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On the Growth of Transcendental Meromorphic Solutions of Certain algebraic Difference Equations

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Abstract. In this article, we investigate the growth of meromorphic solutions of

$$a(z)(\frac{\triangle_c\eta}{\eta})^2+(b_2(z)\eta^2(z)+b_1(z)\eta(z)+b_0(z))\frac{\triangle_c\eta}{\eta}$$
$$=d_4(z)\eta^4(z)+d_3(z)\eta^3(z)+d_2(z)\eta^2(z)+d_1(z)\eta(z)+d_0(z),$$

where $a(z), b_i(z)$ for $i=0,1,2$ and $d_j(z)$ for $j=0,...,4$ are given functions, $\triangle_c\eta=\eta(z+c)-\eta(z)$ with $c\in\mathbb{C}\backslash\{0\}$. In particular, when the $a(z)$, the $b_i(z)$ and the $d_j(z)$ are polynomials, and $d_4(z)\equiv 0$, we shall show that if $\eta(z)$ is a transcendental entire solution of finite order, and either $\deg a(z)\neq\deg d_0(z)+1$, or, $\deg a(z)=\deg d_0(z)+1$ and $\rho(\eta)\neq\frac{1}{2}$, then $\rho(\eta)\geq 1$.

1. Introduction and Main Results

We begin by discussing the case of differential equations, and then move on to difference equations. Concerning the case of first-order differential equations, Malmquist [13] showed a century ago that the only equation of the form

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$$\eta' = R(z, \eta),$$

where R is rational in both arguments, that can have transcendental meromorphic solutions, is the Riccati equation:

$$\eta' = a_0(z) + a_1(z)\eta + a_2(z)\eta^2.$$

In 1954, Wittich [15] obtained the result that if the coefficients $a_j(z)$ are rational functions, then all meromorphic solutions of the Riccati equation are of finite order.

We consider a more general case of the following first-order algebraic differential equation

$$C(z,\eta)(\eta')^2 + B(z,\eta)\eta' + A(z,\eta) = 0, \tag{1.1}$$

where $C(z,\eta) \not\equiv 0, B(z,\eta)$ and $A(z,\eta)$ are polynomials in z and η. In 1980, Steinmetz [14] showed that if (1.1) has a transcendental meromorphic solution, then the equation (1.1) can be reduced to the form

$$\begin{aligned} &a(z)\eta'^2 + (b_2(z)\eta^2 + b_1(z)\eta + b_0(z))\eta' \\ &= d_4(z)\eta^4(z) + d_3(z)\eta^3 + d_2(z)\eta^2 + d_1(z)\eta + d_0(z). \end{aligned} \tag{1.2}$$

where $a(z), b_i(z)$ for $i = 0, 1, 2$ and $d_j(z)$ for $j = 0, ..., 4$ are polynomials.

In this paper, we adopt the standard notation of Nevanlinna theory, as found in [7, 16]. Moreover, the forward difference $\triangle_c\eta$ is defined as $\triangle_c\eta = \eta(z+c) - \eta(z)$. In recent years, there has been tremendous interest in developing the value distribution of meromorphic functions with respect to a difference analogue, see [3, 4]. In 2018, Ishizaki and Korhonen [8] investigated meromorphic solutions of a difference equation of the form

$$\triangle\eta(z)^2 = A(z)(\eta(z)\eta(z+1) - B(z)).$$

They proved that the above difference equation possesses a continuous limit to the difference equation

$$(\eta')^2 = A(z)(\eta^2 - 1),$$

which extends to solutions in certain cases.

For a more general case, next let us consider the difference analogue of (1.2). It is interesting to consider the nature of a meromorphic solution η of

$$\begin{aligned} &a(z)\left(\frac{\triangle_c\eta}{\eta}\right)^2 + (b_2(z)\eta^2(z) + b_1(z)\eta(z) + b_0(z))\frac{\triangle_c\eta}{\eta} \\ &= d_4(z)\eta^4(z) + d_3(z)\eta^3(z) + d_2(z)\eta^2(z) + d_1(z)\eta(z) + d_0(z). \end{aligned} \tag{1.3}$$

Our first theorem is about the growth of meromorphic solutions of (1.3).

