\chi_\Omega)(y) + M(|\nabla f|\chi_\Omega)(y + h_k e_l)) =: C\Gamma_1(y)$$

for almost every $y \in B(x,\delta(x))$. By the L^p bounds of M and Minkowski's inequality, one can easily check that $\|\Gamma_1\|_{L^p(B(x,\delta(x)))} \le C\|\nabla f\|_{L^p(\Omega)}$. Hence, $\Gamma_1 \in L^1(\Omega)$. From the above we have that for a fixed $t \in (0,\delta(x)]$,

$$(3.32)\qquad \lim_{k\to\infty} \int_{B(x,\delta(x)-t)} |f^l_{h_k}(y) - D_l f(y)|dy = 0.$$

By the absolute continuity of the integral, then for every $\epsilon > 0$, there exists $t_0 > 0$ such that for any $t \in (0, t_0)$,

$$\begin{aligned}(3.33)\qquad &\int_{B(x,\delta(x))\setminus B(x,\delta(x)-t)} |f^l_{h_k}(y) - D_l f(y)|dy \\ &\le C \int_{B(x,\delta(x))\setminus B(x,\delta(x)-t)} (|\Gamma_1(y)| + |D_l f(y)|)dy < C\epsilon.\end{aligned}$$

Hence, we get from (3.32) and (3.33) that

$$\Big| \int_{B(x,\delta(x))} F_b(x,y) f^l_{h_k}(y)dy - \int_{B(x,\delta(x))} F_b(x,y) D_l f(y)dy \Big|$$

$$\leq \int_{B(x,\delta(x))} F_b(x,y)|f^l_{h_k}(y) - D_l f(y)|dy$$
$$\leq 2\|b\|_{L^\infty(\Omega)} \int_{B(x,\delta(x))} |f^l_{h_k}(y) - D_l f(y)|dy \to 0 \text{ as } k \to \infty.$$

This gives (3.28).

Now we prove (3.29). Without loss of generality we may assume that all $h_k \in (0,1)$. Note that $|(F_{x,b})^l_{h_k}(y)| \leq \|b\|_{Lip(\Omega)}$ and $|D_{l,y}F_b(x,y)| \leq \|b\|_{Lip(\Omega)}$ for any $x \in \Omega$ and almost every $y \in \Omega$. Moreover, we get by (3.31) that $|f_{h_k,l}(y)| \leq C\Gamma_1(y)|h_k| + |f(y)| \leq C\Gamma_1(y) + |f(y)|$ for almost every $y \in B(x,\delta(x))$. It follows that

$$|(F_{x,b})^l_{h_k}(y)f_{h_k,l}(y) - D_{l,y}F_b(x,y)f(y)| \leq C\|b\|_{Lip(\Omega)}(\Gamma_1(y) + |f(y)|)$$

for almost every $y \in B(x,\delta(x))$. An argument similar to (3.33) shows that for every $\epsilon > 0$, there exists $t_1 > 0$ such that for any $t \in (0,t_1)$,

$$\int_{B(x,\delta(x))\setminus B(x,\delta(x)-t)} |(F_{x,b})^l_{h_k}(y)f_{h_k,l}(y) - D_{l,y}F_b(x,y)f(y)| < C\epsilon. \tag{3.34}$$

Note that $f_{h_k,l} \to f$ in $L^p_{\text{loc}}(\Omega)$. For a fixed $t \in (0,\delta(x)]$, we have

$$\begin{aligned} &\Big|\int_{B(x,\delta(x)-t)} (F_{x,b})^l_{h_k}(y)f_{h_k,l}(y)dy \\ &\quad - \int_{B(x,\delta(x)-t)} (F_{x,b})^l_{h_k}(y)f(y)dy\Big| \\ &\leq \|b\|_{Lip(\Omega)} \int_{B(x,\delta(x)-t)} |f_{h_k,l}(y) - f(y)|dy \to 0 \text{ as } k \to \infty. \end{aligned} \tag{3.35}$$

Moreover, noting that $(F_{x,b})^l_{h_k}(y) \to D_l F_b(x,y)$ as $k \to \infty$ for all $x \in \Omega$ and almost every $y \in \Omega$. Applying the dominated convergence theorem,

$$\lim_{k\to\infty} \int_{B(x,\delta(x)-t)} (F_{x,b})^l_{h_k}(y)f(y)dy = \int_{B(x,\delta(x)-t)} D_l F_b(x,y)f(y)dy. \tag{3.36}$$

Combining (3.36) with (3.34) and (3.35) gives (3.29).

