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A Nonlinear Elliptic Equation of Emden Fowler Type with Convection Term

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Abstract. In this paper we give conditions for the existence of, and describe the asymtotic behavior of, radial positive solutions of the nonlinear elliptic equation of Emden-Fowler type with convection term

$$\Delta_p u + \alpha|u|^{q-1}u + \beta x.\nabla(|u|^{q-1}u) = 0$$

for $x \in \mathbb{R}^N$, where $p > 2$, $q > 1$, $N \geq 1$, $\alpha > 0$, $\beta > 0$ and Δ_p is the p-Laplacian operator. In particular, we determine $\lim_{r\to\infty} r^{\frac{p}{q+1-p}} u(r)$ when $\frac{\alpha}{\beta} > N > p$ and $q \geq \frac{N(p-1)+p}{N-p}$.

1. Introduction

As many common problems in mathematical physics can be formulated as equations of Emden-Fowler type, such equations have been an active topic of research in recent years; they have been approaced with a wide variety of methods and techniques. Interesting results can be found in Napoles [22], [21], [25] and the references therein.

This paper is a contribution to the study of the radial equation of Emden-Fowler type with convection term,

(1.1)
$$(|u'|^{p-2}u')'(r) + \frac{N-1}{r}|u'|t^{p-2}u'(r) + \alpha|u|^{q-1}u(r) + \beta r(|u|^{q-1}u)'(r) = 0, \; r > 0,$$

where $p > 2$, $q > 1$, $N \geq 1$, $\alpha > 0$ and $\beta > 0$.

Equation (1.1) has been well studied in the case of $p = 2$. Indeed, with $\alpha = 1$ and $\beta = 0$, we obtain the classic Emden-Fowler equation studied by Emden [7]

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and Fowler [9]. In [8], [9] and [10], Fowler showed the existence of, and gave a classification of, global solutions. The case $\alpha > 0$, $\beta = 1$ and $q < 1$, (1.1) was studied extensively by Hulshof in [16] . The existence of positive solutions in the case $\alpha = 1$, $\beta = 0$ and $N > 2$ was considered by Aviles [1], Caffarelli, Gidas and Spruck [6], Gidas and Spruck [11], and Lions [18]. Hsu [13], [14] and Hui [15], looked at the case of $\alpha > 0$ and $\beta > 0$.

In the general case of $p > 2$, equation (1.1) was studied with $\alpha = 1$ and $\beta = 0$ by Ni and Serrin [23] and [24], M.F. Bidaut [2] and [3], Guedda and Véron [12]. When $\alpha > 0$ and $\beta = 1$, Leoni [17] studied the existence and the asymptotic behavior of global solutions.

The main feature of this paper is the presence of the convection term that influences the existence and asymptotic behavior of positive solutions of equation (1.1). More precisely, we have improved the result on the asymptotic behavior obtained by M.F. Bidaut [2] by giving equivalent explicit solutions, and their derivatives, at infinity. Our approach is based on the energy methods introduced by M.F. Bidaut [2] and [3], and the oscillation methods of Napoles [20] and [19].

We study equation (1.1) by classical methods developed by Bouzelmate, Gmira and Reyes [4], suitably modified in order to deal with its degenerate character at $r = 0$ as well as at points where $u' = 0$. This is particularly important for local existence, since we are interested in radial solutions and it is natural to impose $u'(0) = 0$. Thus, consider the following 'initial value problem'

$$(P)\begin{cases}(|u'|^{p-2}u')'(r) + \dfrac{N-1}{r}|u'|^{p-2}u'(r) + \alpha|u|^{q-1}u + \beta r(|u|^{q-1}u)' = 0, r > 0,\\ u(0) = a > 0, \quad u'(0) = 0,\end{cases}$$

where $p > 2$, $q > 1$, $N \geq 1$, $\alpha > 0$ and $\beta > 0$.

Existence and uniqueness of solutions of problem (P) has was showed by Bouzelmate and El Hathout in [5]. By reducing the initial value problem (P) to a fixed point problem for a suitable integral operator, we prove that for each $a > 0$, there exists a unique global solution $u(., a, \alpha, \beta)$ of problem (P) such that

$$(|u^{'}|^{p-2}u^{'})^{'}(0) = \frac{-\alpha a^q}{N}. \tag{1.2}$$

The behaviou of equation (1.1) depends strongly of the sign of $N\beta - \alpha$ and the comparison of q with the two critical values $\frac{N(p-1)}{N-p}$ and $\frac{N(p-1)+p}{N-p}$. The case $N\beta - \alpha \geq 0$ was studied in [5]. We now focus on the case $N\beta - \alpha < 0$.

The main theorems are as follows; they will be proved in Sections $3, 4$, and 5.

Theorem 1. [Existence of non-positive solutions]
Assume that $\frac{\alpha}{\beta} > N$. Let u be a solution of problem (P), then u changes sign in the following cases:
(i) $q < p - 1$.