Theorem 1.1. *Let $c \in \mathbb{C} \setminus \{0\}$, let $T(r, a(z)) = S(r, \eta)$, let $T(r, b_i(z)) = S(r, \eta)$ for $i = 0, 1, 2$, let $T(r, d_j(z)) = S(r, \eta)$ for $j = 0, \cdots, 4$ and let $d_4(z) \not\equiv 0$. If $\eta(z)$ is a transcendental meromorphic solution of (1.3), then $\rho(\eta) = \infty$.*

Here $\rho(\eta)$ denotes the order of growth of the meromorphic function $\eta(z)$. In what follows $\lambda(\eta)$ and $\lambda(\frac{1}{\eta})$ denote the exponents of convergence of the zeros and poles of $\eta(z)$, respectively. While the above result was about the case $d_4(z) \not\equiv 0$, the following is about the case $d_4(z) \equiv 0$. Indeed, taking $d_4(z) \equiv 0$, (1.3) becomes

$$\begin{aligned} & a(z)\eta'^2 + (b_2(z)\eta^2 + b_1(z)\eta + b_0(z))\eta' \\ & = d_4(z)\eta^4(z) + d_3(z)\eta^3 + d_2(z)\eta^2 + d_1(z)\eta + d_0(z) \end{aligned} \tag{1.4}$$

Using the method from Liao and Yang [11], we obtain

Theorem 1.2. *Let $c \in \mathbb{C} \setminus \{0\}$, and let $a(z), b_i(z)$ for $i = 0, 1, 2)$, and $d_j(z)$ for $j = 0, 1, 2, 3$ be polynomials. If $\eta(z)$ is a finite order transcendental entire solution of (1.4), and either $\deg a(z) \neq \deg d_0(z) + 1$, or, $\deg a(z) = \deg d_0(z) + 1$ and $\rho(\eta) \neq \frac{1}{2}$, then*

$$\rho(\eta) \geq 1.$$

2. Proof of Theorem 1.1

Let $c_j, j = 1, \cdots, n$, be a finite collection of complex numbers. Then a difference polynomial in $\eta(z)$ is a function which is polynomial in $\eta(z + c_j)$ for $j = 1, \cdots, n$, with meromorphic coefficients $a_\lambda(z)$ such that $T(r, a_\lambda) = S(r, \eta)$ for all λ. As for difference counterparts of the Clunie Lemma [1], see [5, Corollary 3.3]. The following lemma is a more general version. The following lemma due to Laine and Yang [9] is an analogue of a result due to A. Z. Mohon'ko and V. D. Mohon'ko [12] on differential equations. We start by recalling some lemmas.

Lemma 2.1 [9] Let η be a transcendental meromorphic solution of finite order of a difference equation of the form

$$U(z, \eta)P(z, \eta) = Q(z, \eta), \tag{2.1}$$

where $U(z, \eta), P(z, \eta)$, and $Q(z, \eta)$ are difference polynomials such that the total degree $\deg U(z, \eta) = n$ in $\eta(z)$ and its shifts, and $\deg Q(z, \eta) \leq n$. Moreover, we assume that $U(z, \eta)$ contains just one term of maximal total degree in $\eta(z)$ and its shifts. Then

$$m(r, P(z, \eta)) = S(r, \eta).$$

We need one more lemma from [6]. We say that η has more than $S(r, \eta)$ poles of a certain type, if the integrated counting function of these poles is not of type $S(r, \eta)$.

We use the notation $D(z_0, r)$ to denote an open disc of radius r centered at $z_0 \in \mathbb{C}$. Also, ∞^k denotes a pole of η with multiplicity k. Similarly, 0^k and $a + 0^k$ denote a zero and a-point of η, respectively, with the multiplicity k.