It remains to conclude (3.30). Note that $|(F_{y+h_k e_l,b})^l_{h_k}(x)| \leq \|b\|_{Lip(\Omega)}$ and $|D_{l,x}F_b(x,y)| \leq \|b\|_{Lip(\Omega)}$ for almost $x \in \Omega$ and every $y \in \Omega$. By the argument similar to those used to derive (3.34) and (3.35), then for every $\epsilon > 0$, there exists $t_2 > 0$ such that for any $t \in (0,t_2)$,

$$\int_{B(x,\delta(x))\setminus B(x,\delta(x)-t)} |(F_{y+h_k e_l,b})^l_{h_k}(x) - D_{l,x}F_b(x,y)f(y)| < C\epsilon. \tag{3.37}$$

Moreover, for a fixed $t \in (0, \delta(x)]$, we have

$$\begin{aligned} &\Big| \int_{B(x,\delta(x)-t)} (F_{y+h_k e_l,b})^l_{h_k}(x) f_{h_k,l}(y)dy \\ &\quad - \int_{B(x,\delta(x)-t)} (F_{y+h_k e_l,b})^l_{h_k}(x) f(y)dy \Big| \\ &\leq \|b\|_{Lip(\Omega)} \int_{B(x,\delta(x)-t)} |f_{h_k,l}(y) - f(y)|dy \to 0 \quad \text{as} \quad k \to \infty. \end{aligned} \tag{3.38}$$

In view of (3.37) and (3.38), for (3.30) it is enough to show that for a fixed $t \in (0, \min\{\delta(x), t_2\})$,

$$\begin{aligned} &\lim_{k\to\infty} \int_{B(x,\delta(x)-t)} (F_{y+h_k e_l,b})^l_{h_k}(x) f(y)dy \\ &= \int_{B(x,\delta(x)-t)} D_{l,x}F_b(x,y) f(y)dy. \end{aligned} \tag{3.39}$$

By a change of variable,

$$\int_{B(x,\delta(x)-t)} (F_{y+h_k e_l,b})^l_{h_k}(x) f(y)dy = \int_{B(x+h_k e_l,\delta(x)-t)} (F_{y,b})^l_{h_k}(x) f_{h_k,l}(y)dy.$$

Since $h_k \to 0$ as $k \to \infty$, then we may assume that all $h_k < t/2$. Note that $(F_{y,b})^l_{h_k}(x) \to D_{l,x}F_b(x,y)$ as $k \to \infty$ for almost every $x \in \Omega$ and every $y \in \Omega$. Applying the dominated convergence theorem, we have that for almost every $x \in \Omega$,

$$\begin{aligned} &\Big| \int_{B(x+h_k e_l,\delta(x)-t)} (F_{y,b})^l_{h_k}(x) f(y)dy \\ &\quad - \int_{B(x+h_k e_l,\delta(x)-t)} D_{l,x}F_b(x,y) f(y)dy \Big| \\ &\leq \int_{B(x,\delta(x)-t/2)} |(F_{y,b})^l_{h_k}(x) - D_{l,x}F_b(x,y)||f(y)|dy \\ &\to 0 \quad \text{as} \quad k \to \infty. \end{aligned} \tag{3.40}$$