(ii) $N \leq p$.
(iii) $N > p$ and $p-1 \leq q \leq \frac{N(p-1)}{N-p}$.
(iv) $\frac{N(p-1)}{N-p} < q < \frac{N(p-1)+p}{N-p}$ and $\frac{\alpha}{q\beta} > \frac{N(N-p)}{(p-1)[N(p-1)+p-q(N-p)]}$.

Theorem 2. [Existence of positive solutions]
Assume that $\frac{\alpha}{\beta} > N$. Let u be a solution of problem (P). Then u is strictly positive in the following cases:
(i) $\frac{N-p}{p-1} \geq \frac{\alpha}{q\beta}$.
(ii) $0 < \frac{N-p}{p-1} < \frac{\alpha}{q\beta}$ and $q \geq \frac{N(p-1)+p}{N-p}$.

Theorem 3. [Behavior of positive solutions near infinity]
Assume that $\frac{\alpha}{\beta} > N > p$ and $q \geq \frac{N(p-1)+p}{N-p}$. Let u be a solution of problem (P). Then

$$\lim_{r\to\infty} r^{\frac{p}{q+1-p}} u(r) = L \tag{1.3}$$

and

$$\lim_{r\to\infty} r^{\frac{p}{q+1-p}+1} u'(r) = \frac{-p}{q+1-p}\, L \tag{1.4}$$

where

$$L = \left(\frac{(p-1)\left(\frac{N-p}{p-1} - \frac{p}{q+1-p}\right)\left(\frac{p}{q+1-p}\right)^{p-1}}{\alpha - q\beta\frac{p}{q+1-p}} \right)^{\frac{1}{q+1-p}}. \tag{1.5}$$

Our paper is organized as follows. In Section 2, we present some basic tools which will be useful to prove the fundamental theorems above. Existence of non-positive solutions of problem (P) is given in Section 3. In Section 4, we prove the existence of positive solutions. In Section 5 we describe the asymptotic behavior of positive solutions. More precisely, we prove that under some assumptions, the positive solution of problem (P) behaves like $r^{\frac{-p}{q+1-p}}$ at infinity. Finally, in Section 6 we give conclusions that can be used to study future research work based on the obtained results in this paper.

2. Basic Tools

In this section we will give some basic tools which will be useful for proving the main results.

Lemma 2.1. The solution u of problem (P) is strictly decreasing as long as that it is strictly positive.
Proof. We argue by contradiction. Let $r_0 > 0$ be the first zero of u'. Since by

(1.2) $u^{'}(r) < 0$ for $r \sim 0$, there exists, by continuity and the definition of r_0, a left neighborhood $]r_0 - \varepsilon, r_0[$ (for some $\varepsilon > 0$) where u' is strictly increasing and strictly negative. That is $(|u'|^{p-2}u')'(r) > 0$ for any $r \in]r_0 - \varepsilon, r_0[$. Hence by letting $r \to r_0$ we get $(|u'|^{p-2}u')'(r_0) \geq 0$. But by equation (1.1), we have $(|u'|^{p-2}u')'(r_0) = -\alpha|u|^{q-1}u(r_0) < 0$ since $u(r_0) > 0, u'(r_0) = 0$ and $\alpha > 0$, which is a contradiction. This completes the proof. □

Proposition 2.2. Let u be a solution of problem (P). If $N > 1$ or $N = 1$ and u is strictly positive, then

$$\lim_{r\to\infty} u(r) = \lim_{r\to\infty} u'(r) = 0. \tag{2.1}$$

Proof. We distinguish two cases.

Case 1**.** $N > 1$. We define the following energy function,

$$E(r) = \frac{p-1}{p}|u'(r)|^p + \frac{\alpha}{q+1}|u(r)|^{q+1}. \tag{2.2}$$

According to equation (1.1), we get

$$E^{'}(r) = -ru'^2(r)\left[\frac{N-1}{r^2}|u'(r)|^{p-2} + q\beta|u(r)|^{q-1}\right]. \tag{2.3}$$

We show that $\lim_{r\to\infty} E(r) = 0$. Since $E'(r) \leq 0$ and $E(r) \geq 0$ for all $r > 0$, there exists a constant $l \geq 0$ such that $\lim_{r\to\infty} E(r) = l \geq 0$. If $l > 0$, then there exists $r_1 > 0$, such that

$$E(r) \geq \frac{l}{2} \qquad for \quad r \geq r_1. \tag{2.4}$$

Now consider the function

$$D(r) = E(r) + \frac{N-1}{2r}|u^{'}|^{p-2}u^{'}(r)u(r) + \frac{q\beta(N-1)}{2(q+1)}|u(r)|^{q+1}.$$

We get

$$D'(r) = -q\beta r|u(r)|^{q-1}(r)u'^2(r) - \frac{N-1}{2r}\left[|u'(r)|^p + \frac{N}{r}|u^{'}|^{p-2}u^{'}u(r) + \alpha|u(r)|^{q+1}\right].$$

