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# Topology on Semi-Well Ordered Sets

Angela Sunny* and P. Sini
*Department of Mathematics, University of Calicut, Malappuram, Kerala, India*
*e-mail* : angelasunny@uoc.ac.in *and* sini@uoc.ac.in

Abstract. A semi-well ordered set is a partially ordered set in which every non-empty subset of it contains a least element or a greatest element. It is defined as an extension of the concept of well ordered sets. An attempt is made to identify the properties of a semi-well ordered set equipped with the order topology.

## 1. Introduction

In 1985, Ramachandran introduced the concept of semi-well ordered sets during his investigation of anti-homogeneous topological spaces. A partially ordered set is said to be semi-well ordered [11, 12] if every non-empty subset of it has a least element or a greatest element. Sharkovsky's ordering [3] on the set of positive integers introduced by Oleksandr Sharkovsky in his result on the discrete dynamical system is an example of a semi-well ordered set. The concept of semi-well ordered sets is an extension of the concept of well ordered sets. It is easy to see that a semi-well ordered set is linearly ordered. To generalise the idea of well-ordered sets, many attempts have been made in the field of set theory, including those by [5, 8, 7, 4, 17, 9, 10, 13]. One such generalization is the idea of partially well ordered sets [17]. However, the collection of partially well ordered sets and the collection of semi-well ordered sets are two separate extensions of the collection of well ordered sets, whose only intersection is the collection of well ordered sets. A detailed study on semi-well ordered sets and semi-ordinals can be found in [16].

This paper is an attempt to explore the properties of order topology on semi-well ordered sets. Since a semi-well ordered set is linearly ordered, it is Hausdorff, completely normal [2] and Sober [6]. In [14] Sreeja proved that the order topology on a semi-well ordered set is scattered and anti-rigid. Section 2 is a collection of definitions, notations, and basic results which we will use further on. In Section 3, a

* Corresponding Author.

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detailed study on semi-well ordered sets with order topology is made. We prove that a semi-well ordered set together with its order topology is totally disconnected, zero dimensional, and strongly locally compact. Also, we identify the isolated points of a semi-well ordered set. A characterization of compact semi-well ordered sets and infinite homogeneous semi-well ordered sets is also provided. Several other properties of this topological space are also discussed.

## 2. Preliminaries

In this section some basic definitions, notations, and results which we will use further on are collected.

A partially ordered set is said to be *well ordered* if every non-empty subset of it has a least element and it is said to be *co-well ordered* [11] if every non-empty subset of it has a greatest element. A *semi-well ordered set* [11] is a partially ordered set in which every non-empty subset of it has a least element or a greatest element. It is obvious that every well ordered and co-well ordered set is semi-well ordered.

Let $(X, \leq_1)$ and $(Y, \leq_2)$ be two disjoint linearly ordered sets, then $X+Y$ denote the partially ordered set $X \cup Y$, where the order on $X \cup Y$ is $\leq_1 \cup \leq_2 \cup \{(x, y)\colon x \in X \text{ and } y \in Y\}$. In [11] Ramachandran proved that: "Every semi-well ordered set $(X, \leq)$ can be represented as $A+B$, where $A$ and $B$ are well ordered and co-well ordered subsets of $X$ respectively". In this representation $A$ is the set consisting of all elements of $X$ with no immediate predecessor and those elements of $X$ which is an $n^{th}$ successor of an element with no immediate predecessor and $B$ is the complement of $A$ in $X$. The *interval topology* [18] on a partially ordered set $(X, \leq)$ is a topology in which $\{x \in X\colon x \leq a\}$ and $\{x \in X\colon x \geq a\}$ forms a sub-base for the closed sets. Interval topology is equivalent to the order topology on chains [1].

A topological space is said to be *zero dimensional* [15] if it has a basis consisting of clopen subsets. If every point in a topological space is contained in an open set whose closure is compact, the space is said to be strongly locally compact [15]. An element of a subset $A$ of a topological space is said to be *isolated* [15] if it is contained in an open set containing no other points of $A$. A topological space $(X, \mathscr{T})$ is said to be *homogeneous* if for any $x, y \in X$ there exists a homeomorphism $f$ on $X$ such that $f(x) = y$.

