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Bernoulli and Euler Polynomials in Two Variables

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Abstract. In a previous work we studied generalized Stirling numbers of the second kind $S_{a_1,b_1}^{(a_2,b_2,p_2)}(p_1,k)$, where a_1,a_2,b_1,b_2 are given complex numbers, $a_1,a_2\neq 0$, and p_1,p_2 are non-negative integers given. In this work we use these generalized Stirling numbers to define Bernoulli polynomials in two variables $B_{p_1,p_2}(x_1,x_2)$, and Euler polynomials in two variables $E_{p_1,p_2}(x_1,x_2)$. By using results for $S_{1,x_1}^{(1,x_2,p_2)}(p_1,k)$, we obtain generalizations, to the bivariate case, of some well-known properties from the standard case, as addition formulas, difference equations and sums of powers. We obtain some identities for bivariate Bernoulli and Euler polynomials, and some generalizations, to the bivariate case, of several known identities for Bernoulli and Euler numbers and polynomials of the standard case.

1. Introduction

Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ can be defined by the corresponding generating functions

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad\text{and}\quad\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},$$

or by the explicit formulas [3, p. 48]

$$B_p(x)=\sum_{k=0}^{p}\binom{p}{k}B_kx^{p-k}\quad\text{and}\quad E_p(x)=\sum_{k=0}^{p}\binom{p}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{p-k},$$

where $B_k=B_k(0)$ is the k-th Bernoulli number, and $E_k=2^kE_k\left(\frac{1}{2}\right)$ is the k-th Euler number.

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In a previous work [10] we studied a generalization of Stirling Numbers of the second kind (GSN, for short), denoted as $S_{a_1,b_1}^{a_2,b_2,p_2}(p_1,k)$, where $a_j, b_j \in \mathbb{C}$, $a_j \neq 0$, $j = 1, 2$, and p_1, p_2 are non-negative integers, involved in the expansion

$$(1.1)\qquad (a_1 n + b_1)^{p_1}(a_2 n + b_2)^{p_2} = \sum_{k=0}^{p_1+p_2} k! S_{a_1,b_1}^{a_2,b_2,p_2}(p_1,k)\binom{n}{k},$$

($S_{a_1,b_1}^{(a_2,b_2,p_2)}(p_1,k) = 0$ if $k < 0$ or $k > p_1+p_2$). An explicit formula for these numbers is

$$(1.2)\qquad S_{a_1,b_1}^{a_2,b_2,p_2}(p_1,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(a_1(k-j)+b_1)^{p_1}(a_2(k-j)+b_2)^{p_2}.$$

If $p_2 = 0$, we write the corresponding GSN as $S_{a,b}(p,k)$. That is, we have

$$(1.3)\qquad S_{a,b}(p,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(a(k-j)+b)^p.$$

In particular, if $a_1 = a_2 = a$, $b_1 = b_2 = b$, expression (1.2) beomes

$$S_{a,b}^{a,b,p_2}(p_1,k) = S_{a,b}(p_1+p_2,k).$$

Two trivial examples are $S_{1,0}(p,k) = S(p,k)$ (the standard Stirling numbers of the second kind), and $S_{1,1}(p,k) = S(p+1,k+1)$.

We summarize some facts about GSN $S_{1,x_1}^{1,x_2,p_2}(p_1,k)$ (contained in [10]) that we will use in this work.

- Some values

$$(1.4)\qquad \begin{aligned}
S_{1,x_1}^{1,x_2,p_2}(p_1,0) &= x_1^{p_1}x_2^{p_2},\\
S_{1,x_1}^{1,x_2,p_2}(p_1,1) &= (x_1+1)^{p_1}(x_2+1)^{p_2} - x_1^{p_1}x_2^{p_2},\\
S_{1,x_1}^{1,x_2,p_2}(p_1,2) &= \frac{1}{2}(x_1+2)^{p_1}(x_2+2)^{p_2}\\
&\quad -(x_1+1)^{p_1}(x_2+1)^{p_2} + \frac{1}{2}x_1^{p_1}x_2^{p_2},\\
&\ \ \vdots\\
S_{1,x_1}^{1,x_2,p_2}(p_1,p_1+p_2) &= 1.
\end{aligned}$$

- The GSN $S_{1,x_1}^{1,x_2,p_2}(p_1,k)$ can be written in terms of the GSN $S_{1,y_1}^{1,y_2,p_2}(p_1,k)$ as follows

$$(1.5)\qquad \begin{aligned}
&S_{1,x_1}^{1,x_2,p_2}(p_1,k)\\
&= \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-y_1)^{p_1-j_1}(x_2-y_2)^{p_2-j_2}S_{1,y_1}^{1,y_2,j_2}(j_1,k).
\end{aligned}$$

- The GSN $S_{1,x_1}^{1,x_2,p_2}(p_1,k)$ can be written in terms of standard Stirling numbers as follows

$$\begin{aligned} S_{1,x_1}^{1,x_2,p_2}(p_1,k) &= \frac{1}{k!}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-m)^{p_1-j_1}(x_2-m)^{p_2-j_2} \\ &\quad\times\sum_{i=0}^{m}\binom{m}{i}(k+i)!S(j_1+j_2,k+i), \end{aligned} \tag{1.6}$$

where m is a non-negative integer.

- The GSN $S_{1,x_1}^{(1,x_2,p_2)}(p_1,k)$ can be written in terms of standard Stirling numbers as follows

$$\begin{aligned} S_{1,x_1}^{1,x_2,p_2}(p_1,k) &= \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-n)^{p_1-j_1}(x_2-n)^{p_2-j_2} \\ &\quad\times\sum_{i=0}^{n-1}(-1)^i s(n,n-i)S(j_1+j_2+n-i,k+n), \end{aligned} \tag{1.7}$$

where n is a positive integer, and $s(\cdot,\cdot)$ are the Stirling numbers of the first kind (with recurrence $s(q+1,k)=s(q,k-1)+qs(q,k)$). In particular, from (1.6) with $m=0$, and (1.7) with $n=1$, we have

$$S_{1,x_1}^{1,x_2,p_2}(p_1,k) = \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}x_1^{p_1-j_1}x_2^{p_2-j_2}S(j_1+j_2,k) \tag{1.8}$$

$$= \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-1)^{p_1-j_1}(x_2-1)^{p_2-j_2}S(j_1+j_2+1,k+1). \tag{1.9}$$

- The GSN $S_{1,x_1}^{1,x_2,p_2}(p_1,k)$ satisfy the identity

$$S_{1,x_1+1}^{1,x_2+1,p_2}(p_1,k) = S_{1,x_1}^{1,x_2,p_2}(p_1,k) + (k+1)S_{1,x_1}^{1,x_2,p_2}(p_1,k+1). \tag{1.10}$$

- The GSN $S_{1,x_1}^{1,x_2,p_2}(p_1,k)$ satisfy the recurrence

$$S_{1,x_1}^{1,x_2,p_2}(p_1,k) = S_{1,x_1}^{1,x_2,p_2}(p_1-1,k-1) + (k+x_1)S_{1,x_1}^{1,x_2,p_2}(p_1-1,k). \tag{1.11}$$

2. Definitions and Preliminary Results

The relation between Bernoulli numbers and Stirling numbers of the second kind, is an old known story, that dates back to Worpitzky [13]. In fact, we have

$$B_p = \sum_{k=0}^{p} S(p,k) \frac{(-1)^k k!}{k+1} \tag{2.1}$$

(see [7]). In the Euler case, formula (3.3) in [6] shows Euler polynomials written in terms of Stirling numbers of the second kind. In [8] and [11] we used the GSN $S_{1,x}(p,k)$ described above, to write, respectively, Bernoulli polynomials $B_p(x)$ and Euler polynomials $E_p(x)$ as

$$B_p(x) = \sum_{k=0}^{p} S_{1,x}(p,k) \frac{(-1)^k k!}{k+1}, \tag{2.2}$$

$$E_p(x) = \sum_{k=0}^{p} S_{1,x}(p,k) \frac{(-1)^k k!}{2^k}. \tag{2.3}$$

(Formula (2.2) was inspired by (2.1), and (2.3) was inspired by formula (3.3) in [6] mentioned before.)