Since $\beta > 0$, we have

$$D'(r) \leq -\frac{N-1}{2r}\left[|u'(r)|^p + \alpha|u(r)|^{q+1} + \frac{N}{r}|u'|^{p-2}u'u(r)\right].$$

Recalling that u and u' are bounded (because E is bounded), we have

$$\lim_{r\to\infty} \frac{|u^{'}|^{p-2}u^{'}u(r)}{r} = 0.$$

Moreover, by (2.4) we have

$$|u'(r)|^p + \alpha|u(r)|^{q+1} \geq \frac{p-1}{p}|u'(r)|^p + \frac{\alpha}{q+1}|u(r)|^{q+1} = E(r) \geq \frac{l}{2} \quad \text{for} \quad r \geq r_1.$$

Consequently, there exist two constants $c > 0$ and $r_2 \geq r_1$ such that

$$D^{'}(r) \leq -\frac{c}{r} \quad for \quad r \geq r_2.$$

Integrating this last inequality between r_2 and r, we get

$$D(r) \leq D(r_2) - c\ln(\frac{r}{r_2}) \quad for \quad r \geq r_2.$$

In particular we obtain $\lim_{r\to\infty} D(r) = -\infty$. Since

$$E(r) + \frac{N-1}{2r}|u'|^{p-2}u'(r)u(r) \leq D(r),$$

we get $\lim_{r\to\infty} E(r) = -\infty$. This is impossible. So $\lim_{r\to\infty} E(r) = 0$, giving the conclusion.

Case 2. $N = 1$ and u is strictly positive. Let

$$\phi(r) = |u'|^{p-2}u'(r) + \beta r|u|^{q-1}u(r). \tag{2.5}$$

By equation (1.1),

$$\phi'(r) = (\beta - \alpha)|u|^{q-1}u(r). \tag{2.6}$$

Since u is strictly positive, it is strictly decreasing. Therefore $\lim_{r\to\infty} u(r) \in [0, \infty[$. Suppose that $\lim_{r\to\infty} u(r) = L > 0$. Since the energy function E given by (2.2) converges, then necessarily, $\lim_{r\to\infty} u'(r) = 0$. Therefore $\lim_{r\to\infty} \phi(r) = \infty$. Using L'Hopital's rule, we have

$$\lim_{r\to\infty} \phi'(r) = \lim_{r\to\infty} \frac{\phi(r)}{r}.$$

That is

$$(\beta - \alpha)L^q = \beta\, L^q.$$

Therefore, $-\alpha L^q = 0$, contradicting $L > 0$. Hence, $\lim_{r\to\infty} u(r) = 0$. □

Lemma 2.3. Assume that $\frac{\alpha}{\beta} > N$. Let u be a strictly positive solution of (P).
(i) If $N > p$, there exists a constant $C_1 > 0$ such that

$$u(r) \geq C_1 r^{-\frac{N-p}{p-1}} \quad \text{for large } r. \tag{2.7}$$

(ii) If $q > p-1$, there exists a constant $C_2 > 0$ such that

$$u(r) \leq C_2 r^{\frac{-p}{q+1-p}} \quad \text{for large } r. \tag{2.8}$$

Proof.(i) We introduce the following function

$$\varphi(r) = r^{N-1}|u'|^{p-2}u'(r) + \beta r^N u^q(r). \tag{2.9}$$

Then by (1.1), we get

$$\varphi'(r) = (\beta N - \alpha) r^{N-1} u^q(r). \tag{2.10}$$

Since, $u > 0$ and $N\beta < \alpha$, then $\varphi'(r) < 0\ \ \forall r > 0$ and as $\varphi(0) = 0$, we get $\varphi(r) < 0$, $\forall r > 0$. Then $\lim_{r\to\infty} \varphi(r) \in [-\infty, 0[$. Therefore there exists $C > 0$, such that $\varphi(r) < -C$ for large r. This gives

$$r^{N-1}|u^{'}|^{p-2}u^{'}(r) < -C \quad \text{for large } r. \tag{2.11}$$

Consequently

$$u'(r) < -C^{\frac{1}{p-1}}\, r^{\frac{1-N}{p-1}} \quad \text{for large } r. \tag{2.12}$$

Integrating this last inequality on (r, R) for large r and using the fact that $N > p$ and $\lim_{r\to\infty} u(r) = 0$, we deduce by letting $R \to \infty$, that there exists a constant $C_1 > 0$ satisfying (2.7).