Lemma 2.2 [6] Let η be a meromorphic function having more than $S(r,\eta)$ poles, and let $a_s, s = 1, \cdots, n$, be small meromorphic functions with respect to η. Denote by m_j the maximum order of zeros and poles of the functions a_s at z_j. Then for any $\varepsilon > 0$, there are at most $S(r,\eta)$ points z_j such that q

$$\eta(z_j) = \infty^{k_j},$$

where $m_j \geq \varepsilon k_j$.

Proof of Theorem 1.1. Let η be a meromorphic solution of (1.3). We assume that $\rho(\eta) = \rho < \infty$. (1.3) can be written as follows

$$d_4(z)\eta^6(z) = Q(z,\eta), \tag{2.1}$$

where $Q(z,\eta) = a(z)(\triangle_c\eta)^2 + (b_2(z)\eta^2(z) + b_1(z)\eta(z) + b_0(z))\triangle_c \eta\eta - d_3(z)\eta^5(z) - d_2(z)\eta^4(z) - d_1(z)\eta^3(z) - d_0(z)\eta^2(z)$. Since the total degree of $Q(z,\eta)$ as a polynomial in $\eta(z)$ and its shifts, $\deg Q(z,\eta) \leq 5$, by Lemma 2.1 and (2.1), we have

$$m(r,\eta) = S(r,\eta).$$

So, η has more than $S(r,\eta)$ poles, counting multiplicities. Using z_j to denote points in the pole sequence. By Lemma 2.2, we obtain that there exist more than $S(r,\eta)$ points such that $\eta(z_j) = \infty^{k_j}$, where $\varepsilon k_j > m_j$. Here m_j refers to the coefficients $a(z), b_i(z)(i = 0,1,2), d_j(z)(j = 0,1,2,3)$. Denoting the sequence of such poles by $z_{1,j}$, we take this sequence as our starting point. For $\varepsilon < \frac{1}{8}$, (1.3) implies that $\eta(z_{1,j} + c) = \infty^{k_{2,j}}$, where $k_{2,j} \geq (2-\varepsilon)k_{1,j}$. Lemma 2.2 implies that η has more than $S(r,\eta)$ such points $z_{2,j}$ such that $\eta(z_{2,j}) = \infty^{k_{2,j}}$, where $\varepsilon k_{2,j} > m_{2,j}$. Then we only pick one of these points and denote it by $z_{2,j}$. Continuing to the next phase. By (1.3), we deduce that $z_{3,j} := z_{2,j} + c$ is a pole of η of multiplicity $k_{3,j}$, where

$$k_{3,j} \geq (2-\varepsilon)k_{2,j} \geq (2-\varepsilon)^2 k_{1,j}.$$

Following the steps above, we can find a sequence z_n of poles of η, the multiplicity of which is k_n, and $k_n \geq (2-\varepsilon)^{n-1}k_1 \geq (2-\varepsilon)^{n-1}$.
By a simple geometric observation, we have

$$z_n \in D(z_1, (n-1)|c|) \subset D(0, |z_1| + (n-1)|c|) = D(0, r_n).$$

As $n \to \infty$, we have $r_n \leq 2(n-1)|c|$. So,

$$n(r_n, f) \geq (2-\varepsilon)^{n-1} > \left(\frac{15}{8}\right)^{n-1}.$$

Hence, we have $\lambda(\eta)=\infty$, a contradiction. So $\rho(\eta)=\infty$.