## 3. Topology on Semi-Well Ordered Sets

In the sequel, we analyse the order topology on a semi-well ordered set and abbreviate semi-well ordered as *swo*.

**Proposition 3.1.** *A swo set is connected if and only if it is empty or singleton.*

*Proof.* Let $(X, \leq)$ be a *swo* set such that $X{=}A+B$. Suppose that $X$ is connected. Assume the contrary that $X$ has more than one element. Then $B$ should be non-empty. Let $b$ be the greatest element of $B$. Then $(b\text{-}1,\ b]=\{b\}$ is an open set in $X$, also it is closed in $X$. Hence $\{b\}$ is a clopen subset of $X$, a contradiction. The converse part is trivial. □

**Remark 3.1.** *A non-empty semi-well ordered set is totally disconnected.*

**Lemma 3.1.** *Let $X = A + B$ be a swo set with $A$ non-empty. Then,*

$(a)$ *for each $x \in A$ where $x$ is neither the least nor the greatest element of $A$ and open set $I$ containing $x$, there exists an open set in $X$ of the form $(y,\ x]$ where $y \in A$ such that $x \in (y,\ x] \subset I$.*

$(b)$ *for each $y \in B$ where $y$ is not the greatest element of $B$ and open set $I$ containing $y$, there exists an open set in $X$ of the form $[y,\ z)$ where $z \in B$ such that $y \in [y,\ z) \subset I$.*

*Proof.*

$(a)$ Let $x_0$ be the least element of $X$ and $x \in A$. Then,

- (i) $x \in [x_0,\ y)$ implies $x \in (x_0,\ x] \subset [x_0,\ y)$.
- (ii) $x \in (y_1,\ y_2)$ implies $x \in (y_1,\ x] \subset (y_1,\ y_2)$.
- (iii) If $X$ has a greatest element say $y_0$, then $x \in (y,\ y_0]$ implies $x \in (y,\ x] \subset (y,\ y_0]$.

$(b)$ Let $y_0$ be the greatest element of $X$ and $y \in B$. Then,

- (i) $y \in (a,\ y_0]$ implies $y \in [y,\ y_0) \subset (a,\ y_0]$.
- (ii) $y \in (a,\ b)$ implies $y \in [y,\ b) \subset (a,\ b)$.
- (iii) $y \in [x_0,\ b)$ implies $x \in$ [y,b) $\subset [x_0,\ b)$. □

We know that for a well ordered set $(X, \leq)$ the isolated points are exactly the least element, and those elements of $X$ that has an immediate predecessor. Now we have the following theorem:

**Theorem 3.1.** *Let $(X{=}A{+}B,\ \leq)$ be a swo set, where $A$ has no greatest element. Then the isolated points of $X$ are exactly the least element of $A$(if it exists), the greatest element of $B$(if it exists), the elements of $A$ with an immediate predecessor, and the elements of $B$ with an immediate successor.*

*Proof.* To prove this we have the following cases:

(i) $A$ is non-empty and $B$ is empty.
Then $X$ is a well ordered set and the theorem follows.

(ii) $A$ is empty and $B$ is non-empty.
Then $X = C$ is a co-well ordered set. Let $x_0$ be the greatest element of $X$. Then $\{x_0\}$ is open in $X$, $x_0$ is an isolated point. Let $x$ be an element of $X$ with an immediate successor say $a$ then $\{x\}$=$(x_{-1},\, a)$ is an open set in $X$, where $x_{-1}$ is the immediate predecessor of $x$. So $x$ is an isolated point of $X$. Again let $y$ be an element with no immediate successor, then by Lemma 3.1, any open set containing $y$ contains a subset of the form $[y,\, b)$. So $y$ is not an isolated point.

(iii) Both $A$ and $B$ are non-empty.
Let $x_0$ and $y_0$ be the least and greatest elements of $X$ respectively, then $\{x_0\}$ and $\{y_0\}$ are both open in $X$, so both $x_0$ and $y_0$ are isolated points of $X$. Let $x \neq x_0$ be an element in $A$ with no immediate predecessor, then by Lemma 3.1 it is not an isolated point, and if $x$ has an immediate predecessor, then $x$ is an isolated point. Also, if $y \neq y_0$ is an element in $B$ with an immediate successor, then $y$ is an isolated point of $X$ and if $y$ has no immediate successor, then by Lemma 3.1, it is not an isolated point of $X$. □

**Remark 3.2.** *In Theorem 3.1, if $A$ has the greatest element $a$ and $B$ is non-empty, then $a$ is not an isolated point of $X$.*

A well ordered set with order topology is compact if and only if it has the greatest element. But a semi-well-ordered set need not be compact, even if it has the largest element. Compact semi-well ordered sets are characterised by the following theorem.