(ii) Using the fact that $\varphi(r) < 0$ and $u'(r) < 0$, $\forall r > 0$, we obtain

$$u'(r) u^{\frac{-q}{p-1}} \leq -\beta^{\frac{1}{p-1}} r^{\frac{1}{p-1}}. \tag{2.13}$$

Integrating this last inequality on $(0, r)$ and taking into account $q > p-1$, we deduce that there exists a constant $C_2 > 0$ satisfying (2.8). □

Now for any $c > 0$, define the function

$$E_c(r) = cu(r) + ru'(r), \quad r > 0. \tag{2.14}$$

It is clear that

$$(r^c u(r))' = r^{c-1} E_c(r), \quad r > 0. \tag{2.15}$$

Hence, using (1.1), we have for any $r > 0$ such that $u'(r) \neq 0$,

$$\begin{aligned}(p-1)|u'|^{p-2}(r)E_c'(r) &= (p-1)(c - \frac{N-p}{p-1})|u'|^{p-2}u'(r) - \alpha r|u|^{q-1}u \\ &\quad - q\beta r^2|u|^{q-1}u'(r) \\ &= (p-1)(c - \frac{N-p}{p-1})|u'|^{p-2}u'(r) - q\beta r|u|^{q-1}E_{\frac{\alpha}{q\beta}}(r).\end{aligned} \tag{2.16}$$

Consequently, if $E_c(r_0) = 0$ for some $r_0 > 0$, equation (1.1) gives

$$(2.17)\quad \begin{aligned}(p-1)|u'|^{p-2}(r_0)E_c'(r_0) = r_0|u|^{q-1}u(r_0)\Bigg[&(q\beta c-\alpha)+ \\ &(p-1)c^{p-1}\left(\frac{N-p}{p-1}-c\right)\frac{|u|^{p-q-1}(r_0)}{r_0^p}\Bigg].\end{aligned}$$

From which the sign of $E_c(r)$ for large r can be obtained.

Lemma 2.4. Assume that $\frac{\alpha}{\beta} > N$. Let u be a strictly positive solution of (P).
(i) If $\frac{N-p}{p-1} \geq \frac{\alpha}{q\beta}$, $E_{\frac{\alpha}{q\beta}}(r) > 0$ for any $r > 0$.
(ii) If $0 < \frac{N-p}{p-1} < \frac{\alpha}{q\beta}$, $E_{\frac{N-p}{p-1}} > 0$ for any $r > 0$.

Proof.
(i) We distinguish two cases.
Case 1. $\frac{N-p}{p-1} > \frac{\alpha}{q\beta}$. We have $E_{\frac{\alpha}{q\beta}}(0) = \frac{\alpha}{q\beta}u(0) > 0$. Let $r_0 > 0$ be the first zero of $E_{\frac{\alpha}{q\beta}}(r)$. Therefore $E_{\frac{\alpha}{q\beta}}(r) > 0$ in $[0, r_0[$, $E_{\frac{\alpha}{q\beta}}(r_0) = 0$ and $E'_{\frac{\alpha}{q\beta}}(r_0) \leq 0$. But using the fact that $u(r_0) > 0$ and $\frac{N-p}{p-1} > \frac{\alpha}{q\beta}$, we have by (2.17), $E'_{\frac{\alpha}{q\beta}}(r_0) > 0$, which is a contradiction.

Case 2. $\frac{N-p}{p-1} = \frac{\alpha}{q\beta}$. By (2.16), we have

$$(2.18)\qquad (p-1)|u'|^{p-2}E'_{\frac{\alpha}{q\beta}}(r) = -q\beta r|u|^{q-1}E_{\frac{\alpha}{q\beta}}(r).$$

Let $r_0 > 0$. We introduce the following function

$$(2.19)\qquad f(r) = \frac{q\beta}{p-1}\int_{r_0}^{r} s|u'(s)|^{2-p}|u(s)|^{q-1}ds.$$

By (2.18), we obtain

$$(2.20)\qquad E'_{\frac{\alpha}{q\beta}}(r) + f'(r)E_{\frac{\alpha}{q\beta}}(r) = 0.$$

Hence,

$$(2.21)\qquad \left(e^{f(r)}E_{\frac{\alpha}{q\beta}}(r)\right)' = 0.$$

Integrating this last equality from r_0 to r, we obtain

$$(2.22)\qquad E_{\frac{\alpha}{q\beta}}(r) = E_{\frac{\alpha}{q\beta}}(r_0)e^{-f(r)} \qquad \forall r > r_0.$$

Since $E_{\frac{\alpha}{q\beta}}(r_0) > 0$ for any $r_0 > 0$ close to 0, then $E_{\frac{\alpha}{q\beta}}(r) > 0$ for any $r > 0$.
(ii) We will show the result in two steps.

Step 1. $E_{\frac{\alpha}{q\beta}}(r) \neq 0$ for large r. Assume that there exists a large r_0 such that $E_{\frac{\alpha}{q\beta}}(r_0) = 0$. Using the fact that $u > 0$ and $\frac{N-p}{p-1} < \frac{\alpha}{q\beta}$, we get from (2.17), $E'_{\frac{\alpha}{q\beta}}(r_0) < 0$ and thereby $E_{\frac{\alpha}{q\beta}}(r) \neq 0$ for large r.