Note that $|D_{l,x}F_b(x,y)| \leq \|b\|_{Lip(\Omega)}$ and $f_{h_k,l} \to f$ in $L^p_{\text{loc}}(\Omega)$. Thus, we have

$$\begin{aligned} &\Big| \int_{B(x+h_k e_l,\delta(x)-t)} D_{l,x}F_b(x,y) f_{h_k,l}(y)dy \\ &\quad - \int_{B(x+h_k e_l,\delta(x)-t)} D_{l,x}F_b(x,y) f(y)dy \Big| \\ &\leq \|b\|_{Lip(\Omega)} \int_{B(x,\delta(x)-t/2)} |f_{h,k_l}(y) - f(y)|dy \to 0 \quad \text{as} \quad k \to \infty. \end{aligned} \tag{3.41}$$

We also note that $\lim_{k\to\infty} \chi_{B(x+h_k e_l,\delta(x)-t)}(y) = 1$ for all $y \in B(x, \delta(x) - t)$. This together with the fact that $|D_{l,x}F_b(x,y)f(y)| \leq \|b\|_{Lip(\Omega)}|f(y)| \in$

$L^1(B(x,\delta(x)-t/2))$ and the dominated convergence theorem implies that

$$\begin{aligned}(3.42)\qquad &\lim_{k\to\infty}\int_{B(x+h_ke_l,\delta(x)-t)} D_{l,x}F_b(x,y)f(y)dy\\ &= \int_{B(x,\delta(x)-t)} D_{l,x}F_b(x,y)f(y)dy.\end{aligned}$$

Then (3.39) follows from (3.40)-(3.42). □

Applying the arguments similar to those used to derive Lemma 3.5, we can get the following result. The details are omitted.

Lemma 3.6. *Let $f \in W^{1,p}(\Omega)$ with $1 < p < \infty$ and $b \in \mathrm{Lip}(\Omega)$. Let $\{h_k\}_{k\geq 1} \subset (0,\infty)$ be such that $\lim_{k\to\infty} h_k = 0$ and $l \in \{1, 2, \dots, n\}$. Assume that $|\Omega| < \infty$ and $\delta(x) \geq \delta(x+h_ke_l)$ for almost every $x \in \Omega$ and all $k \geq 1$. Then, for almost every $x \in \Omega$,*

$$\begin{aligned}&\lim_{k\to\infty}\int_{B(x,\delta(x+h_ke_l))} \frac{F_b(x+h_ke_l,y+h_ke_l)f_{h_k,l}(y)-F_b(x,y)f(y)}{h_k}dy\\ &= \int_{B(x,\delta(x))}(D_{l,y}F_b(x,y)+D_{l,x}F_b(x,y))f(y)dy\\ &\quad+\int_{B(x,\delta(x))}F_b(x,y)D_lf(y)dy.\end{aligned}$$

Lemma 3.7. *Let $f \in W^{1,p}(\Omega)$ for some $p \in (1,\infty)$ and $\{f_j\}_{j=1}^{\infty} \subsetneq W^{1,p}(\Omega)$ such that $f_j \to f$ in $W^{1,p}(\Omega)$ as $j \to \infty$. Assume $b \in \mathrm{Lip}(\Omega)$ and $|\Omega| < \infty$. If $\alpha = 0$ or $\alpha \in [1, n/p)$, then for any $K \subset\subset \Omega$ and all $l \in \{1, 2, \dots, n\}$, we have*

$$\lim_{j\to\infty}\|D_lM_{b,\alpha,\Omega}f_j - D_lM_{b,\alpha,\Omega}f\|_{L^{q_\alpha}(K_j)} = 0,$$

where

$$K_j := \{x \in K : \delta(x) \in I_{b,\alpha,f_j}(x) \cap I_{b,\alpha,f}(x)\}$$

and

$$q_\alpha = \begin{cases} p, & \text{if } \alpha = 0;\\ np/(n-(\alpha-1)p), & \text{if } \alpha \in [1,n/p).\end{cases}$$

Proof. By [35, Lemma 2.11], [29, Lemmas 5.1-5.2] and the arguments similar to those used to derive [32, Lemma 3.8], one can get the desire conclusion of Lemma 3.7. The details are omitted. □

3.2. Proof of Theorem 1.8

Applying Lemmas 3.1-3.7, [35, Lemma 2.11], [35, Corollary 2.7] and the arguments similar to those used to derive the proof of [32, Theorem 1.2], the conclusion of Theorem 1.8 can be proved. The details are omitted.