3. Proof of Theorem 1.2

Lemma 3.1 [2] Let η be a transcendental entire function of order $\rho(\eta)=\rho<1$, let $0<\varepsilon<\frac{1}{8}$ and z be such that $|z|=r$, where

$$|\eta(z)|>M(r,\eta)(\nu(r,\eta))^{-\frac{1}{8}+\varepsilon}$$

holds. Then for each positive integer k, there exists a set $E\subset(1,\infty)$ that has finite logarithmic measure, such that for all $r\notin E\cup[0,1]$,

$$\frac{\triangle_c\eta}{\eta}=c\frac{\nu(r,\eta)}{z}(1+o(1)).$$

Lemma 3.2 [10] Suppose that $\eta(z)$ is a transcendental entire function of finite order $\rho(\eta)=\rho<\infty$, and that a set $E_r\subset R^+$ has a finite logarithmic measure. Then, there exists a sequence of positive numbers r_k satisfying $r_k\notin E_r$ and $r_k\to\infty$ such that for given $\varepsilon>0$, as r_k sufficiently lager, we have $r_k^{\rho-\varepsilon}<\nu(r_k,\eta)<r_k^{\rho+\varepsilon}$ and $\exp r_k^{\rho-\varepsilon}<M(r_k,\eta)<\exp r_k^{\rho+\varepsilon}$.

Proof of Theorem 1.2. Suppose that $\eta(z)$ is a transcendental entire function. Suppose, contrary to the assertion, that $\rho(\eta)=\rho<1$. If $d_3(z)\not\equiv 0$, then we can write (1.4) in this form

$$\begin{aligned}d_3(z)=&\frac{a(z)}{\eta^3}\left(\frac{\triangle_c\eta}{\eta}\right)^2+\frac{b_2(z)}{\eta}\frac{\triangle_c\eta}{\eta}+\frac{b_1(z)}{\eta^2}\frac{\triangle_c\eta}{\eta}\\&+\frac{b_0(z)}{\eta^3}\frac{\triangle_c\eta}{\eta}-\frac{d_2(z)}{\eta}-\frac{d_1(z)}{\eta^2}-\frac{d_0(z)}{\eta^3}.\end{aligned}\tag{4.1}$$

By Lemma 3.1, we know that there exists a set $H\subset(1,\infty)$ of finite logarithmic measure, such that

$$\frac{\triangle_c\eta}{\eta}=c\frac{\nu(r,\eta)}{z}(1+o(1)),\ \ |z|=r\notin H,\tag{4.2}$$

where z satisfy $|z|=r$ and $|\eta(z)|=M(r,\eta)$, $\nu(r,\eta)$ is the central index of $\eta(z)$. By Lemma 3.2, we see that there exist some infinite sequence of points z_k such that $|\eta(z_k)|=M(r_k,\eta)$, and such that for any given $\varepsilon(0<\varepsilon<\frac{1-\rho}{2})$, as $r_k\to\infty$, and $|z_k|=r_k\notin H_1\cup H\cup[0,1]$,where $H_1\subset(1,\infty)$ is a subset with finite logarithmic measure, we have

$$\frac{\nu(r_k,\eta)}{r_k}<r_k^{\rho+\varepsilon-1}\to 0.\tag{4.3}$$

Thus, by (4.1)-(4.3), we deduce that as z_k satisfy $|\eta(z_k)| = M(r_k, \eta), |z_k| = r_k \notin H_1 \cup H \cup [0,1], r_k \to \infty$

$$
\begin{aligned}
(4.4) \qquad |d_3(z_k)| &\leq |\frac{a(z_k)}{M(r_k,\eta)^3}(\frac{\triangle_c\eta}{\eta})^2| + |\frac{b_2(z_k)}{M(r_k,\eta)}\frac{\triangle_c\eta}{\eta}| \\
&+ |\frac{b_1(z_k)}{(M(r_k,\eta)^2)}\frac{\triangle_c\eta}{\eta}| + |\frac{b_0(z_k)}{M(r_k,\eta)^3}\frac{\triangle_c\eta}{\eta}| \\
&+ |\frac{d_2(z_k)}{M(r_k,\eta)}| + |\frac{d_1(z_k)}{M(r_k,\eta)^2}| + |\frac{d_0(z_k)}{M(r_k,\eta)^3}| \\
&= |\frac{a(z_k)}{M(r_k,\eta)^3}||(c\frac{\nu(r_k,\eta)}{r_k}(1+o(1)))^2| \\
&+ |\frac{b_2(z_k)}{M(r_k,\eta)}||c\frac{\nu(r_k,\eta)}{r_k}(1+o(1))| \\
&+ |\frac{b_1(z_k)}{(M(r_k,\eta))^2}||c\frac{\nu(r_k,\eta)}{r_k}(1+o(1))| \\
&+ |\frac{b_0(z_k)}{M(r_k,\eta)^3}||c\frac{\nu(r_k,\eta)}{r_k}(1+o(1))| \\
&+ |\frac{d_2(z_k)}{M(r_k,\eta)}| + |\frac{d_1(z_k)}{M(r_k,\eta)^2}| + |\frac{d_0(z_k)}{M(r_k,\eta)^3}| \to 0.
\end{aligned}
$$