Step 2. $E_{\frac{N-p}{p-1}}(r) > 0 \;\; \forall r > 0$. We have $E_{\frac{N-p}{p-1}}(0) > 0$. Let $r_0 > 0$ be the first zero of $E_{\frac{N-p}{p-1}}$. Then by (2.17), $E'_{\frac{N-p}{p-1}}(r_0) < 0$. Therefore $E_{\frac{N-p}{p-1}}(r) < 0$ $\forall r > r_0$. On the other hand by Lemma 2.3, we get $r^{\frac{\alpha}{q\beta}}u(r) \geq C_1 r^{\frac{\alpha}{q\beta}-\frac{N-p}{p-1}}$, hence $\lim_{r\to\infty} r^{\frac{\alpha}{q\beta}}u(r) = \infty$. Since $E_{\frac{\alpha}{q\beta}}(r) \neq 0$ for large r by step 1 , then necessarily $E_{\frac{\alpha}{q\beta}}(r) > 0$ for large r. Moreover, by (2.16), we have $E'_{\frac{N-p}{p-1}}(r) < 0$ for large r. As $E_{\frac{N-p}{p-1}}(r) < 0 \; \forall r > r_0$ we have $\lim_{r\to\infty} E_{\frac{N-p}{p-1}}(r) \in [-\infty, 0[$, which implies that $\lim_{r\to\infty} ru'(r) \in [-\infty, 0[$ (because $\lim_{r\to\infty} u(r) = 0$), but this is impossible since u is positive and bounded. Then $E_{\frac{N-p}{p-1}}(r) > 0 \;\; \forall r > 0$. This completes the proof of Lemma. □

3. Proof of Theorem 1.

Proof. Assume that u is strictly positive. We distinguish seven cases:

Case 1. $q < p - 1$. Since $\varphi(r) < 0$ and $u'(r) < 0$, for any $r > 0$, we have estimate (2.13), and therefore

$$\left(\frac{u^{(p-1-q)/(p-1)}}{\frac{p-1-q}{p-1}}\right)' < -\beta^{1/(p-1)}\left(\frac{r^{p/(p-1)}}{\frac{p}{p-1}}\right)'. \tag{3.1}$$

Integrating this last inequality twice from 0 to r and letting $r \to \infty$, we obtain $\lim_{r\to\infty} u^{(p-1-q)/(p-1)} = -\infty$, which contradicts the fact that u is strictly positive.

Case 2. $N < p$. Since $u > 0, u' < 0, \varphi(r) < 0$ and $\varphi'(r) < 0$, for any $r > 0$, we obtain (2.12). By integrating it on (r_0, r) for large r_0 and letting $r \to \infty$, we obtain $\lim_{r\to\infty} u(r) = -\infty$, which is a contradiction with the fact that u is strictly positive.

Case 3. $N = p$. Since $N = p$, inequality (2.12) is equivalent to

$$u'(r) < -C_1 r^{-1} \quad \text{for large } r. \tag{3.2}$$

Integrating (3.2) on (r_0, r) for large r_0 and letting $r \to \infty$, we obtain a contradiction with the fact that u is strictly positive.

Case 4. $N > p$ and $p - 1 < q < \frac{N(p-1)}{N-p}$. In this case we have, $0 < \frac{N-p}{p-1} < \frac{p}{q+1-p}$.

By Lemma 2.3, we obtain

$$C_1 r^{\frac{p}{q+1-p}-\frac{N-p}{p-1}} \le C_2 \quad \text{for large } r. \tag{3.3}$$

Letting $r \to \infty$ in this last inequality we obtain a contradiction with the fact that

$$\frac{N-p}{p-1} < \frac{p}{q+1-p}.$$

Case 5. $N > p$ and $q = p-1$. By (2.13), we obtain

$$\frac{u'(r)}{u(r)} < -\beta^{\frac{1}{p-1}} r^{\frac{1}{p-1}}. \tag{3.4}$$

Integrating in $(0, r)$ for $r > 0$, we get

$$u(r) < u(0) e^{-\frac{p-1}{p}\beta^{\frac{1}{p-1}} r^{\frac{p}{p-1}}}. \tag{3.5}$$

Then, $\lim_{r\to\infty} r^{\frac{N-p}{p-1}} u(r) = 0$. But this contradicts (2.7).

Case 6. $N > p$ and $q = \frac{N(p-1)}{N-p}$. Then $\frac{N-p}{p-1} = \frac{p}{q+1-p}$. We have by (2.7), $u(r) \ge C_1 r^{\frac{p-N}{p-1}} = C_1 r^{\frac{-p}{q+1-p}}$ for large r. Since, $N\beta - \alpha < 0$, we have by (2.10),

$$\varphi'(r) \le (\beta N - \alpha) C_1^q r^{N-1-\frac{pq}{q+1-p}}, \tag{3.6}$$

Since $N = \frac{pq}{q+1-p}$ (*because* $\frac{N-p}{p-1} = \frac{p}{q+1-p}$), then by (3.6)

$$\varphi'(r) \le (\beta N - \alpha) C_1^q r^{-1} \qquad \text{for large } r. \tag{3.7}$$

Integrating (3.7), we get $\lim_{r\to\infty} \varphi(r) = -\infty$, which implies by (2.9) that $\lim_{r\to\infty} r^{\frac{N-1}{p-1}} u'(r) = -\infty$. Using Hopital's rule we obtain, $\lim_{r\to\infty} r^{\frac{N-p}{p-1}} u(r) = \infty$. But this contradicts the fact that $u(r) \le C_2 r^{\frac{-p}{q+1-p}} = C_2 r^{\frac{p-N}{p-1}}$.