References

[1] M. Agcayazi, A. Gogatishvili, K. Koca, and R. Mustafayev, *A note on maximal commutators and commutators of maximal functions*, J. Math. Soc. Japan **67** (2015), no. 2, 581–593. https://doi.org/10.2969/jmsj/06720581

[2] J. M. Aldaz and J. Pérez Lázaro, *Functions of bounded variation, the derivative of the one dimensional maximal function, and applications to inequalities*, Trans. Amer. Math. Soc. **359** (2007), no. 5, 2443–2461. https://doi.org/10.1090/S0002-9947-06-04347-9

[3] A. M. Alphonse, *An end point estimate for maximal commutators*, J. Fourier Anal. Appl. **6** (2000), no. 4, 449–456. https://doi.org/10.1007/BF02510149

[4] D. Beltran, C. González-Riquelme, J. Madrid, and J. Weigt, *Continuity of the gradient of the fractional maximal operator on $W^{1,1}(\mathbb{R}^d)$*, to appear in Math. Res. Lett, arXiv:2102.10206v1.

[5] D. Beltran and J. Madrid, *Regularity of the centered fractional maximal function on radial functions*, J. Funct. Anal. **279** (2020), no. 8, 108686, 28 pp. https://doi.org/10.1016/j.jfa.2020.108686

[6] D. Beltran and J. Madrid, *Endpoint Sobolev continuity of the fractional maximal function in higher dimensions*, Int. Math. Res. Not. IMRN **2021** (2021), no. 22, 17316–17342. https://doi.org/10.1093/imrn/rnz281

[7] E. Carneiro, C. González-Riquelme, and J. Madrid, *Sunrise strategy for the continuity of maximal operators*, J. Anal. Math. **148** (2022), no. 1, 37–84. https://doi.org/10.1007/s11854-022-0222-7

[8] E. Carneiro and J. Madrid, *Derivative bounds for fractional maximal functions*, Trans. Amer. Math. Soc. **369** (2017), no. 6, 4063–4092. https://doi.org/10.1090/tran/6844

[9] E. Carneiro, J. Madrid, and L. B. Pierce, *Endpoint Sobolev and BV continuity for maximal operators*, J. Funct. Anal. **273** (2017), no. 10, 3262–3294. https://doi.org/10.1016/j.jfa.2017.08.012

[10] E. Carneiro and D. Moreira, *On the regularity of maximal operators*, Proc. Amer. Math. Soc. **136** (2008), no. 12, 4395–4404. https://doi.org/10.1090/S0002-9939-08-09515-4

[11] P. Chen, X. T. Duong, J. Li, and Q. Wu, *Compactness of Riesz transform commutator on stratified Lie groups*, J. Funct. Anal. **277** (2019), no. 6, 1639–1676. https://doi.org/10.1016/j.jfa.2019.05.008

[12] F. Deringoz, V. S. Guliyev, and S. G. Hasanov, *Commutators of fractional maximal operator on generalized Orlicz-Morrey spaces*, Positivity **22** (2018), no. 1, 141–158. https://doi.org/10.1007/s11117-017-0504-y

[13] F. Deringoz, V. S. Guliyev, and S. G. Samko, *Vanishing generalized Orlicz-Morrey spaces and fractional maximal operator*, Publ. Math. Debrecen **90** (2017), no. 1-2, 125–147.