In this work we use the GSN $S_{1,x_1}^{1,x_2,p_2}(p_1,k)$ to define bivariate Bernoulli polynomials $B_{p_1,p_2}(x_1,x_2)$ and bivariate Euler polynomials $E_{p_1,p_2}(x_1,x_2)$, as

$$B_{p_1,p_2}(x_1,x_2) = \sum_{k=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2}(p_1,k) \frac{(-1)^k k!}{k+1}, \tag{2.4}$$

$$E_{p_1,p_2}(x_1,x_2) = \sum_{k=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2}(p_1,k) \frac{(-1)^k k!}{2^k}. \tag{2.5}$$

Explicitly, we have

$$B_{p_1,p_2}(x_1,x_2) = \sum_{k=0}^{p_1+p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1}\binom{p_2}{j_2} x_1^{p_1-j_1} x_2^{p_2-j_2} S(j_1+j_2,k) \frac{(-1)^k k!}{k+1}, \tag{2.6}$$

$$E_{p_1,p_2}(x_1,x_2) = \sum_{k=0}^{p_1+p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1}\binom{p_2}{j_2} x_1^{p_1-j_1} x_2^{p_2-j_2} S(j_1+j_2,k) \frac{(-1)^k k!}{2^k}. \tag{2.7}$$

Clearly we have that

$$\begin{aligned} B_{0,0}(x_1,x_2) &= 1, \\ B_{p_1,0}(x_1,x_2) &= B_{p_1}(x_1), \\ B_{0,p_2}(x_1,x_2) &= B_{p_2}(x_2), \\ B_{p_1,p_2}(x,x) &= B_{p_1+p_2}(x), \end{aligned}$$

and similar results for E replacing B.

Some examples are the following

$$\begin{aligned} B_{1,1}(x_1,x_2) &= x_1x_2-\frac{1}{2}x_1-\frac{1}{2}x_2+\frac{1}{6}, \\ E_{1,1}(x_1,x_2) &= x_1x_2-\frac{1}{2}x_1-\frac{1}{2}x_2, \\ B_{1,2}(x_1,x_2) &= x_1x_2^2-x_1x_2+\frac{1}{6}x_1-\frac{1}{2}x_2^2+\frac{1}{3}x_2, \\ E_{1,2}(x_1,x_2) &= x_1x_2^2-x_1x_2-\frac{1}{2}x_2^2+\frac{1}{4}. \end{aligned}$$

From (1.6) and (1.7) we can write the following families of formulas for bivariate Bernoulli and Euler polynomials

$$\begin{aligned} B_{p_1,p_2}(x_1,x_2) \\ = \sum_{k=0}^{p_1+p_2}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-m)^{p_1-j_1}(x_2-m)^{p_2-j_2} \\ \times\sum_{i=0}^{m}\binom{m}{i}S(j_1+j_2,k+i)\frac{(-1)^k(k+i)!}{k+1}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} E_{p_1,p_2}(x_1,x_2) \\ = \sum_{k=0}^{p_1+p_2}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-m)^{p_1-j_1}(x_2-m)^{p_2-j_2} \\ \times\sum_{i=0}^{m}\binom{m}{i}S(j_1+j_2,k+i)\frac{(-1)^k(k+i)!}{2^k}, \end{aligned} \tag{2.9}$$

where m is a non-negative integer (the case $m=0$ of (2.8) and (2.9) are (2.6) and (2.7), respectively), and

$$\begin{aligned} B_{p_1,p_2}(x_1,x_2) &= \sum_{k=0}^{p_1+p_2}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-n)^{p_1-j_1}(x_2-n)^{p_2-j_2} \\ &\quad\times\sum_{i=0}^{n-1}(-1)^i s(n,n-i)S(j_1+j_2+n-i,k+n)\frac{(-1)^k k!}{k+1}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} E_{p_1,p_2}(x_1,x_2) &= \sum_{k=0}^{p_1+p_2}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-n)^{p_1-j_1}(x_2-n)^{p_2-j_2} \\ &\quad\times\sum_{i=0}^{n-1}(-1)^i s(n,n-i)S(j_1+j_2+n-i,k+n)\frac{(-1)^k k!}{2^k}, \end{aligned} \tag{2.11}$$

where n is a positive integer.

3. Properties

We begin this section with the bivariate version of addition formulas for Bernoulli and Euler polynomials.

Proposition 3.1. *We have*

$$P_{p_1,p_2}(x_1,x_2)=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-y_1)^{p_1-j_1}(x_2-y_2)^{p_2-j_2}P_{j_1,j_2}(y_1,y_2), \tag{3.1}$$

where P is B or E.

Proof. In the case $P=B$, by using (1.5) we have

$$\begin{aligned}
&B_{p_1,p_2}(x_1,x_2)\\
&=\sum_{k=0}^{p_1+p_2}S_{1,x_1}^{1,x_2,p_2}(p_1,k)\frac{(-1)^k k!}{k+1}\\
&=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-y_1)^{p_1-j_1}(x_2-y_2)^{p_2-j_2}\sum_{k=0}^{j_1+j_2}S_{1,y_1}^{(1,y_2,j_2)}(j_1,k)\frac{(-1)^k k!}{k+1}\\
&=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-y_1)^{p_1-j_1}(x_2-y_2)^{p_2-j_2}B_{j_1,j_2}(y_1,y_2),
\end{aligned}$$

as desired. The proof of the Euler case is similar. □

We have the following particular case of (3.1), corresponding to $y_1=y_2=y$,

$$P_{p_1,p_2}(x_1,x_2)=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-y)^{p_1-j_1}(x_2-y)^{p_2-j_2}P_{j_1+j_2}(y). \tag{3.2}$$

By setting $y=\frac{1}{2}$ in the case $P=E$ of (3.2), we have also the following expression for $E_{p_1,p_2}(x_1,x_2)$ involving Euler numbers

$$\begin{aligned}
&E_{p_1,p_2}(x_1,x_2)\\
&\qquad=2^{-p_1-p_2}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(2x_1-1)^{p_1-j_1}(2x_2-1)^{p_2-j_2}E_{j_1+j_2}.
\end{aligned} \tag{3.3}$$

From the case $P=B$ of (3.2) with $y=x_1,x_2$, we see that

$$\begin{aligned}
B_{p_1,p_2}(x_1,x_2)&=\sum_{j_2=0}^{p_2}\binom{p_2}{j_2}(x_2-x_1)^{p_2-j_2}B_{p_1+j_2}(x_1)\\
&=\sum_{j_1=0}^{p_1}\binom{p_1}{j_1}(x_1-x_2)^{p_1-j_1}B_{j_1+p_2}(x_2).
\end{aligned} \tag{3.4}$$

In particular, with $x_1 = 1$ and $x_2 = 0$ we obtain from (3.4) that

$$B_{p_1,p_2}(1,0) = (-1)^{p_1+p_2} \sum_{j_2=0}^{p_2} \binom{p_2}{j_2} B_{p_1+j_2} = \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} B_{j_1+p_2}. \tag{3.5}$$

That is, we have the identity

$$(-1)^{p_2} \sum_{j_2=0}^{p_2} \binom{p_2}{j_2} B_{p_1+j_2} = (-1)^{p_1} \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} B_{j_1+p_2}. \tag{3.6}$$

Formula (3.6) is the well-known Carlitz identity [1].

Similarly, we have from the case $P = E$ of (3.2) with $y = x_1, x_2$

$$\begin{aligned} E_{p_1,p_2}(x_1,x_2) &= \sum_{j_2=0}^{p_2} \binom{p_2}{j_2} (x_2 - x_1)^{p_2-j_2} E_{p_1+j_2}(x_1) \\ &= \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} (x_1 - x_2)^{p_1-j_1} E_{j_1+p_2}(x_2). \end{aligned} \tag{3.7}$$

By setting $x_1 = 1, x_2 = 0$, we obtain the following Euler versions of Carlitz identity (3.6):

$$(-1)^{p_2} \sum_{j_2=0}^{p_2} \binom{p_2}{j_2} E_{p_1+j_2}(0) = (-1)^{p_1} \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} E_{j_1+p_2}(0), \tag{3.8}$$

$$\sum_{j_2=0}^{p_2} \binom{p_2}{j_2} (-1)^{j_2} E_{p_1+j_2}(1) = \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} (-1)^{j_1} E_{j_1+p_2}(1). \tag{3.9}$$

Combining formulas for $B_{p_1,p_2}(x_1,x_2)$ and $E_{p_1,p_2}(x_1,x_2)$ in (3.4) and (3.7), we can write the following expressions for the difference $B_{p_1,p_2}(x_1,x_2) - E_{p_1,p_2}(x_1,x_2)$:

$$\begin{aligned} &B_{p_1,p_2}(x_1,x_2) - E_{p_1,p_2}(x_1,x_2) \\ &= \sum_{j_2=0}^{p_2} \binom{p_2}{j_2} (x_2 - x_1)^{p_2-j_2} \left(B_{p_1+j_2}(x_1) - E_{p_1+j_2}(x_1)\right) \\ &= \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} (x_1 - x_2)^{p_1-j_1} \left(B_{j_1+p_2}(x_2) - E_{j_1+p_2}(x_2)\right). \end{aligned} \tag{3.10}$$

Moreover, we can use Cheon's formula

$$B_n(x) - E_n(x) = \sum_{k=2}^{n} \binom{n}{k} B_k E_{n-k}(x), \tag{3.11}$$

where $n>0$ (see [2]), to write (3.10) as

$$B_{p_1,p_2}(x_1,x_2)-E_{p_1,p_2}(x_1,x_2) \tag{3.12}$$
$$=\sum_{j_2=0}^{p_2}\binom{p_2}{j_2}(x_2-x_1)^{p_2-j_2}\sum_{k=2}^{p_1+j_2}\binom{p_1+j_2}{k}B_kE_{p_1+j_2-k}(x_1)$$
$$=\sum_{j_1=0}^{p_1}\binom{p_1}{j_1}(x_1-x_2)^{p_1-j_1}\sum_{k=2}^{j_1+p_2}\binom{j_1+p_2}{k}B_kE_{j_1+p_2-k}(x_2),$$

where $p_1,p_2>0$.