**Theorem 3.2.** *Let $(X, \leq)$ be a non-empty semi-well ordered set where $X = A + B$, then $X$ is compact if and only if $A$ has a greatest element.*

*Proof.* Suppose that $X$ is compact. Again suppose $A$ is non-empty and has no greatest element. Clearly then $B$ is non-empty. Let $\{x_0\}$ and $\{y_0\}$ be the least and greatest element of $X$ respectively. Let $U$=$\{[x_0,\, x)\colon\ x \in A\}$ and $V$=$\{(y,\, y_0]\colon$ $y \in B\}$. Clearly $U \cup V$ is an open cover of $X$. In particular $U$ and $V$ are open covers of $A$ and $B$ respectively with open sets in $X$ . Now $X$ is compact so $U \cup V$ has a finite sub-cover. Since $U$ and $V$ are nested intervals there exist single sets $[x_0,\, x')$ and $(y',\, y_0]$ such that $A$= $[x_0,\, x')$ and $B$= $(y',\, y_0]$. So $X$=$[x_0,\, x')$∪$(y',\, y_0]$, which implies $x'$ and $y'$ does not belong to $X$, a contradiction. So $A$ has a greatest element.

Conversely, suppose that $A$ has a greatest element, say $a$. If $B$ is empty, then $X$ is a well ordered set with a greatest element. So it is compact. Suppose $B$ is non-empty. Let $x_0$ and $y_0$ be the least and greatest elements of $X$ respectively. Let $\mathscr{U}$ be a open cover of $X$ by basic open sets. Then we have the following cases:

(i) The open set in $\mathscr{U}$ containing $a$ is the basic open set $(x,\, y)$, where $x \in A$ and $y \in B$.

Assume without loss of generality that all elements of $\mathscr{U}$ other than $(x, y)$ is the basic open sets of the form $\{x_0\}$, $\{y_0\}$, $(x, x']\subset A$ or $[y, y')\subset B$. Clearly then $\{\{x_0\}, \{y_0\}, (x, y)\}\subset \mathscr{U}$. If $x \neq x_0$, then let $b_{-1} = x$. If this is the case, then there exists $a_{-1} \in A$ such that $(a_{-1}, b_{-1}]\in \mathscr{U}$. If $a_{-1} = x_0$, then $\{\{x_0\}, (a_{-1}, b_{-1}], (x, y)\}$ covers $A$. If $a_{-1} \neq x_0$ put $a_{-1} = b_{-2}$ and then there exists $(a_{-2}, b_{-2}]\in \mathscr{U}$ such that $a_{-1} \in(a_{-2}, b_{-2}]$. If $a_{-2} = x_0$ then we are done. If not, the process continues. Now $a_{-n} = x_0$ for some $n$ because otherwise $\{a_{-1}, a_{-2}, a_{-3}, \cdots\}$ will be a strictly decreasing sequence in $A$, which is not possible.

Similarly, if $y \neq y_0$ put $y = b_1$, then there exists $a_1 \in B$ with $[b_1, a_1)\in \mathscr{U}$. If $a_1 = y_0$, $\{\{x_0\}, (x_{-1}, y_{-1}], \cdots, (x_{-n}, y_{-n}], (x, y), [y, y_0), \{y_0\}\}$ is a finite sub-cover of $\mathscr{U}$. If $a_1 \neq y_0$ put $a_1 = b_2$ and then there exists $[b_2, a_2)\in \mathscr{U}$ such that $a_1 \in[b_2, a_2)$ and $a_1 < a_2$. If $a_2 = y_0$ we are done. Otherwise, this process continues. Now $a_m = y_0$ for some $m$ because otherwise, $\{a_1, a_2, \cdots\}$ will be a strictly increasing sequence in $C$ which is not possible. So $\{\{x_0\}, (x_{-1}, y_{-1}], \cdots, (x_{-n}, y_{-n}], (x, y), [b_1, a_1), \cdots, [b_m, a_m), \{y_0\}\}$ is a finite sub-cover of $\mathscr{U}$.