Case 7. $\frac{N(p-1)}{N-p} < q < \frac{N(p-1)+p}{N-p}$ and $\frac{\alpha}{q\beta} > \frac{N(N-p)}{(p-1)[N(p-1)+p-q(N-p)]}$. Using the Pohozaev identity, we put

$$G(r) = r^N \left(\frac{p-1}{p} |u'|^p + \frac{\alpha(N-p)}{Np} u^{q+1} \right) + \frac{N-p}{p} r^{N-1} |u'|^{p-2} u' u, \quad r > 0. \tag{3.8}$$

Then

$$G'(r) = q\beta r^N u^{q-1} |u'| E_\gamma(r), \quad r > 0 \tag{3.9}$$

where $\gamma = \frac{\alpha}{q\beta} \frac{N(p-1)+p-q(N-p)}{Np} + \frac{N-p}{p}$.

As $\frac{\alpha}{q\beta} > \frac{N(N-p)}{(p-1)[N(p-1)+p-q(N-p)]}$, we have $\gamma > \frac{N-p}{p-1} > 0$, therefore by Lemma 2.4,

$E_\gamma(r) > 0$ for any $r > 0$, hence $G'(r) > 0$ for any $r > 0$. As $G(0) = 0$ we obtain $G(r) > 0$ for any $r > 0$, then $\lim_{r\to\infty} G(r) \in]0, \infty]$, which implies that there exists a constant $C > 0$ such that $G(r) \geq C$ for large r. This gives by (3.8) and the fact that u' is negative that

$$\frac{p-1}{p}|u'|^p \geq Cr^{-N} - \frac{\alpha(N-p)}{Np}u^{q+1} \quad \text{for large } r. \tag{3.10}$$

Using (2.8), we obtain

$$\frac{p-1}{p}|u'|^p \geq r^{-N}\left[C - C_2^{q+1}\frac{\alpha(N-p)}{Np}r^{N-\frac{p(q+1)}{q+1-p}}\right] \quad \text{for large } r. \tag{3.11}$$

Since $N - \frac{p(q+1)}{q+1-p} = \frac{q(N-p)-N(p-1)-p}{q+1-p} < 0$, we have $\lim_{r\to\infty} r^{N-\frac{p(q+1)}{q+1-p}} = 0$. Consequently by (3.11), there exists a constant $K > 0$ such that

$$\frac{p-1}{p}|u'|^p \geq Kr^{-N} \quad \text{for large } r.$$

This gives, since $u'(r) < 0$, that

$$u'(r) \leq -\left(\frac{Kp}{p-1}\right)^{1/p} r^{-\frac{N}{p}} \quad \text{for large } r. \tag{3.12}$$

Integrating this last inequality on (R, r) and letting $R \to \infty$ we see that there exists a constant $M > 0$ such that

$$u(r) \geq Mr^{\frac{p-N}{p}} \quad \text{for large } r. \tag{3.13}$$

This gives that $\lim_{r\to\infty} r^{\frac{p}{q+1-p}}u(r) = \infty$ since $\frac{p}{q+1-p} > \frac{N-p}{p}$ (because $q < \frac{N(p-1)+p}{N-p}$). But this contradicts the fact that $r^{\frac{p}{q+1-p}}u(r)$ is bounded by Lemma 2.3.

We deduce that in the seven cases, u is not strictly positive. Let $r_0 > 0$ be the first zero of u. Then, $u(r) > 0$, $u'(r) < 0$, for any $r \in (0, r_0)$ and $u'(r_0) \leq 0$. Suppose that $u'(r_0) = 0$, hence by (2.9), $\varphi(r_0) = 0$. Since $\varphi'(r) < 0$ $\forall r \in (0, r_0)$, then $\varphi(r_0) < \varphi(r) < \varphi(0) = 0$. A contradiction arises, consequently $u'(r_0) < 0$ and so u changes sign. This completes the proof of theorem. □

4. Proof of Theorem 2.

The proof requires the following result.