This is impossible. Hence $d_3(z) \equiv 0$. Now we may write (1.4) as follows

$$
(4.5) \quad d_2(z) - b_2(z)\frac{\triangle_c\eta}{\eta} = \frac{a(z)}{\eta^2}(\frac{\triangle_c\eta}{\eta})^2 + \frac{b_1(z)}{\eta}\frac{\triangle_c\eta}{\eta} + \frac{b_0(z)}{\eta^2}\frac{\triangle_c\eta}{\eta} - \frac{d_1(z)}{\eta} - \frac{d_0(z)}{\eta^2}.
$$

By (4.2), (4.3), and (4.5), we know that

$$
\begin{aligned}
(4.6) \qquad & ||d_2(z_k)| - |b_2(z_k)c\frac{\nu(r_k,\eta)}{r_k}(1+o(1))|| \\
&\leq |d_2(z_k) - b_2(z_k)\frac{\triangle_c\eta}{\eta}| \\
&= |\frac{a(z_k)}{\eta^2}(\frac{\triangle_c\eta}{\eta})^2 + \frac{b_1(z_k)}{\eta}\frac{\triangle_c\eta}{\eta} + \frac{b_0(z_k)}{\eta^2}\frac{\triangle_c\eta}{\eta} - \frac{d_1(z_k)}{\eta} - \frac{d_0(z_k)}{\eta^2}| \\
&= |\frac{a(z_k)}{M(r_k,\eta)^2}||(c\frac{\nu(r_k,\eta)}{r_k}(1+o(1)))^2| \\
&+ |\frac{b_1(z_k)}{M(r_k,\eta)}||c\frac{\nu(r_k,\eta)}{r_k}(1+o(1))| \\
&+ |\frac{b_0(z_k)}{M(r_k,\eta)^2}||c\frac{\nu(r_k,\eta)}{r_k}(1+o(1))| \\
&+ |\frac{d_1(z_k)}{M(r_k,\eta)}| + |\frac{d_0(z_k)}{M(r_k,\eta)^2}| \to 0.
\end{aligned}
$$

We divide the proof into the following two cases

Case 1. If $b_2(z) \equiv 0$, then (4.6) implies that $d_2(z) \equiv 0$, hence (1.4) can be written as following

$$a(z)\left(\frac{\triangle_c\eta}{\eta}\right)^2 + (b_1(z)\eta(z) + b_0(z))\frac{\triangle_c\eta}{\eta} = d_1(z)\eta(z) + d_0(z). \tag{4.7}$$

By computing (4.7), we have

$$\begin{aligned} & ||d_1(z)| - |b_1(z)\frac{\triangle_c\eta}{\eta}|| \\ & \leq |d_1(z) - b_1(z)\frac{\triangle_c\eta}{\eta}| \\ & = |\frac{a(z)}{\eta}(\frac{\triangle_c\eta}{\eta})^2 + \frac{b_0(z)}{\eta}\frac{\triangle_c\eta}{\eta} - \frac{d_0(z)}{\eta}| \\ & \leq |\frac{a(z)}{\eta}(\frac{\triangle_c\eta}{\eta})^2| + |\frac{b_0(z)}{\eta}\frac{\triangle_c\eta}{\eta}| + |\frac{d_0(z)}{\eta}|. \end{aligned} \tag{4.8}$$