(ii) The open set in $\mathscr{U}$ containing $a$ is the basic open set $(x, y_0]$, where $x \in A$.

(iii) The open set in $\mathscr{U}$ containing $a$ is the basic open set $[x_0, y)$, where $y \in B$.

The cases (ii) and(iii) can be proved easily. So we get a finite sub-cover of $\mathscr{U}$. Since $\mathscr{U}$ is arbitrary $X$ is compact. □

**Proposition 3.2.** *If $(X, \leq)$ is a swo set, where $X = A+B$, then $B$ is always open in $X$.*

*Proof.* Let $x_0 \in B$ be the greatest element. Then $\{x_0\}$ is an open set in $X$ contained in $B$. Now for all $x \in B$ and $x \neq x_0$, $[x, y)$ where $y \in B$ is an open set in $X$ containing $x$ contained in $B$. So $B$ is open in $X$. □

**Remark 3.3.** *If $X = A + B$ is a swo set, then $A$ need not be open in $X$. For example, consider $X$=\{1, 2, 3, $\cdots$, 0, $\cdots$, -3, -2, -1\} then $A$=\{1, 2, 3, $\cdots$, 0\} and is not open in $X$ since int(A)=\{1, 2, 3, $\cdots$\}.*

**Proposition 3.3.** *Every swo set is zero dimensional.*

*Proof.* Let $(X, \leq)$ be a *swo* set where $X = A+B$. Now we have the following three cases:

(i) $A$ is non-empty and $B$ is empty.

Let $a_0$ be the least element of $A$. Then $\{\{a_0\}\}\cup\{(a, x]\colon a, x \in A, x \neq a_0\}$ is a clopen basis for $X$.

(ii) $A$ is empty and $B$ is non-empty.

Let $b_0$ be the greatest element of $B$. Then $\{\{b_0\}\}\cup\{[y,\, b)\colon\, y,\, b \in B,\, y \neq b_0\}$ is a clopen basis for $X$.

(iii) A is a non-empty set with no greatest element and $B$ is non-empty.

Let $a_0$ and $b_0$ be the least and greatest elements of $X$ respectively. Then $\{\{a_0\}\}\cup\{(x_1,\, x_2]\colon\, x_1,\, x_2 \in A,\, x_2 \neq a_0\}\cup\{\{b_0\}\}\cup\{[y_1,\, y_2)\colon\, y_1,\, y_2 \in B,\, y_1 \neq b_0\}$ is a clopen basis for $X$.

(iv) A is a non-empty set with a greatest element say $a$ and $B$ is non-empty.

Here, if $a \neq a_0$ then $\{\{a_0\}\}\cup\{(x_1,\, x_2]\colon\, x_1,\, x_2 \in A,\, x_2 \neq a_0,\, x_2 \neq a\}\cup\{\{b_0\}\}\cup\{[y_1,\, y_2)\colon\, y_1,\, y_2 \in B,\, y_1 \neq b_0\}\cup\{(a_1,\, b_1)\colon\, a_1 \in A,\, b_1 \in B,\, a_1 < a\}$ is a clopen basis for $X$. If $a = a_0$, then $\{[a,\, b_1)\colon\, b_1 \in B\}\cup\{\{b_0\}\}\cup\{[y_1,\, y_2)\colon\, y_1,\, y_2 \in B,\, y_1 \neq b_0\}$ is a clopen basis for $X$.

Hence in all the cases $X$ has a basis with clopen subsets of $X$, so $X$ is zero dimensional. □

**Proposition 3.4.** *Every swo set is strongly locally compact.*

*Proof.* Let $X = A+B$ be a *swo* set, where $A$ is a well ordered set and $B$ is a co-well ordered set. If $B$ is empty, then $X$ is well ordered and hence it will be strongly locally compact.