Next we consider properties related to symmetries, difference equations and sums of powers for bivariate Bernoulli and Euler polynomials. We will need some preliminary results, contained in the following lemma.

Lemma 3.2. (a) *For non-negative integer p and $x\in\mathbb{R}$, we have*

$$\sum_{k=0}^{p-1}S_{1,x}(p,k+1)(-1)^kk!=px^{p-1}. \tag{3.13}$$

(b) *For non-negative integers p_1,p_2, and $x_1,x_2\in\mathbb{R}$, we have*

$$\sum_{k=0}^{p_1+p_2}S_{1,x_1}^{1,x_2,p_2}(p_1,k+1)(-1)^kk!=p_1x_1^{p_1-1}x_2^{p_2}+p_2x_1^{p_1}x_2^{p_2-1}. \tag{3.14}$$

Proof. (a) We proceed by induction on p. For $p=0$ the result is clear. If the formula works for a given $p\in\mathbb{N}$, then, by using the recurrence (1.11) and the value $S_{1,x}(p,0)=x^p$, we have

$$\begin{aligned}
&\sum_{k=0}^{p}S_{1,x}(p+1,k+1)(-1)^kk!\\
=&\sum_{k=0}^{p}\left(S_{1,x}(p,k)+(k+1+x)S_{1,x}(p,k+1)\right)(-1)^kk!\\
=&\sum_{k=0}^{p}S_{1,x}(p,k)(-1)^kk!+\sum_{k=0}^{p-1}S_{1,x}(p,k+1)(-1)^k(k+1)!\\
&+x\sum_{k=0}^{p-1}S_{1,x}(p,k+1)(-1)^kk!\\
=&\sum_{k=0}^{p}S_{1,x}(p,k)(-1)^kk!-\sum_{k=1}^{p}S_{1,x}(p,k)(-1)^kk!+px^p\\
=&S_{1,x}(p,0)+px^p\\
=&(p+1)x^p,
\end{aligned}$$

as desired.

(b) We proceed by induction on p_2. If $p_2 = 0$ the result is true by (3.13). If formula (3.14) is true for a given $p_2 \in \mathbb{N}$, then, by using the recurrence (1.11) and the value $S_{1,x_1}^{1,x_2,p_2}(p_1,0) = x_1^{p_1} x_2^{p_2}$, we have that

$$\begin{aligned}
&\sum_{k=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2+1}(p_1,k+1)(-1)^k k! \\
= &\sum_{k=0}^{p_1+p_2} \left(S_{1,x_2}^{1,x_1,p_1}(p_2,k) + (k+1+x_2) S_{1,x_2}^{1,x_1,p_1}(p_2,k+1) \right)(-1)^k k! \\
= &\sum_{k=0}^{p_1+p_2} S_{1,x_2}^{1,x_1,p_1}(p_2,k)(-1)^k k! + \sum_{k=0}^{p_1+p_2-1} S_{1,x_2}^{1,x_1,p_1}(p_2,k+1)(-1)^k (k+1)! \\
&+ x_2 \sum_{k=0}^{p_1+p_2-1} S_{1,x_2}^{1,x_1,p_1}(p_2,k+1)(-1)^k k! \\
= &\sum_{k=0}^{p_1+p_2} S_{1,x_2}^{1,x_1,p_1}(p_2,k)(-1)^k k! \\
&- \sum_{k=1}^{p_1+p_2} S_{1,x_2}^{1,x_1,p_1}(p_2,k)(-1)^k k! + x_2 \left(p_1 x_2^{p_2} x_1^{p_1-1} + p_2 x_1^{p_1} x_2^{p_2-1} \right) \\
= &S_{1,x_1}^{1,x_2,p_2}(p_1,0) + x_2 \left(p_1 x_2^{p_2} x_1^{p_1-1} + p_2 x_1^{p_1} x_2^{p_2-1} \right) \\
= &x_1^{p_1} x_2^{p_2} + p_1 x_2^{p_2+1} x_1^{p_1-1} + p_2 x_1^{p_1} x_2^{p_2} \\
= &p_1 x_2^{p_2+1} x_1^{p_1-1} + (p_2+1) x_1^{p_1} x_2^{p_2},
\end{aligned}$$

as desired. □

Proposition 3.3. *The bivariate Bernoulli and Euler polynomials have the following properties* (*when possible, we write P to denote B or E*).

(a) (Symmetry) *We have*

$$P_{p_1,p_2}(1-x_1, 1-x_2) = (-1)^{p_1+p_2} P_{p_1,p_2}(x_1,x_2). \tag{3.15}$$

(b) (Difference equation) *We have*

$$B_{p_1,p_2}(x_1+1, x_2+1) - B_{p_1,p_2}(x_1,x_2) = p_1 x_1^{p_1-1} x_2^{p_2} + p_2 x_1^{p_1} x_2^{p_2-1}, \tag{3.16}$$

$$E_{p_1,p_2}(x_1+1, x_2+1) + E_{p_1,p_2}(x_1,x_2) = 2 x_1^{p_1} x_2^{p_2}. \tag{3.17}$$

(c) (Sum of powers) *If r is a positive integer, then*

$$\begin{aligned}
&B_{p_1,p_2}(x_1+r, x_2+r) - B_{p_1,p_2}(x_1,x_2) \\
&= p_1 \sum_{l=0}^{r-1} (x_1+l)^{p_1-1} (x_2+l)^{p_2} + p_2 \sum_{l=0}^{r-1} (x_1+l)^{p_1} (x_2+l)^{p_2-1},
\end{aligned} \tag{3.18}$$

(3.19)
$$\begin{aligned} & E_{p_1,p_2}(x_1+r,x_2+r)-(-1)^r E_{p_1,p_2}(x_1,x_2) \\ = \; & 2\sum_{l=0}^{r-1}(-1)^{r-1-l}(x_1+l)^{p_1}(x_2+l)^{p_2}. \end{aligned}$$

(d) (Derivatives) *We have*

(3.20)
$$\frac{\partial}{\partial x_1}P_{p_1,p_2}(x_1,x_2) = p_1 P_{p_1-1,p_2}(x_1,x_2),$$

(3.21)
$$\frac{\partial}{\partial x_2}P_{p_1,p_2}(x_1,x_2) = p_2 P_{p_1,p_2-1}(x_1,x_2).$$

(e) (Integrals) *We have*

(3.22)
$$\begin{aligned} & \int_0^1\int_0^1 P_{p_1,p_2}(x_1,x_2)dx_1dx_2 \\ = \; & \frac{1+(-1)^{p_1+p_2}}{(p_1+1)(p_2+1)}\left(P_{p_1+1,p_2+1}(1,1)-P_{p_1+1,p_2+1}(1,0)\right). \end{aligned}$$

Proof. (a) Let us consider the case $P=B$. In (3.2) replace x_1 by $1-x_1$ and x_2 by $1-x_2$, and set $y=1$, to obtain

$$B_{p_1,p_2}(1-x_1,1-x_2)=\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(-x_1)^{p_1-j_1}(-x_2)^{p_2-j_2}B_{j_1+j_2}(1),$$

from where (3.15) follows. The proof in the case $P=E$ is similar.

(b) By using (1.10) and (3.14) we have

$$\begin{aligned} & B_{p_1,p_2}(x_1+1,x_2+1) \\ & \quad = \sum_{k=0}^{p_1+p_2} S_{1,x_1+1}^{(1,x_2+1,p_2)}(p_1,k)\frac{(-1)^k k!}{k+1} \\ & \quad = \sum_{k=0}^{p_1+p_2}\left(S_{1,x_1}^{(1,x_2,p_2)}(p_1,k)+(k+1)S_{1,x_1}^{(1,x_2,p_2)}(p_1,k+1)\right)\frac{(-1)^k k!}{k+1} \\ & \quad = \sum_{k=0}^{p_1+p_2} S_{1,x_1}^{(1,x_2,p_2)}(p_1,k)\frac{(-1)^k k!}{k+1}+\sum_{k=0}^{p_1+p_2} S_{1,x_1}^{(1,x_2,p_2)}(p_1,k+1)(-1)^k k! \\ & \quad = B_{p_1,p_2}(x_1,x_2)+p_1x_2^{p_2}x_1^{p_1-1}+p_2x_1^{p_1}x_2^{p_2-1}, \end{aligned}$$

as desired. Similarly, by using (1.10) we have

$$
\begin{aligned}
&E_{p_1,p_2}(x_1+1,x_2+1)\\
&\quad=\sum_{k=0}^{p_1+p_2}\left(S_{1,x_1}^{(1,x_2,p_2)}(p_1,k)+(k+1)\,S_{1,x_1}^{(1,x_2,p_2)}(p_1,k+1)\right)\frac{(-1)^k k!}{2^k}\\
&\quad=E_{p_1,p_2}(x_1,x_2)-2\sum_{k=1}^{p_1+p_2}S_{1,x_1}^{(1,x_2,p_2)}(p_1,k)\,\frac{(-1)^k k!}{2^k}\\
&\quad=E_{p_1,p_2}(x_1,x_2)-2\left(E_{p_1,p_2}(x_1,x_2)-x_1^{p_1}x_2^{p_2}\right)\\
&\quad=-E_{p_1,p_2}(x_1,x_2)+2x_1^{p_1}x_2^{p_2},
\end{aligned}
$$

as claimed.