By (4.1)-(4.3) and (4.8), we obtain that as z_k satisfy $|\eta(z_k)| = M(r_k, \eta), |z_k| = r_k \notin H_1 \cup H \cup [0, 1], r_k \to \infty$

$$\begin{aligned} & ||d_1(z_k)| - |b_1(z_k)c\frac{\nu(r_k,\eta)}{r_k}(1 + o(1))|| \\ & \leq |\frac{a(z_k)}{M(r_k,\eta)}(c\frac{\nu(r_k,\eta)}{r_k}(1 + o(1)))^2| \\ & + |\frac{b_0(z_k)}{M(r_k,\eta)}c\frac{\nu(r_k,\eta)}{r_k}(1 + o(1))| \\ & + |\frac{d_0(z_k)}{M(r_k,\eta)}| \to 0. \end{aligned} \tag{4.9}$$

If $b_1(z) \not\equiv 0$, then by (4.9),

$$\frac{||d_1(z_k)| - |b_1(z_k)c\frac{\nu(r_k,\eta)}{r_k}(1 + o(1))||}{b_1(z_k)} \to 0. \tag{4.10}$$

(4.3) and (4.10) imply that

$$\frac{d_1(z_k)}{b_1(z_k)} \to 0, \tag{4.11}$$

as $k \to \infty$. Since $d_1(z)$ and $b_1(z)$ are polynomials, we obtain that by (4.11)

$$\frac{z_k d_1(z_k)}{b_1(z_k)} \to q, \tag{4.12}$$

as $k\to\infty$, and q is a finite constant. Suppose that $d_1(z)\not\equiv 0$. Then we deduce that from (4.10) and (4.12)

$$|q|=\lim_{k\to\infty}|\frac{z_kd_1(z_k)}{b_1(z_k)}|=|c|\lim_{k\to\infty}\nu(r_k,\eta)(1+o(1))=\infty.$$

This is impossible. Hence $d_1(z)\equiv 0$. (1.4) can be reduced into

$$a(z)(\frac{\triangle_c\eta}{\eta})^2+(b_1(z)\eta(z)+b_0(z))\frac{\triangle_c\eta}{\eta}=d_0(z), \tag{4.13}$$

By the above assumption, we know $b_1(z)\not\equiv 0$, then (4.13) implies that

$$\begin{aligned}&|\triangle_c\eta|\\&=|\frac{d_0(z_k)}{b_1(z_k)}-\frac{a(z_k)(\frac{\triangle_c\eta}{\eta})^2}{b_1(z_k)}-\frac{b_0(z_k)\frac{\triangle_c\eta}{\eta}}{b_1(z_k)}|\\&\le|\frac{d_0(z_k)}{b_1(z_k)}|+|\frac{a(z_k)(\frac{\triangle_c\eta}{\eta})^2}{b_1(z_k)}|+|\frac{b_0(z_k)\frac{\triangle_c\eta}{\eta}}{b_1(z_k)}|\\&\le Mr_k^N.\end{aligned} \tag{4.14}$$

where M and N are some finite constants. On the other hand, we know that

$$|\frac{\triangle_c\eta}{Mr_k^N}|=|c|\frac{\nu(r_k,\eta)(1+o(1))M(r_k,\eta)}{|M|r_k^{N+1}}\to\infty, \tag{4.15}$$

as $k\to\infty$, a contradiction. Hence $b_1(z)\equiv 0$. By (4.9), we also get $d_1(z)\equiv 0$. Hence, we can write (1.4) as follows

$$a(z)(\frac{\triangle_c\eta}{\eta})^2+b_0(z)\frac{\triangle_c\eta}{\eta}=d_0(z). \tag{4.16}$$