Proposition 4.1. Let u be a solution of problem (P). Assume that the first zero R of u is positive. For $0 < k < m$, we have

$$\int_0^R u^q|u'|^k s^{m-1}\,ds \leq \frac{q+k}{m-k}\int_0^R u^{q-1}|u'|^{k+1}s^m\,ds. \tag{4.1}$$

Proof. By Holder's inequality we have

$$\int_0^R u^q |u'|^k s^{m-1}\,ds \le \left(\int_0^R u^{q+k} s^{m-1-k}\,ds\right)^{\frac{1}{k+1}} \times \left(\int_0^R u^{q-1}|u'|^{k+1} s^m\,ds\right)^{\frac{k}{k+1}}. \tag{4.2}$$

On the other hand, using the fact that $u(R)=0$, we obtain

$$\int_0^R \left(u^{q+k} s^{m-1-k}\right)' s\,ds = -\int_0^R u^{q+k} s^{m-1-k}\,ds. \tag{4.3}$$

Therefore

$$\begin{aligned}(q+k)\int_0^R u' u^{q+k-1} s^{m-k}\,ds + (m-1-k)\int_0^R u^{q+k} s^{m-1-k}\,ds \\ = -\int_0^R u^{q+k} s^{m-1-k}\,ds.\end{aligned} \tag{4.4}$$

Using the fact that $u'<0$ in $(0,R)$, we get

$$\int_0^R u^{q+k} s^{m-1-k}\,ds = \frac{q+k}{m-k}\int_0^R |u'| u^{q+k-1} s^{m-k}\,ds. \tag{4.5}$$

Applying Holder's inequality again we obtain

$$\int_0^R u^{q+k} s^{m-1-k}\,ds \le \frac{q+k}{m-k}\left(\int_0^R u^{q+k} s^{m-1-k}\,ds\right)^{\frac{k}{k+1}} \times \left(\int_0^R u^{q-1}|u'|^{k+1} s^m\,ds\right)^{\frac{1}{k+1}}. \tag{4.6}$$

Therefore,

$$\left(\int_0^R u^{q+k} s^{m-1-k}\,ds\right)^{1-\frac{k}{k+1}} \le \frac{q+k}{m-k}\left(\int_0^R u^{q-1}|u'|^{k+1} s^m\,ds\right)^{\frac{1}{k+1}}. \tag{4.7}$$

Combining (4.2) and (4.7), we obtain easily the estimation (4.1). This completes the proof of proposition. □

Now, we turn to the proof of Theorem 2.

Proof. We argue by contradiction and assume that the first zero r_0 of u exists and is positive. Then, $u(r) > 0\ \forall r \in [0, r_0[$, $u'(r) < 0\ \forall r \in (0, r_0)$ and $u'(r_0) \leq 0$. We distinguish three cases:

Case 1. $\frac{N-p}{p-1} > \frac{\alpha}{q\beta}$. We have $E_{\frac{\alpha}{q\beta}}(r) > 0\ \ \forall r \in [0, r_0[$. Indeed, suppose there exists $r_1 \in]0, r_0[$ such that $E_{\frac{\alpha}{q\beta}}(r_1) = 0$ (r_1 is the first zero because $E_{\frac{\alpha}{q\beta}}(0) > 0$). Since $u(r_1) > 0$, then $u'(r_1) < 0$ and therefore $E'_{\frac{\alpha}{q\beta}}(r_1)$ exists and $E'_{\frac{\alpha}{q\beta}}(r_1) \leq 0$. On the other hand, we have by (2.17)

$$(4.8)\quad (p-1)|u'(r_1)|^{p-2}E'_{\frac{\alpha}{q\beta}}(r_1) = (p-1)\left(\frac{\alpha}{q\beta}\right)^{p-1}\left(\frac{N-p}{p-1} - \frac{\alpha}{q\beta}\right)\frac{|u|^{p-2}u(r_1)}{r_1^{p-1}}.$$

Then $E'_{\frac{\alpha}{q\beta}}(r_1) > 0$. This is a contradiction. Hence, $E_{\frac{\alpha}{q\beta}}(r) > 0\ \forall r \in [0, r_0[$. Recall (2.15), this gives $\left(r^{\frac{\alpha}{q\beta}}u(r)\right)' > 0$ in $]0, r_0[$ and consequently for $0 < r < r_0$, we have $r^{\frac{\alpha}{q\beta}}u(r) < r_0^{\frac{\alpha}{q\beta}}u(r_0) = 0\ \ \forall r \in]0, r_0[$. Which is impossible.

Case 2. $\frac{N-p}{p-1} = \frac{\alpha}{q\beta}$. As $u > 0$ and $u' < 0$ on $(0, r_0)$, then $E'_{\frac{\alpha}{q\beta}}(r)$ exists in $(0, r_0)$ and we have

$$(4.9)\qquad (p-1)|u'|^{p-2}E'_{\frac{\alpha}{q\beta}}(r) = -q\beta r u^{q-1}E_{\frac{\alpha}{q\beta}}(r) \quad \forall r \in (0, r_0).$$

Since $E_{\frac{\alpha}{q\beta}}(0) > 0$, then for $r_1 > 0$ near to 0, we have $E_{\frac{\alpha}{q\beta}}(r_1) > 0$ and we obtain by (4.9)

$$(4.10)\quad E_{\frac{\alpha}{q\beta}}(r) = E_{\frac{\alpha}{q\beta}}(r_1)e^{-\frac{q\beta}{p-1}\int_{r_1}^{r} s|u'(s)|^{2-p}u^{q-1}(s)\,ds}, \quad \forall r \in (r_1, r_0).$$

Therefore $E_{\frac{\alpha}{q\beta}}(r) > 0$ for all $r \in (r_1, r_0)$, which is equivalent to $\left(r^{\frac{\alpha}{q\beta}}u(r)\right)' > 0$ on (r_1, r_0), but this is impossible since $u(r_0) = 0$.