(c) The case $r=1$ of (3.18) is (3.16). If we suppose that (3.18) is valid for a positive integer $r>1$, then by using (3.16) we have

$$
\begin{aligned}
&B_{p_1,p_2}(x_1+r+1,x_2+r+1)-B_{p_1,p_2}(x_1,x_2)\\
=\;&B_{p_1,p_2}(x_1+r+1,x_2+r+1)-B_{p_1,p_2}(x_1+1,x_2+1)\\
&+B_{p_1,p_2}(x_1+1,x_2+1)-B_{p_1,p_2}(x_1,x_2)\\
=\;&p_1\sum_{l=0}^{r-1}(x_1+l+1)^{p_1-1}(x_2+l+1)^{p_2}+p_2\sum_{l=0}^{r-1}(x_1+l+1)^{p_1}(x_2+l+1)^{p_2-1}\\
&+p_1x_1^{p_1-1}x_2^{p_2}+p_2x_1^{p_1}x_2^{p_2-1}\\
=\;&p_1\sum_{l=0}^{r}(x_1+l)^{p_1-1}(x_2+l)^{p_2}+p_2\sum_{l=0}^{r}(x_1+l)^{p_1}(x_2+l)^{p_2-1},
\end{aligned}
$$

as desired. The proof of (3.19) is similar.

(d) Formulas (3.20) and (3.21) say that for p_1 or p_2 fixed, the sequences of bivariate Bernoulli and Euler polynomials are Appel sequences, and this fact is equivalent to addition formula (3.1). However, a direct proof follows from the following easy observation:

$$
\begin{aligned}
\frac{\partial}{\partial x_1}S_{1,x_1}^{1,x_2,p_2}(p_1,k) &= \frac{\partial}{\partial x_1}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}x_1^{p_1-j_1}x_2^{p_2-j_2}S(j_1+j_2,k)\\
&= p_1\sum_{j_1=0}^{p_1-1}\sum_{j_2=0}^{p_2}\binom{p_1-1}{j_1}\binom{p_2}{j_2}x_1^{p_1-1-j_1}x_2^{p_2-j_2}S(j_1+j_2,k)\\
&= p_1S_{1,x_1}^{1,x_2,p_2}(p_1-1,k).
\end{aligned}
$$

Similarly, we have $\frac{\partial}{\partial x_2}S_{1,x_1}^{1,x_2,p_2}(p_1,k)=p_2S_{1,x_1}^{1,x_2,p_2-1}(p_1,k)$. Then (3.20) and (3.21) follows directly from the definitions (2.4) and (2.5), and the observation above.

(e) We use (3.20) and (3.21) to write

$$\begin{aligned}
&\int_0^1\int_0^1 P_{p_1,p_2}(x_1,x_2)dx_1dx_2\\
=\ &\frac{1}{p_1+1}\int_0^1\left(P_{p_1+1,p_2}(1,x_2)-P_{p_1+1,p_2}(0,x_2)\right)dx_2\\
=\ &\frac{1}{(p_1+1)(p_2+1)}\left(P_{p_1+1,p_2+1}(1,1)-P_{p_1+1,p_2+1}(1,0)\right.\\
&\qquad\left.-P_{p_1+1,p_2+1}(0,1)+P_{p_1+1,p_2+1}(0,0)\right)\\
=\ &\frac{1+(-1)^{p_1+p_2}}{(p_1+1)(p_2+1)}\left(P_{p_1+1,p_2+1}(1,1)-P_{p_1+1,p_2+1}(1,0)\right),
\end{aligned}$$

which proves (3.22). In the last step we used (3.15) to write $P_{p_1+1,p_2+1}(0,1) = (-1)^{p_1+p_2}P_{p_1+1,p_2+1}(1,0)$ and $P_{p_1+1,p_2+1}(1,1) = (-1)^{p_1+p_2}P_{p_1+1,p_2+1}(0,0)$. □

An immediate consequence of (3.22) is the following:

Corollary 3.4. *If p_1+p_2 is odd, we have*

$$\int_0^1\int_0^1 B_{p_1,p_2}(x_1,x_2)dx_1dx_2=\int_0^1\int_0^1 E_{p_1,p_2}(x_1,x_2)dx_1dx_2=0.$$

4. Some Identities

In this section we obtain some identities involving bivariate Bernoulli and/or Euler polynomials.

Proposition 4.1. *We have the following identities*

(1)

$$\begin{aligned}
&\frac{1}{2}\left(p_1E_{p_1-1,p_2}(x_1,x_2)+p_2E_{p_1,p_2-1}(x_1,x_2)\right)\\
&\qquad=B_{p_1,p_2}(x_1,x_2)-2^{p_1+p_2}B_{p_1,p_2}\left(\frac{x_1}{2},\frac{x_2}{2}\right).
\end{aligned}\tag{4.1}$$

(2)

$$\begin{aligned}
&p_1E_{p_1-1,p_2}(x_1,x_2)+p_2E_{p_1,p_2-1}(x_1,x_2)\\
&\qquad=2^{p_1+p_2}\left(B_{p_1,p_2}\left(\frac{x_1+1}{2},\frac{x_2+1}{2}\right)-B_{p_1,p_2}\left(\frac{x_1}{2},\frac{x_2}{2}\right)\right).
\end{aligned}\tag{4.2}$$

(3) *For $m\in\mathbb{N}$,*

$$B_{p_1,p_2}(mx_1,mx_2)=m^{p_1+p_2-1}\sum_{k=0}^{m-1}B_{p_1,p_2}\left(x_1+\frac{k}{m},x_2+\frac{k}{m}\right).\tag{4.3}$$

(4) *For* $m = 1, 3, 5 \ldots,$

$$E_{p_1,p_2}(mx_1, mx_2) = m^{p_1+p_2} \sum_{k=0}^{m-1} (-1)^k E_{p_1,p_2}\left(x_1 + \frac{k}{m}, x_2 + \frac{k}{m}\right). \tag{4.4}$$

(5) *For* $m = 2, 4, 6, \ldots,$

$$\begin{aligned} &p_1 E_{p_1-1,p_2}(mx_1, mx_2) + p_2 E_{p_1,p_2-1}(mx_1, mx_2) \\ &\qquad = -2m^{p_1+p_2-1} \sum_{k=0}^{m-1} (-1)^k B_{p_1,p_2}\left(x_1 + \frac{k}{m}, x_2 + \frac{k}{m}\right). \end{aligned} \tag{4.5}$$

Proof. We show the proofs of (4.1), (4.3) and (4.5).

(1) Beginning with the right-hand side of (4.1), we have

$$\begin{aligned} & B_{p_1,p_2}(x_1, x_2) - 2^{p_1+p_2} B_{p_1,p_2}\left(\frac{x_1}{2}, \frac{x_2}{2}\right) \\ =\ & \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1}\binom{p_2}{j_2} (x_1 - y)^{p_1-j_1} (x_2 - y)^{p_2-j_2} B_{j_1+j_2}(y) \\ & -2^{p_1+p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1}\binom{p_2}{j_2} \left(\frac{x_1}{2} - \frac{y}{2}\right)^{p_1-j_1} \left(\frac{x_2}{2} - \frac{y}{2}\right)^{p_2-j_2} B_{j_1+j_2}\left(\frac{y}{2}\right) \\ =\ & \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1}\binom{p_2}{j_2} (x_1 - y)^{p_1-j_1} (x_2 - y)^{p_2-j_2} \times \\ & \quad \times \left(B_{j_1+j_2}(y) - 2^{j_1+j_2} B_{j_1+j_2}\left(\frac{y}{2}\right)\right). \end{aligned} \tag{4.6}$$