We assume that $a(z)\not\equiv 0$, next we consider the following two subcases.
Subcase I. $\deg b_0(z)\ge\deg a(z)$, we have by (4.16), (4.2) and (4.3),

$$|lb_0(z_k)\frac{\nu(r_k,\eta)}{r_k}(1+o(1))|=|d_0(z_k)| \tag{4.17}$$

where l is a finite nonzero constant. So

$$\lim_{k\to\infty}|\frac{d_0(z_k)}{b_0(z_k)}|=\lim_{k\to\infty}|l|\frac{\nu(r_k,\eta)}{r_k}(1+o(1))=0 \tag{4.18}$$

(4.18) implies that

$$\lim_{k\to\infty}\frac{z_kd_0(z_k)}{b_0(z_k)}\to l_1, \tag{4.19}$$

where l_1 is some finite constant. By (4.19) and (4.17), as $k \to \infty$, we obtain

$$\nu(r_k, \eta) \le |\frac{d_0(z_k)}{b_0(z_k)} r_k| \to l_1.$$

We can get a contradiction, since $\nu(r_k, \eta) \to \infty$, as $k \to \infty$.
Subcase 2. $\deg b_0(z) < \deg a(z)$, we have by (4.16), (4.2) and (4.3),

$$|\frac{d_0(z_k)}{a(z_k)}| \le (c\frac{\nu(r_k, \eta)}{r_k})^2(1 + o(1))| + |\frac{b_0(z_k)}{a(z_k)} c\frac{\nu(r_k, \eta)}{r_k}(1 + o(1))| \to 0, \tag{4.20}$$

as $k \to \infty$. We assume that $d_0(z) \not\equiv 0$. If $\deg a(z) = \deg b_0(z) + 1$, then as $k \to \infty$

$$\frac{r_k d_0(z_k)}{a(z_k)} = l_2, \quad \frac{r_k b_0(z_k)}{a(z_k)} \to l_3 \tag{4.21}$$

where l_2 is a finite nonzero constant, and l_2 is a finite constant. By Lemma 3.2, we have

$$r_n^{\rho(\eta)-\varepsilon} < \nu(r_n, \eta) < r_n^{\rho(\eta)+\varepsilon}. \tag{4.22}$$

If $\rho(\eta) < \frac{1}{2}$, then by (4.16), (4.21) and (4.22), for any given $\varepsilon(0 < \varepsilon < \frac{1-2\rho}{2})$, we have

$$|l_2| = |\frac{r_k d_0(z_k)}{a(z_k)}| \le |\frac{\nu(r_k, \eta)^2}{r_k}| + |\frac{\nu(r_k, \eta)}{r_k} \frac{r_k b_0(z_k)}{a(z_k)}| \le r_k^{2\rho+2\varepsilon-1} + |l_3| r_k^{\rho+\varepsilon-1} \to 0,$$

a contradiction. If $\rho(\eta) > \frac{1}{2}$, then by (4.16), (4.21) and (4.22), for any given $\varepsilon(0 < \varepsilon < \frac{1-2\rho}{2})$, we have

$$r_k^{2\rho-1-\varepsilon} \le |\frac{(\nu(r_k, \eta)^2)}{r_k}| \le |\frac{(\nu(r_k, \eta))}{r_k}||\frac{r_k b_0(z_k)}{a(z_k)}| + |\frac{r_k d_0(z_k)}{a(z_k)}| \le l_4,$$

where l_4 is some finite constant. This is impossible, since $r_k^{2\rho-1-\varepsilon} \to \infty$, as $k \to \infty$.
If $\deg a(z) > \deg d_0(z) + 1$, as $k \to \infty$, we have

$$\frac{r_k b_0(z_k)}{a(z_k)} \to l_3, \quad \frac{r_k^2 d_0(z_k)}{a(z_k)} \to l_5, \tag{4.23}$$

where l_5 are some finite constants. By (4.23) and (4.16), as $k \to \infty$, we have

$$\nu(r_k, \eta) \le |\frac{r_k b_0(z_k)}{a(z_k)}| + |\frac{r_k^2 d_0(z_k)}{a(z_k)} \frac{1}{\nu(r_k, \eta)}| \to l_6,$$

where l_6 is some finite constant, we can get a contradiction, since $\nu(r_k, \eta) \to \infty$. So $d_0(z) \equiv 0$. By (4.16), we have

$$\nu(r_k, \eta) \le |\frac{r_k b_0(z_k)}{a(z_k)}| \to l_3,$$

a contradiction.