Now suppose that $B$ is non-empty. If $A$ has a greatest element, then $X$ is compact and hence strongly locally compact. Suppose that $A$ has no greatest element. Let $a_0$ be the least element of $A$(hence of $X$) and $b_0$ be the greatest element of $B$( and hence of $X$). For the element $a_0$ of $X$, $\{a_0\}$ is an open set in $X$ whose closure $\{a_0\}$ itself is compact. Again let $a \in A$ and $a \neq a_0$, then $(x,\, a]$ where $x \in A$, is a clopen neighbourhood of $a$ which is compact. Now for $b_0$, $\{b_0\}$ is a clopen set which is compact. Again, for $b \in B$ and $b \neq b_0$ $[b,\, y)$, $y \in B$ is a clopen neighbourhood of $b$ which is compact. So $X$ is strongly locally compact. □

**Remark 3.4.** *Since every strongly locally compact space is locally compact, every swo set is locally compact.*

**Theorem 3.3.** *Let $(X,\, \leq)$ be a swo set, where $X = A+B$. Then $X$ is complete if and only if $A$ has a greatest element.*

*Proof.* Suppose that $A$ has a greatest element. If $B$ is empty, then $A$ is well ordered and so $X$ is complete. Suppose that $B$ is non-empty. Let $Y$ be a non-empty subset of $X$ which is bounded above. Then $Y = W + C$, where $W \subset A$ and $C \subset B$. If $C$ is empty, then $W \subset A$ has a least upper bound. If $C$ is non-empty, then $C$ has a greatest element which is the least upper bound of $W + C$. So $X$ is complete.

Conversely, suppose that $X$ is complete. If $B$ is empty, then since $A$ is well ordered it has a greatest element. Suppose that $B$ is non-empty. If $A = \emptyset$, then $B =[0,\, \alpha)^*$, for some limit ordinal $\alpha$, where $[0,\, \alpha)^*$ is the dual of $[0,\, \alpha)$. Now $\alpha$ is an upper bound for $B$. So $B = [0,\, \beta]^*$, but this is not possible by the definition of $B$. So $A$

should be non-empty. So, $B$ has lower bound by elements of $A$. If this lower bounds has a supremum, then it should belong to $A$ and so it is the greatest element of $A$. □

We have the following theorem for an arbitrary partially ordered set.

**Theorem 3.4.** [18] *Let $X$ be a partially ordered set, $\mathscr{T}$ be the interval topology on $X$ and $\{S_n\colon n \in D\}$ a net in $X$. If $\{S_n\}$ converges to $y$ in $(X, \mathscr{T})$ and $S_n \leq y$ for all $n \in D$, then $y$=l.u.b(range$S_n$).*

Analogous to this theorem we have the following theorem:

**Theorem 3.5.** *Let $X$ be a partially ordered set, $\mathscr{T}$ be the interval topology on $X$ and $\{S_n : n \in D\}$ a net in $X$. If $\{S_n\}$ converges to $y$ in $(X, \mathscr{T})$ and $S_n \geq y$ for all $n \in D$, then $y$=g.l.b(range$S_n$).*

*Proof.* Let $A$ be the range set of $\{S_n\colon n \in D\}$. Suppose that $y \neq$g.l.b(range$S_n$). Let $P$=$\{x \in X\colon x \leq a$ for all $a \in A\}$. Since $y \neq$g.l.b(range$S_n$), we can find $z \in P$ such that $y \notin \{x \in X\colon z \leq x\}(=Q)$. Now $Q$ is closed, $X \setminus Q$ is an open set containing $y$. So $\{S_n\}$ is eventually in $X \setminus Q$. Which implies $z \notin P$, a contradiction. □

**Theorem 3.6.** *Let $(X, \leq)$ be a swo set where $X = A + B$, $B \neq \phi$. If $\{S_n\}$ is a net in $X$ which converges to $y \in X$, then*

*(i) if $y \in A$ and $y$ is not the greatest element of $A$, then $y$=l.u.b$\{S_n\colon m \leq n\}$ for some $m \in \mathbb{N}$.*

*(ii) if $y \in B$, then $y$=g.l.b$\{S_n\colon m \leq n\}$ for some $m \in \mathbb{N}$.*

*Proof.*

(i) Let $K$=$\{x \in X\colon x \leq y\}$. Since $y$ is not the greatest element of $A$, $K$ is an open set containing $y$. So $S_n \in K$ for all $n \geq m$ for some $m \in \mathbb{N}$. Hence, $S_n \leq y$ for all $n \geq m$. So by Theorem 3.4, $y$=l.u.b$\{S_n\colon m \leq n\}$.