By using the identity $E_{n-1}(y) = \frac{2}{n}\left(B_n(y) - 2^n B_n\left(\frac{y}{2}\right)\right)$ (see [9, Eq. 24.4.22]), we have

$$\begin{aligned} (4.6) \ =\ & \frac{1}{2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1}\binom{p_2}{j_2} (x_1 - y)^{p_1-j_1} (x_2 - y)^{p_2-j_2} (j_1 + j_2) E_{j_1+j_2-1}(y) \\ =\ & \frac{p_1}{2} \sum_{j_1=0}^{p_1-1} \sum_{j_2=0}^{p_2} \binom{p_1-1}{j_1}\binom{p_2}{j_2} (x_1 - y)^{p_1-1-j_1} (x_2 - y)^{p_2-j_2} E_{j_1+j_2}(y) \\ & + \frac{p_2}{2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2-1} \binom{p_1}{j_1}\binom{p_2-1}{j_2} (x_1 - y)^{p_1-j_1} (x_2 - y)^{p_2-1-j_2} E_{j_1+j_2}(y) \\ =\ & \frac{1}{2}\left(p_1 E_{p_1-1,p_2}(x_1, x_2) + p_2 E_{p_1,p_2-1}(x_1, x_2)\right), \end{aligned}$$

as desired

(2) Beginning with the left-hand side of (4.3), we have

$$\begin{aligned} (4.7) \qquad & B_{p_1,p_2}(mx_1, mx_2) \\ &= \sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(mx_1-my)^{p_1-j_1}(mx_2-my)^{p_2-j_2}B_{j_1+j_2}(my). \end{aligned}$$

By using the identity $B_n(my) = m^{n-1}\sum_{k=0}^{m-1}B_n\left(y+\frac{k}{m}\right)$ (see [9, Eq. 24.4.18]) we obtain

$$\begin{aligned} (4.7) &= m^{p_1+p_2-1}\sum_{k=0}^{m-1}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}\left(x_1+\frac{k}{m}-\left(y+\frac{k}{m}\right)\right)^{p_1-j_1}\times \\ &\qquad \times\left(x_2+\frac{k}{m}-\left(y+\frac{k}{m}\right)\right)^{p_2-j_2}B_{j_1+j_2}\left(y+\frac{k}{m}\right) \\ &= m^{p_1+p_2-1}\sum_{k=0}^{m-1}B_{p_1,p_2}\left(x_1+\frac{k}{m}, x_2+\frac{k}{m}\right), \end{aligned}$$

as desired.

(3) Beginning with the right-hand side of (4.5), we have

$$\begin{aligned} (4.8) \qquad & -2m^{p_1+p_2-1}\sum_{k=0}^{m-1}(-1)^k B_{p_1,p_2}\left(x_1+\frac{k}{m}, x_2+\frac{k}{m}\right) \\ &= -2m^{p_1+p_2-1}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-y)^{p_1-j_1}(x_2-y)^{p_2-j_2}\times \\ &\qquad \times\sum_{k=0}^{m-1}(-1)^k B_{j_1+j_2}\left(y+\frac{k}{m}\right). \end{aligned}$$

By using the identity $E_n(my) = -\frac{2m^n}{n+1}\sum_{k=0}^{m-1}(-1)^k B_{n+1}\left(y+\frac{k}{m}\right)$ for $m = 2, 4, 6, \ldots$ (see [9, Eq. 24.4.19]) we obtain

$$\begin{aligned} (4.8) &= -2m^{p_1+p_2-1}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(x_1-y)^{p_1-j_1}(x_2-y)^{p_2-j_2}\times \\ &\qquad \times\left(-\frac{j_1+j_2}{2m^{j_1+j_2-1}}E_{j_1+j_2-1}(my)\right) \\ &= p_1\sum_{j_1=0}^{p_1-1}\sum_{j_2=0}^{p_2}\binom{p_1-1}{j_1}\binom{p_2}{j_2}(mx_1-my)^{p_1-1-j_1}(mx_2-my)^{p_2-j_2}E_{j_1+j_2}(my) \\ &+ p_2\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2-1}\binom{p_1}{j_1}\binom{p_2-1}{j_2}(mx_1-my)^{p_1-j_1}(mx_2-my)^{p_2-1-j_2}E_{j_1+j_2}(my) \\ &= p_1 E_{p_1-1,p_2}(mx_1, mx_2) + p_2 E_{p_1,p_2-1}(mx_1, mx_2), \end{aligned}$$

as desired.

The proofs of (4.2) and (4.4) are similar, using the identities

$$E_{n-1}(y) = \frac{2^n}{n}\left(B_n\left(\frac{y+1}{2}\right) - B_n\left(\frac{y}{2}\right)\right) \text{ (see [9, Eq. 24.4.23])}$$

and

$$E_n(my) = m^n \sum_{k=0}^{m-1} (-1)^k E_n\left(y + \frac{k}{m}\right) \text{ for } m = 1, 3, 5, \ldots \text{ (see [9, Eq. 24.4.20]).}$$

□

Now we consider a different kind of identities involving Bernoulli and Euler numbers and polynomials, which are based in the following result.

Proposition 4.2. *The polynomial identity*

$$\sum_{l=0}^{n} a_{n,l}(x+\alpha)^l = \sum_{l=0}^{n} b_{n,l}(x+\beta)^l, \tag{4.9}$$

implies the Bernoulli and Euler polynomials identities

$$\sum_{l=0}^{n} a_{n,l}B_l(x+\alpha) = \sum_{l=0}^{n} b_{n,l}B_l(x+\beta), \tag{4.10}$$

$$\sum_{l=0}^{n} a_{n,l}E_l(x+\alpha) = \sum_{l=0}^{n} b_{n,l}E_l(x+\beta). \tag{4.11}$$

Proof. Observe that the polynomial identity (4.9) comes together with the identities of the corresponding derivatives

$$\sum_{l=0}^{n} a_{n,l}\binom{l}{r}(x+\alpha)^{l-r} = \sum_{l=0}^{n} b_{n,l}\binom{l}{r}(x+\beta)^{l-r}, \tag{4.12}$$

for $0 \leq r \leq l \leq n$. Beginning with the left-hand side of (4.10), and using (4.12) in the third step, we have

$$\begin{aligned}
\sum_{l=0}^{n} a_{n,l} B_l\left(x+\alpha\right) &= \sum_{l=0}^{n} a_{n,l} \sum_{k=0}^{l} S_{1,x+\alpha}\left(l,k\right) \frac{(-1)^k k!}{k+1} \\
&= \sum_{k=0}^{n} \sum_{j=0}^{n} \left(\sum_{l=0}^{n} a_{n,l} \binom{l}{j} \left(x+\alpha\right)^{l-j} \right) S\left(j,k\right) \frac{(-1)^k k!}{k+1} \\
&= \sum_{k=0}^{n} \sum_{j=0}^{n} \left(\sum_{l=0}^{n} b_{n,l} \binom{l}{j} \left(x+\beta\right)^{l-j} \right) S\left(j,k\right) \frac{(-1)^k k!}{k+1} \\
&= \sum_{l=0}^{n} b_{n,l} \sum_{k=0}^{l} S_{1,x+\beta}\left(l,k\right) \frac{(-1)^k k!}{k+1} \\
&= \sum_{l=0}^{n} b_{n,l} B_l\left(x+\beta\right),
\end{aligned}$$

which shows (4.10). The proof of (4.11) is similar. □

For example, from (3.2) we can write

$$\begin{aligned}
(4.13)\qquad & \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} \left(x_1-y\right)^{p_1-j_1} \left(x_2-y\right)^{p_2-j_2} B_{j_1+j_2}\left(y\right) \\
&= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} \left(x_1-z\right)^{p_1-j_1} \left(x_2-z\right)^{p_2-j_2} B_{j_1+j_2}\left(z\right),
\end{aligned}$$

where y, z are arbitrary parameters. By using Proposition 4.2 in (4.13) we obtain identities of the form

$$\begin{aligned}
(4.14)\qquad & \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} P_{p_1-j_1}\left(x_1-y\right) Q_{p_2-j_2}\left(x_2-y\right) B_{j_1+j_2}\left(y\right) \\
&= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} P_{p_1-j_1}\left(x_1-z\right) Q_{p_2-j_2}\left(x_2-z\right) B_{j_1+j_2}\left(z\right),
\end{aligned}$$

where P, Q are Bernoulli or Euler polynomials.