By Subcase 1 and Subcase 2, we have $a(z) \equiv 0$. So (1.4) can be reduced into

$$b_0(z)\frac{\triangle_c \eta}{\eta} = d_0(z) \tag{4.24}$$

Together (4.24) and (4.2), we obtain

$$b_0(z_k)c\frac{\nu(r_k, \eta)}{r_k} = d_0(z_k). \tag{4.25}$$

(4.25) implies that either $\lim_{k\to\infty} \nu(r_k, \eta) = l_7$, where l_7 is a finite constant, or $\nu(r_k, \eta) \geq l_8 r_k^n$, where l_8 is a finite nonzero constant, and n is a positive integer. This is a contradiction.

Case 2. If $b_2(z) \not\equiv 0$, then by (4.6)

$$\frac{||d_2(z_k)| - |b_2(z_k)c\frac{\nu(r_k,\eta)}{r_k}(1 + o(1))||}{b_2(z_k)} \to 0. \tag{4.26}$$

By (4.26), we have

$$\frac{d_2(z_k)}{b_2(z_k)} \to 0, \tag{4.27}$$

as $k \to \infty$. Since $d_2(z)$ and $b_2(z)$ are polynomials, we get by (4.27)

$$\frac{z_k d_2(z_k)}{b_2(z_k)} \to l_9, \tag{4.28}$$

as $k \to \infty$, and l_9 is a finite constant. Suppose that $d_2(z) \not\equiv 0$. Then we deduce that from (4.28)

$$l_9 = \lim_{k\to\infty} |\frac{z_k d_2(z_k)}{b_2(z_k)}| = |c| \lim_{k\to\infty} \nu(r_k, \eta)(1 + o(1)) = \infty. \tag{4.29}$$

This is impossible, since l_9 is a finite constant. Hence $d_2(z) \equiv 0$. (1.4) can be reduced into

$$\begin{aligned}
| \triangle_c \eta| &= |\frac{d_1(z_k)}{b_2(z_k)} + \frac{d_0(z_k)}{b_2(z_k)}\frac{1}{\eta(z)} - \frac{b_1(z_k)\frac{\triangle_c\eta}{\eta}}{b_2(z_k)} - \frac{a(z_k)(\frac{\triangle_c\eta}{\eta})^2}{b_2(z_k)}\frac{1}{\eta} - \frac{b_0(z_k)\frac{\triangle_c\eta}{\eta}}{b_2(z_k)}\frac{1}{\eta}| \\
&\leq |\frac{d_1(z_k)}{b_2(z_k)}| + |\frac{d_0(z_k)}{b_2(z_k)}\frac{1}{\eta(z)}| + |\frac{b_1(z_k)(\frac{\triangle_c\eta}{\eta})^2}{b_2(z_k)}| + |\frac{a(z_k)(\frac{\triangle_c\eta}{\eta})^2}{b_2(z_k)}\frac{1}{\eta}| + |\frac{b_0(z_k)\frac{\triangle_c\eta}{\eta}}{b_2(z_k)}\frac{1}{\eta}| \\
&\leq l_{10} r_k^{l^{11}}
\end{aligned}$$

where l_{10} and l_{11} are some finite constants. On the other hand, we know that

$$\frac{\triangle_c \eta}{l_{10} r_k^{l_{11}}} = |c| \frac{\nu(r_k, \eta)(1 + o(1))M(r_k, \eta)}{l_{10} r_k^{l_{11}+1}} \to \infty, \tag{4.30}$$

as $k \to \infty$. This is a contradiction, $\frac{\triangle_c \eta}{l_{10} r_k^{l_{11}}} < 1$

By Case 1 and Case 2, we know $\rho(\eta) \geq 1$.

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