(ii) Let $Q$=$\{x \in X\colon x \geq y\}$. Clearly $Q$ is open and contains $y$. So $\{S_n\}$ is eventual in $Q$. So $y$=g.l.b$\{S_n\colon m \leq n\}$ for some $m \in \mathbb{N}$. □

For a subset $Y$ of a semi-well ordered set $X$, the subspace topology on $Y$ need not be equivalent to its order topology. For example, let $X$ be a set with ordering $1 < 2 < 3 < \cdots < 1^0 < 2^0 < \cdots < 1' < 2' < \cdots < \cdots < -3' < -2' < -1' < \cdots < -3 < -2 < -1$ and $Y$=$\{2^0\}$+$\{\cdots$, -3, -2, -1$\}$ with the ordering inherited from $X$. Then $\{2^0\}$=$(1^0$,$3^0)\cap Y$ is an open set in the relative topology on $Y$ but not open in the order topology on $Y$.

Since every *swo* set is linearly ordered, from Theorem 4 of [1], we have the following theorem.

**Theorem 3.7.** [1] *Let $(X, \leq)$ be a swo set and $Y \subset X$, then the order topology on $Y$ is equivalent to the relative topology on $Y$ as a subspace of $X$ if and only if* *for any $y \in Y$ and $x \in X \setminus Y$, there exists an element $a \in Y$ between $x$ and $y$, or if $y < x$, the elements greater than $x$ in $Y$ have the least element, or if $x < y$, the element less than $x$ in $Y$ has the greatest element.*

**Corollary 3.1.** *Let $X = A + B$ be a swo set, where $A$ has no greatest element. Let $Y$ be a non-empty swo set. Then $X + Y$ is a swo set and the subspace topology on $X$ as a subspace of $X + Y$ is equivalent to the order topology on $X$.*

*Proof.* Let $Y$=$C + D$, where $C$ is well ordered and $D$ is co-well ordered. Then $X + Y$= $(A + C)$+$(D$+$B)$. Now $x \in X$ and $y \in (X + Y) \setminus X$ implies that there exists an element between $x$ and $y$ (since $A$ has no greatest element and $B$ has no least element). Hence by Theorem 3.7, the corollary follows. □

**Proposition 3.5.** *Let $(X, \leq)$ be an infinite swo set where $X$=$A + B$. Then, $X$ is homogeneous if and only if the least element (if any) of $X$ has an immediate successor, greatest element (if any) of $X$ has an immediate predecessor and all other elements has both an immediate predecessor and an immediate successor.*

*Proof.* The sufficient part trivially follows since then the order topology coincides with the discrete topology on $X$. Now for the necessary part suppose $X$ has a least element $a_0$ with no immediate successor. Then clearly $B$ is non-empty and has an element say $b$ with an immediate successor and an immediate predecessor. So there exists a homeomorphism $f : X \to X$ such that $f(a_0)$=$b$. But $\{b\}$ is open in $X$ and $f^{-1}(\{b\})$=$\{a_0\}$ is not open in $X$, a contradiction. So if $X$ has a least element, then it should have an immediate successor.

Again suppose that $X$ has a greatest element $b_0$ with no immediate predecessor. Then clearly $b_0 \in A$ and $B$ is empty. Since $A$ is infinite it has an element, say $a$ with an immediate predecessor and successor. So there exists a homeomorphism $g : X \to X$ such that $g(b_0) = a$. Now $\{a\}$ is open in $X$ but $g^{-1}(\{a\})$=$\{b_0\}$ is not open in $X$, a contradiction.

Now if $X$ contains an element say $x$ with no immediate predecessor and no immediate successor, then $x \in A$ and $B$ is non-empty. So $B$ contains an element say $y$ with an immediate predecessor and successor. So there exists a homeomorphism $h : X \to X$ such that $f(x)$=$y$. But $\{y\}$ is open in $X$ and $f^{-1}(\{y\})$=$\{x\}$ is not open in $X$, a contradiction. Hence the theorem follows. □

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