In the case $P = Q = B$, with $x_1 = x_2 = z = 1$ and $y = 0$, we obtain that

$$\begin{aligned}
(4.15)\qquad & (-1)^{p_1+p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (-1)^{j_1+j_2} B_{p_1-j_1} B_{p_2-j_2} B_{j_1+j_2} \\
&= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (-1)^{j_1+j_2} B_{p_1-j_1} B_{p_2-j_2} B_{j_1+j_2},
\end{aligned}$$

from where we conclude that, if $p_1 + p_2$ is odd, then

$$\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (-1)^{j_1+j_2} B_{p_1-j_1} B_{p_2-j_2} B_{j_1+j_2} = 0. \tag{4.16}$$

With similar arguments it is possible to show that if $p_1 + p_2$ is odd, then

$$\begin{aligned}
\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} B_{p_1-j_1} B_{p_2-j_2} E_{j_1+j_2}(1) &= 0, \\
\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (-1)^{j_1+j_2} E_{p_1-j_1}(0) E_{p_2-j_2}(0) B_{j_1+j_2} &= 0, \\
\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} E_{p_1-j_1} E_{p_2-j_2} E_{j_1+j_2} &= 0,
\end{aligned}$$

$$\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (-1)^{j_1+j_2} E_{p_1-j_1} E_{p_2-j_2} E_{j_1+j_2} = 0. \tag{4.17}$$

Now let us consider the difference equations (3.16) and (3.17). Write (3.16) as

$$\begin{aligned}
&\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} \left((x_1+1)^{p_1-j_1} (x_2+1)^{p_2-j_2} - x_1^{p_1-j_1} x_2^{p_2-j_2} \right) B_{j_1+j_2} \\
&= p_1 x_1^{p_1-1} x_2^{p_2} + p_2 x_2^{p_2-1} x_1^{p_1}.
\end{aligned} \tag{4.18}$$

By using Proposition 4.2, we obtain from (4.18) the identities

$$\begin{aligned}
&\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} \times \\
&\quad \times \left(B_{p_1-j_1}(x_1+1) B_{p_2-j_2}(x_2+1) - B_{p_1-j_1}(x_1) B_{p_2-j_2}(x_2) \right) B_{j_1+j_2} \\
&= p_1 B_{p_1-1}(x_1) B_{p_2}(x_2) + p_2 B_{p_1}(x_1) B_{p_2-1}(x_2),
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
&\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} \times \\
&\quad \times \left(E_{p_1-j_1}(x_1+1) E_{p_2-j_2}(x_2+1) - E_{p_1-j_1}(x_1) E_{p_2-j_2}(x_2) \right) B_{j_1+j_2} \\
&= p_1 E_{p_1-1}(x_1) E_{p_2}(x_2) + p_2 E_{p_1}(x_1) E_{p_2-1}(x_2),
\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}
&\sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} \times \\
&\quad \times \left(B_{p_1-j_1}(x_1+1) E_{p_2-j_2}(x_2+1) - B_{p_1-j_1}(x_1) E_{p_2-j_2}(x_2) \right) B_{j_1+j_2} \\
&= p_1 B_{p_1-1}(x_1) E_{p_2}(x_2) + p_2 B_{p_1}(x_1) E_{p_2-1}(x_2).
\end{aligned}$$

Similarly, by writing (3.17) as

$$\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}\left((x_1+1)^{p_1-j_1}(x_2+1)^{p_2-j_2}+x_1^{p_1-j_1}x_2^{p_2-j_2}\right)E_{j_1+j_2}(0) = 2x_1^{p_1}x_2^{p_2}, \tag{4.21}$$

and using Proposition 4.2, we obtain from (4.21) identities of the form

$$\begin{aligned}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}&\binom{p_1}{j_1}\binom{p_2}{j_2}\times\\ &\times\left(P_{p_1-j_1}(x_1+1)\,Q_{p_2-j_2}(x_2+1)+P_{p_1-j_1}(x_1)\,Q_{p_2-j_2}(x_2)\right)E_{j_1+j_2}(0)\\ =\ & 2P_{p_1}(x_1)\,Q_{p_2}(x_2)\,,\end{aligned} \tag{4.22}$$

where P and Q are Bernoulli or Euler polynomials.

To end this section, we show some particular cases of the previous results.

By setting $p_2=2$ and $x_1=x_2=0$ in (4.19), we obtain (after some simplifications) the following identity for $p>0$:

$$\sum_{j=1}^{p+1}\binom{2p+1}{2j-1}B_{2p-2j+2}B_{2j}=-(p+1/2)B_{2p+2}.$$

Similarly, from (4.20) with $p_2=1$ and $x_1=x_2=0$, we get the following identity for $p>0$:

$$2\sum_{j=1}^{p}\binom{2p}{2j-1}E_{2p-2j+1}(1)\,B_{2j}=-B_{2p}-pE_{2p-1}(0)\,.$$

By using Proposition 4.2 in the case $m=2$ of (4.3), we obtain that

$$\begin{aligned}&2^{1-p_1-p_2}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}2^{p_1+p_2-j_1-j_2}B_{p_1-j_1}(x_1)B_{p_2-j_2}(x_2)B_{j_1+j_2}\\ =\ &\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}\times\\ &\times\left(B_{p_1-j_1}(x_1)B_{p_2-j_2}(x_2)+B_{p_1-j_1}(x_1+1/2)B_{p_2-j_2}(x_2+1/2)\right)B_{j_1+j_2}.\end{aligned} \tag{4.23}$$

Set $x_1=x_2=0$ and use that $B_n\left(\frac{1}{2}\right)=\left(2^{1-n}-1\right)B_n$, to get from (4.23) that

$$\begin{aligned}&2\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}2^{-j_1-j_2}B_{p_1-j_1}B_{p_2-j_2}B_{j_1+j_2}\\ =&\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}\left(1+\left(2^{1+j_1-p_1}-1\right)\left(2^{1+j_2-p_2}-1\right)\right)B_{p_1-j_1}B_{p_2-j_2}B_{j_1+j_2}.\end{aligned} \tag{4.24}$$

The case $p_2 = 1$ of (4.24) produces the identity

$$\sum_{j=0}^{p-1} \binom{2p}{2j} \left(1 - 2^{1-2j}\right) B_{2j} B_{2p-2j} = \left(2^{1-2p} - 1\right) 2p B_{2p}, \tag{4.25}$$

which can be written (by using the Euler's identity $\sum_{j=0}^{p} \binom{2p}{2j} B_{2j} B_{2p-2j} = -(2p-1) B_{2p}$ for $p > 1$, see [4]) as

$$\sum_{j=0}^{p} \binom{2p}{2j} 2^{-2j} B_{2j} B_{2p-2j} = (1 - 2p)\, 2^{-2p} B_{2p}.$$

With similar procedures it is possible to obtain also the identities

$$\sum_{j=0}^{p-1} \binom{2p}{2j} 2^{2j} B_{2j} E_{2p-2j} = 2\left(1 - 2^{2p}\right) B_{2p},$$

$$\sum_{j=0}^{p-1} \binom{2p-1}{2j} \left(1 - 2^{1-2j}\right) B_{2j} E_{2p-1-2j}(0) = 2^{1-2p} (2p-1) E_{2p-2}.$$

5. Generalized Recurrences

In this section we show some recurrences for bivariate Bernoulli and Euler polynomials.

Proposition 5.1. *For non-negative integers p_1, p_2, q, we have the recurrences*

$$\begin{aligned} \sum_{k=0}^{q} B_{p_1+k,p_2}(x_1,x_2) \frac{(-1)^k}{k!} \frac{d^k}{dx_1^k} \prod_{j=0}^{q-1} (x_1 + j) &= \sum_{k=0}^{q} B_{p_1,p_2+k}(x_1,x_2) \frac{(-1)^k}{k!} \frac{d^k}{dx_2^k} \prod_{j=0}^{q-1} (x_2 + j) \\ &= \sum_{k=0}^{p_1+p_2} S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,k) \frac{(-1)^k (k+q)!}{k+q+1}, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \sum_{k=0}^{q} E_{p_1+k,p_2}(x_1,x_2) \frac{(-1)^k}{k!} \frac{d^k}{dx_1^k} \prod_{j=0}^{q-1} (x_1 + j) &= \sum_{k=0}^{q} E_{p_1,p_2+k}(x_1,x_2) \frac{(-1)^k}{k!} \frac{d^k}{dx_2^k} \prod_{j=0}^{q-1} (x_2 + j) \\ &= \sum_{k=0}^{p_1+p_2} S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,k) \frac{(-1)^k (k+q)!}{2^{k+q}}. \end{aligned} \tag{5.2}$$

Proof. Formulas in (5.1) are included as particular cases of Proposition 4 in [12]. We prove the identity

$$\sum_{k=0}^{q} E_{p_1+k,p_2}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_1^k}\prod_{j=0}^{q-1}(x_1+j) = \sum_{k=0}^{p_1+p_2} S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,k)\frac{(-1)^k(k+q)!}{2^{k+q}}, \tag{5.3}$$

contained in (5.2) by induction on q. The case $q=0$ is the definition (2.5). If we suppose that (5.3) is true for a given $q\in\mathbb{N}$, then

$$\begin{aligned} &\sum_{k=0}^{q+1} E_{p_1+k,p_2}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_1^k}\prod_{j=0}^{q}(x_1+j) \\ =\; &\sum_{k=0}^{q+1} E_{p_1+k,p_2}(x_1,x_2)\frac{(-1)^k}{k!}\left((x_1+q)\frac{d^k}{dx_1^k}\prod_{j=0}^{q-1}(x_1+j)+k\frac{d^{k-1}}{dx_1^{k-1}}\prod_{j=0}^{q-1}(x_1+j)\right) \\ =\; &(x_1+q)\sum_{k=0}^{q} E_{p_1+k,p_2}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_1^k}\prod_{j=0}^{q-1}(x_1+j) \\ &-\sum_{k=0}^{q} E_{p_1+1+k,p_2}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_1^k}\prod_{j=0}^{q-1}(x_1+j). \end{aligned} \tag{5.4}$$