[14] X. Duong, M. Lacey, J. Li, B. Wick, and Q. Wu, *Commutators of Cauchy-Szego type integrals for domains in $\mathbb{C}^n$ with minimal smoothness*, Indiana Univ. Math. J. **70** (2021), no. 4, 1505–1541. https://doi.org/10.1512/iumj.2021.70.8573

[15] Z. W. Fu, S. L. Gong, S. Z. Lu, and W. Yuan, *Weighted multilinear Hardy operators and commutators*, Forum Math. **27** (2015), no. 5, 2825–2851. https://doi.org/10.1515/forum-2013-0064

[16] J. García-Cuerva, E. Harboure, C. Segovia, and J. Torrea, *Weighted norm inequalities for commutators of strongly singular integrals*, Indiana Univ. Math. J. **40** (1991), no. 4, 1397–1420. https://doi.org/10.1512/iumj.1991.40.40063

[17] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, second edition, Springer, Berlin, 1983. https://doi.org/10.1007/978-3-642-61798-0

[18] C. González-Riquelme, *On the continuity of maximal operators of convolution type at the derivative level*, Israel J. Math. **253** (2023), no. 2, 745–759. https://doi.org/10.1007/s11856-022-2375-6

[19] C. González-Riquelme and D. Kosz, *BV continuity for the uncentered Hardy-Littlewood maximal operator*, J. Funct. Anal. **281** (2021), no. 2, Paper No. 109037, 20 pp. https://doi.org/10.1016/j.jfa.2021.109037

[20] V. S. Guliyev and F. Deringoz, *Some characterizations of Lipschitz spaces via commutators on generalized Orlicz-Morrey spaces*, Mediterr. J. Math. **15** (2018), no. 4, Paper No. 180, 19 pp. https://doi.org/10.1007/s00009-018-1226-5

[21] P. Hajłasz and J. Malý, *On approximate differentiability of the maximal function*, Proc. Amer. Math. Soc. **138** (2010), no. 1, 165–174. https://doi.org/10.1090/S0002-9939-09-09971-7

[22] P. Hajłasz and J. Onninen, *On boundedness of maximal functions in Sobolev spaces*, Ann. Acad. Sci. Fenn. Math. **29** (2004), no. 1, 167–176.

[23] J. Hart, F. Liu, and Q. Xue, *Regularity and continuity of local multilinear maximal type operators*, J. Geom. Anal. **31** (2021), no. 4, 3405–3454. https://doi.org/10.1007/s12220-020-00400-7

[24] T. Heikkinen, J. Kinnunen, J. Korvenpää, and H. Tuominen, *Regularity of the local fractional maximal function*, Ark. Mat. **53** (2015), no. 1, 127–154. https://doi.org/10.1007/s11512-014-0199-2

[25] J. Kinnunen, *The Hardy-Littlewood maximal function of a Sobolev function*, Israel J. Math. **100** (1997), 117–124. https://doi.org/10.1007/BF02773636

[26] J. Kinnunen and P. Lindqvist, *The derivative of the maximal function*, J. Reine Angew. Math. **503** (1998), 161–167.

[27] J. Kinnunen and E. Saksman, *Regularity of the fractional maximal function*, Bull. London Math. Soc. **35** (2003), no. 4, 529–535. https://doi.org/10.1112/S0024609303002017

[28] E. H. Lieb and M. P. Loss, *Analysis*, second edition, Graduate Studies in Mathematics, 14, Amer. Math. Soc., Providence, RI, 2001. https://doi.org/10.1090/gsm/014

[29] F. Liu and G. Wang, *Regularity of commutators of maximal operators with Lipschitz symbols*, Taiwanese J. Math. **25** (2021), no. 5, 1007–1039. https://doi.org/10.11650/tjm/210301

[30] F. Liu and H. Wu, *On the regularity of the multisublinear maximal functions*, Canad. Math. Bull. **58** (2015), no. 4, 808–817. https://doi.org/10.4153/CMB-2014-070-7

[31] F. Liu and S. Xi, *Sobolev regularity for commutators of the fractional maximal functions*, Banach J. Math. Anal. **15** (2021), no. 1, Paper No. 5, 36 pp. https://doi.org/10.1007/s43037-020-00095-6