The induction hypothesis gives us from (5.4), that

$$\begin{aligned} &\sum_{k=0}^{q+1} E_{p_1+k,p_2}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_1^k}\prod_{j=0}^{q}(x_1+j) \\ =\; &(x_1+q)\sum_{k=0}^{p_1+p_2} S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,k)\frac{(-1)^k(k+q)!}{2^{k+q}} \\ &-\sum_{k=0}^{p_1+p_2+1} S_{1,x_1+q}^{1,x_2+q,p_2}(p_1+1,k)\frac{(-1)^k(k+q)!}{2^{k+q}}. \end{aligned} \tag{5.5}$$

According to the recurrence (1.11), we have from (5.5) that

$$\begin{aligned}
&\sum_{k=0}^{q+1} E_{p_1+k,p_2}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_1^k}\prod_{j=0}^{q}(x_1+j)\\
&=(x_1+q)\sum_{k=0}^{p_1+p_2} S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,k)\,\frac{(-1)^k\,(k+q)!}{2^{k+q}}\\
&-\sum_{k=0}^{p_1+p_2+1}\left(S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,k-1)+(k+x_1+q)S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,k)\right)\frac{(-1)^k\,(k+q)!}{2^{k+q}}\\
&=-\sum_{k=0}^{p_1+p_2+1}\left(S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,k-1)+kS_{1,x_1+q}^{1,x_2+q,p_2}(p_1,k)\right)\frac{(-1)^k\,(k+q)!}{2^{k+q}}\\
&=\sum_{k=0}^{p_1+p_2}\left(S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,k)+(k+1)S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,k+1)\right)\frac{(-1)^k\,(k+q+1)!}{2^{k+q+1}}\\
&=\sum_{k=0}^{p_1+p_2} S_{1,x_1+q+1}^{1,x_2+q+1,p_2}(p_1,k)\,\frac{(-1)^k\,(k+q+1)!}{2^{k+q+1}},
\end{aligned}$$

as desired. In the last step we used (1.10).

The proof of the identity

$$\sum_{k=0}^{q} E_{p_1,p_2+k}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_2^k}\prod_{j=0}^{q-1}(x_2+j)=\sum_{k=0}^{p_1+p_2} S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,k)\,\frac{(-1)^k\,(k+q)!}{2^{k+q}}.$$

is similar. □

Proposition 5.2. *For non-negative integers $p_1, p_2, q,$ we have the recurrences*

$$\begin{aligned}
\sum_{k=0}^{q} B_{p_1+k,p_2}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_1^k}\prod_{j=1}^{q}(x_1-j)\\
&=\sum_{k=0}^{q} B_{p_1,p_2+k}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_2^k}\prod_{j=1}^{q}(x_2-j)\\
&=q!\sum_{k=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2}(p_1,k)\,\frac{(-1)^{k+q}\,k!}{k+q+1},
\end{aligned}\tag{5.6}$$

$$\begin{aligned}(5.7)\qquad &\sum_{k=0}^{q}E_{p_1+k,p_2}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_1^k}\prod_{j=1}^{q}(x_1-j)\\ &=\sum_{k=0}^{q}E_{p_1,p_2+k}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_2^k}\prod_{j=1}^{q}(x_2-j)\\ &=\sum_{k=0}^{p_1+p_2}S_{1,x_1}^{1,x_2,p_2}(p_1,k)\frac{(-1)^{k+q}(k+q)!}{2^{k+q}}.\end{aligned}$$

Proof. We prove the identity

$$\begin{aligned}(5.8)\qquad &\sum_{k=0}^{q}E_{p_1,p_2+k}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_2^k}\prod_{j=1}^{q}(x_2-j)\\ &=\sum_{k=0}^{p_1+p_2}S_{1,x_1}^{1,x_2,p_2}(p_1,k)\frac{(-1)^{k+q}(k+q)!}{2^{k+q}},\end{aligned}$$

contained in (5.7) by induction on q. The case $q=0$ of (5.8) is the definition (2.5). If we suppose (5.8) is true for a given $q\in\mathbb{N}$, then

$$\begin{aligned}(5.9)\qquad &\sum_{k=0}^{q+1}E_{p_1,p_2+k}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_2^k}\prod_{j=1}^{q+1}(x_2-j)\\ =\;&\sum_{k=0}^{q+1}E_{p_1,p_2+k}(x_1,x_2)\frac{(-1)^k}{k!}\times\\ &\times\left((x_2-q-1)\frac{d^k}{dx_2^k}\prod_{j=1}^{q}(x_2-j)+k\frac{d^{k-1}}{dx_2^{k-1}}\prod_{j=1}^{q}(x_2-j)\right)\\ =\;&(x_2-q-1)\sum_{k=0}^{q}E_{p_1,p_2+k}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_2^k}\prod_{j=1}^{q}(x_2-j)\\ &-\sum_{k=0}^{q}E_{p_1,p_2+k+1}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_2^k}\prod_{j=1}^{q}(x_2-j).\end{aligned}$$

We use induction hypothesis to write (5.9) as

$$\begin{aligned}&\sum_{k=0}^{q+1}E_{p_1,p_2+k}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_2^k}\prod_{j=1}^{q+1}(x_2-j)\\ =\;&(x_2-q-1)\sum_{k=0}^{p_1+p_2}S_{1,x_1}^{1,x_2,p_2}(p_1,k)\frac{(-1)^{k+q}(k+q)!}{2^{k+q}}\\ &-\sum_{k=0}^{p_1+p_2+1}S_{1,x_1}^{1,x_2,p_2+1}(p_1,k)\frac{(-1)^{k+q}(k+q)!}{2^{k+q}}.\end{aligned}$$

The recurrence (1.11) gives us

$$\begin{aligned}
&\sum_{k=0}^{q+1} E_{p_1,p_2+k}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_2^k}\prod_{j=1}^{q+1}(x_2-j)\\
= \ &(x_2-q-1)\sum_{k=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2}(p_1,k)\,\frac{(-1)^{k+q}(k+q)!}{2^{k+q}}\\
&-\sum_{k=0}^{p_1+p_2+1}\left(S_{1,x_1}^{1,x_2,p_2}(p_1,k-1)+(k+x_2)S_{1,x_1}^{1,x_2,p_2}(p_1,k)\right)\frac{(-1)^{k+q}(k+q)!}{2^{k+q}}\\
= \ &-\sum_{k=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2}(p_1,k)\,\frac{(-1)^{k+q+1}(k+q+1)!}{2^{k+q+1}}\\
&-\sum_{k=0}^{p_1+p_2}(k+q+1)S_{1,x_1}^{1,x_2,p_2}(p_1,k)\,\frac{(-1)^{k+q}(k+q)!}{2^{k+q}}\\
= \ &-\sum_{k=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2}(p_1,k)\,\frac{(-1)^{k+q+1}(k+q+1)!}{2^{k+q+1}}\\
. \ &-\sum_{k=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2}(p_1,k)\,\frac{(-1)^{k+q}(k+q+1)!}{2^{k+q}}\\
= \ &\sum_{k=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2}(p_1,k)\,\frac{(-1)^{k+q+1}(k+q+1)!}{2^{k+q+1}},
\end{aligned}$$

as desired. The proof of the identity

$$\sum_{k=0}^{q} E_{p_1+k,p_2}(x_1,x_2)\frac{(-1)^k}{k!}\frac{d^k}{dx_1^k}\prod_{j=1}^{q}(x_1-j)=\sum_{k=0}^{p_1+p_2} S_{1,x_1}^{1,x_2,p_2}(p_1,k)\,\frac{(-1)^{k+q}(k+q)!}{2^{k+q}},$$

and (5.6) are similar. □

We can write recurrence (5.7) as

$$\begin{aligned}
&\sum_{k=0}^{q} E_{p_1+k,p_2}(x_1+q,x_2+q)\frac{(-1)^k}{k!}\frac{d^k}{dx_1^k}\prod_{j=0}^{q-1}(x_1+j)\\
= \ &\sum_{k=0}^{q} E_{p_1,p_2+k}(x_1+q,x_2+q)\frac{(-1)^k}{k!}\frac{d^k}{dx_2^k}\prod_{j=0}^{q-1}(x_2+j)\\
= \ &\sum_{k=0}^{p_1+p_2} S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,k)\,\frac{(-1)^{k+q}(k+q)!}{2^{k+q}}.
\end{aligned}\tag{5.10}$$

Thus, from (5.2) and (5.10) we have

$$
\begin{aligned}
(5.11)\qquad & \sum_{k=0}^{q} E_{p_1+k,p_2}(x_1+q,x_2+q)\frac{(-1)^{k+q}}{k!}\frac{d^k}{dx_1^k}\prod_{j=0}^{q-1}(x_1+j) \\
= & \sum_{k=0}^{q} E_{p_1,p_2+k}(x_1+q,x_2+q)\frac{(-1)^{k+q}}{k!}\frac{d^k}{dx_2^k}\prod_{j=0}^{q-1}(x_2+j) \\
= & \sum_{k=0}^{q} E_{p_1+k,p_2}(x_1,x_2)\frac{(-1)^{k}}{k!}\frac{d^k}{dx_1^k}\prod_{j=0}^{q-1}(x_1+j) \\
= & \sum_{k=0}^{q} E_{p_1,p_2+k}(x_1,x_2)\frac{(-1)^{k}}{k!}\frac{d^k}{dx_2^k}\prod_{j=0}^{q-1}(x_2+j) \\
= & \sum_{k=0}^{p_1+p_2} S_{1,x_1+q}^{1,x_2+q,p_2}(p_1,k)\frac{(-1)^k(k+q)!}{2^{k+q}}.
\end{aligned}
$$

From (5.1) with $x_1=1, x_2=0$ we have

$$
\begin{aligned}
(5.12)\quad \sum_{k=0}^{q}(-1)^k s(q+1,k+1)B_{p_1+k,p_2}(1,0) &= \sum_{k=0}^{q}(-1)^k s(q,k)B_{p_1,p_2+k}(1,0) \\
&= \sum_{k=0}^{p_1+p_2} S_{1,q+1}^{1,q,p_2}(p_1,k)\frac{(-1)^k(k+q)!}{k+q+1}.
\end{aligned}
$$

We use Carlitz identity (3.5) to obtain the following explicit enriched version of (5.12),

$$
\begin{aligned}
(5.13)\qquad & \sum_{k=0}^{q}\sum_{j_1=0}^{p_1+k}\binom{p_1+k}{j_1}(-1)^k s(q+1,k+1)B_{j_1+p_2} \\
= & (-1)^{p_1+p_2}\sum_{k=0}^{q}\sum_{j_2=0}^{p_2}\binom{p_2}{j_2}s(q+1,k+1)B_{p_1+k+j_2} \\
= & \sum_{k=0}^{q}\sum_{j_1=0}^{p_1}\binom{p_1}{j_1}(-1)^k s(q,k)B_{j_1+p_2+k} \\
= & (-1)^{p_1+p_2}\sum_{k=0}^{q}\sum_{j_2=0}^{p_2+k}\binom{p_2+k}{j_2}s(q,k)B_{p_1+j_2} \\
= & \sum_{k=0}^{p_1+p_2}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(q+1)^{p_1-j_1}q^{p_2-j_2}S(j_1+j_2,k)\frac{(-1)^k(k+q)!}{k+q+1}.
\end{aligned}
$$

Similarly, from (5.6) we obtain

$$\begin{aligned}
&\sum_{k=0}^{q}\sum_{j_1=0}^{p_1+k}\binom{p_1+k}{j_1}s(q,k)B_{j_1+p_2}\\
=\;&(-1)^{p_1+p_2}\sum_{k=0}^{q}\sum_{j_2=0}^{p_2}\binom{p_2}{j_2}(-1)^k s(q,k)B_{p_1+k+j_2}\\
=\;&\sum_{k=0}^{q}\sum_{j_1=0}^{p_1}\binom{p_1}{j_1}s(q+1,k+1)B_{j_1+p_2+k}\\
=\;&(-1)^{p_1+p_2}\sum_{k=0}^{q}\sum_{j_2=0}^{p_2+k}\binom{p_2+k}{j_2}(-1)^k s(q+1,k+1)B_{p_1+j_2}\\
=\;&q!\sum_{k=0}^{p_1+p_2}\sum_{j_1=0}^{p_1}\binom{p_1}{j_1}S(j_1+p_2,k)\frac{(-1)^k k!}{k+q+1},
\end{aligned}$$

and from (5.11) we get

$$\begin{aligned}
&(-1)^q\sum_{k=0}^{q}\sum_{j_1=0}^{p_1+k}\sum_{j_2=0}^{p_2}\binom{p_1+k}{j_1}\binom{p_2}{j_2}(-1)^k s(q+1,k+1)(q+1)^{p_1+k-j_1}q^{p_2-j_2}E_{j_1+j_2}(0)\\
&=(-1)^q\sum_{k=0}^{q}\sum_{j_1=0}^{p_1+k}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2+k}{j_2}(-1)^k s(q,k)(q+1)^{p_1-j_1}q^{p_2+k-j_2}E_{j_1+j_2}(0)\\
&=\sum_{k=0}^{q}\sum_{j_1=0}^{p_1+k}\binom{p_1+k}{j_1}(-1)^k s(q+1,k+1)E_{j_1+p_2}(0)\\
&=\sum_{k=0}^{q}\sum_{j_1=0}^{p_1}\binom{p_1}{j_1}(-1)^k s(q,k)E_{j_1+p_2+k}(0)\\
&=\sum_{k=0}^{p_1+p_2}\sum_{j_1=0}^{p_1}\sum_{j_2=0}^{p_2}\binom{p_1}{j_1}\binom{p_2}{j_2}(q+1)^{p_1-j_1}q^{p_2-j_2}S(j_1+j_2,k)\frac{(-1)^k(k+q)!}{2^{k+q}}.
\end{aligned}$$

Proposition 5.3. *We have the following identities for bivariate Euler polynomials*

$$\sum_{j=0}^{q}\binom{q}{j}(-1)^j E_{p_1+j,p_2+q-j}(x_1,x_2)=(x_2-x_1)^q E_{p_1,p_2}(x_1,x_2), \tag{5.14}$$

$$\sum_{j=0}^{q}\binom{q}{j}(x_2-x_1)^j E_{p_1+q-j,p_2}(x_1,x_2)=E_{p_1,p_2+q}(x_1,x_2). \tag{5.15}$$

Proof. Let us prove (5.14) by induction on q. The case $q=0$ is a trivial identity.

If (5.14) is valid for a given $q \in \mathbb{N}$, then

$$\begin{aligned}
&\sum_{j=0}^{q+1}\binom{q+1}{j}(-1)^j E_{p_1+j,p_2+q+1-j}(x_1,x_2)\\
= &\sum_{j=0}^{q}\binom{q}{j}(-1)^j E_{p_1+j,p_2+1+q-j}(x_1,x_2) - \sum_{j=0}^{q}\binom{q}{j}(-1)^j E_{p_1+1+j,p_2+q-j}(x_1,x_2).
\end{aligned}$$

The induction hypothesis gives us

$$\begin{aligned}
&\sum_{j=0}^{q+1}\binom{q+1}{j}(-1)^j E_{p_1+j,p_2+q+1-j}(x_1,x_2)\\
= &(x_2-x_1)^q E_{p_1,p_2+1}(x_1,x_2) - (x_2-x_1)^q E_{p_1+1,p_2}(x_1,x_2)\\
= &(x_2-x_1)^q \left(E_{p_1,p_2+1}(x_1,x_2) - E_{p_1+1,p_2}(x_1,x_2)\right)\\
= &(x_2-x_1)^{q+1} E_{p_1,p_2}(x_1,x_2),
\end{aligned}$$

as desired. In the last step we used the identity

$$E_{p_1+1,p_2}(x_1,x_2) - E_{p_1,p_2+1}(x_1,x_2) = (x_1-x_2)E_{p_1,p_2}(x_1,x_2), \tag{5.16}$$

which is included in the case $q = 1$ of (5.2) (or (5.7)).

Let us prove (5.15) by induction on q. The case $q = 0$ is a trivial identity. If we suppose (5.15) is valid for a given $q \in \mathbb{N}$, then

$$\begin{aligned}
&\sum_{j=0}^{q+1}\binom{q+1}{j}(x_2-x_1)^j E_{p_1+q+1-j,p_2}(x_1,x_2)\\
= &\sum_{j=0}^{q}\binom{q}{j}(x_2-x_1)^j E_{p_1+q+1-j,p_2}(x_1,x_2)\\
&+(x_2-x_1)\sum_{j=0}^{q}\binom{q}{j}(x_2-x_1)^j E_{p_1+q-j,p_2}(x_1,x_2)\\
= &E_{p_1+1,p_2+q}(x_1,x_2) + (x_2-x_1)E_{p_1,p_2+q}(x_1,x_2)\\
= &E_{p_1,p_2+q+1}(x_1,x_2),
\end{aligned}$$

as desired. We used (5.16) in the last step. □

There are similar results to (5.14) and (5.15) in the case of bivariate Bernoulli polynomials. These results appear in [12] in the context of bivariate poly-Bernoulli polynomials.

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