[32] F. Liu, Q. Y. Xue, and K. Yabuta, *Sobolev boundedness and continuity for commutators of the local Hardy-Littlewood maximal function*, Ann. Fenn. Math. **47** (2022), no. 1, 203–235. https://doi.org/10.54330/afm.113296

[33] F. Liu, Q. Y. Xue, and P. Zhang, *Regularity and continuity of commutators of the Hardy-Littlewood maximal function*, Math. Nachr. **293** (2020), no. 3, 491–509. https://doi.org/10.1002/mana.201900013

[34] H. Luiro, *Continuity of the maximal operator in Sobolev spaces*, Proc. Amer. Math. Soc. **135** (2007), no. 1, 243–251. https://doi.org/10.1090/S0002-9939-06-08455-3

[35] H. Luiro, *On the regularity of the Hardy-Littlewood maximal operator on subdomains of* $\mathbb{R}^n$, Proc. Edinb. Math. Soc. (2) **53** (2010), no. 1, 211–237. https://doi.org/10.1017/S0013091507000867

[36] H. Luiro, *The variation of the maximal function of a radial function*, Ark. Mat. **56** (2018), no. 1, 147–161. https://doi.org/10.4310/ARKIV.2018.v56.n1.a9

[37] H. Luiro and J. Madrid, *The variation of the fractional maximal function of a radial function*, Int. Math. Res. Not. IMRN **2019** (2019), no. 17, 5284–5298. https://doi.org/10.1093/imrn/rnx277

[38] J. Madrid, *Endpoint Sobolev and BV continuity for maximal operators, II*, Rev. Mat. Iberoam. **35** (2019), no. 7, 2151–2168. https://doi.org/10.4171/rmi/1115

[39] C. Segovia and J. L. Torrea, *Weighted inequalities for commutators of fractional and singular integrals*, Publ. Mat. **35** (1991), no. 1, 209–235. https://doi.org/10.5565/PUBLMAT_35191_09

[40] C. Segovia and J. L. Torrea, *Higher order commutators for vector-valued Calderón-Zygmund operators*, Trans. Amer. Math. Soc. **336** (1993), no. 2, 537–556. https://doi.org/10.2307/2154362

[41] S. Shi, Z. Fu, and S. Z. Lu, *On the compactness of commutators of Hardy operators*, Pacific J. Math. **307** (2020), no. 1, 239–256. https://doi.org/10.2140/pjm.2020.307.239

[42] H. Tanaka, *A remark on the derivative of the one-dimensional Hardy-Littlewood maximal function*, Bull. Austral. Math. Soc. **65** (2002), no. 2, 253–258. https://doi.org/10.1017/S0004972700020293

[43] S. Yang, D. Chang, D. Yang, and Z. Fu, *Gradient estimates via rearrangements for solutions of some Schrödinger equations*, Anal. Appl. (Singap.) **16** (2018), no. 3, 339–361. https://doi.org/10.1142/S0219530517500142

[44] M. H. Yang, Z. Fu, and S. Liu, *Analyticity and existence of the Keller-Segel-Navier-Stokes equations in critical Besov spaces*, Adv. Nonlinear Stud. **18** (2018), no. 3, 517–535. https://doi.org/10.1515/ans-2017-6046

[45] M. H. Yang, Z. Fu, and J. Sun, *Existence and Gevrey regularity for a two-species chemotaxis system in homogeneous Besov spaces*, Sci. China Math. **60** (2017), no. 10, 1837–1856. https://doi.org/10.1007/s11425-016-0490-y

[46] P. Zhang, *Characterization of Lipschitz spaces via commutators of the Hardy-Littlewood maximal function*, C. R. Math. Acad. Sci. Paris **355** (2017), no. 3, 336–344. https://doi.org/10.1016/j.crma.2017.01.022

[47] P. Zhang, *Characterization of boundedness of some commutators of maximal functions in terms of Lipschitz spaces*, Anal. Math. Phys. **9** (2019), no. 3, 1411–1427. https://doi.org/10.1007/s13324-018-0245-5

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