Case 3. $0 < \frac{N-p}{p-1} < \frac{\alpha}{q\beta}$ and $q \geq \frac{N(p-1)+p}{N-p}$. Since $u > 0$ and $u' < 0$ on $(0, r_0)$, then for any $r \in (0, r_0)$,

$$(4.11)\quad \left(r^N\left(\frac{p-1}{p}|u'|^p + \frac{\alpha}{q+1}u^{q+1}\right) + \frac{N}{q+1}r^{N-1}|u'|^{p-2}u'u\right)' = \left(\frac{N}{q+1} - \frac{N-p}{p}\right)r^{N-1}|u'|^p + \frac{q\beta N}{q+1}r^N u^q|u'| - q\beta r^{N+1}u^{q-1}u'^2.$$

Integrating this last inequality on $(0,r)$ for $0<r<r_0$, we obtain

$$r^N\left(\frac{p-1}{p}|u'|^p+\frac{\alpha}{q+1}u^{q+1}\right)+\frac{N}{q+1}r^{N-1}|u'|^{p-2}u'u= \tag{4.12}$$
$$\left(\frac{N}{q+1}-\frac{N-p}{p}\right)\int_0^r s^{N-1}|u'|^p ds+\frac{q\beta N}{q+1}\int_0^r s^N u^q|u'(s)|\,ds-q\beta\int_0^r s^{N+1}u^{q-1}u'^2(s)\,ds.$$

Since $u(r_0)=0$, then by Proposition 4.1, we have

$$\int_0^{r_0} s^N u^q|u'(s)|\,ds\le\frac{q+1}{N}\int_0^r s^{N+1}u^{q-1}u'^2(s)\,ds. \tag{4.13}$$

Letting $r\to r_0$ in (4.12), we obtain

$$\frac{p-1}{p}r_0^N|u'(r_0)|^p=\left(\frac{N}{q+1}-\frac{N-p}{p}\right)\int_0^{r_0} s^{N-1}|u'|^p\,ds+ \tag{4.14}$$
$$\frac{q\beta N}{q+1}\int_0^{r_0} s^N u^q|u'(s)|\,ds-q\beta\int_0^{r_0} s^{N+1}u^{q-1}u'^2(s)\,ds.$$

Then by (4.13) we have

$$\frac{p-1}{p}r_0^N|u'(r_0)|^p\le\left(\frac{N}{q+1}-\frac{N-p}{p}\right)\int_0^{r_0} s^{N-1}|u'|^p ds. \tag{4.15}$$

As $q\ge\frac{N(p-1)+p}{N-p}$, then $\frac{N}{q+1}\le\frac{N-p}{p}$, hence $u'(r_0)=0$ and so $\varphi(r_0)=0$, where φ is defined by (2.9). But $\varphi'(r)<0$ for any $r\in(0,r_0)$, this gives $\varphi(r_0)<\varphi(r)<\varphi(0)=0$, which is a contradiction.
In conclusion, u is strictly positive. The proof is complete. □

5. Proof of Theorem 3.

We need this classic result of Gidas and Spruck [11]; we recall its proof.

Lemma 5.1. Let W be a positive differentiable function satisfying
(i) $\int_{t_0}^{\infty} W(t)\,dt<\infty$ for large t_0.

(ii) $W'(t)$ is bounded for large t.

Then $\lim_{t\to\infty} W(t)=0$.

Proof. Suppose that $\lim_{t\to\infty} W(t)\ne 0$. Then, there exist $\varepsilon>0$ and a sequence $t_j\to\infty$ satisfying $W(t_j)\ge 2\varepsilon$. Since $W'(t)$ is bounded for large t, then there exists a constant $K>0$ such that $|W'(t)|\le K$ for large t. Then, $W(t)>\varepsilon$ for $|t-t_j|<\frac{\varepsilon}{K}$. Now we give a subsequence t'_j such that $t'_0>t_0$ and $t'_j>t'_{j-1}+\frac{2\varepsilon}{K}\,t'_0$ for $j>1$. Then,

$$\sum_{j=1}^N\int_{t'_{j-1}}^{t'_j} W(t)\,dt>\sum_{j=1}^N\int_{t'_{j-1}}^{t'_{j-1}+\frac{\varepsilon}{K}} W(t)\,dt>\frac{\varepsilon^2}{K}N\to\infty\ \ as\ \ N\to\infty. \tag{5.1}$$

Which implies,

$$\int_{t_0}^{\infty} W(t)\,dt = \infty. \tag{5.2}$$

This contradiction completes the proof. □

Now, we turn to the proof of Theorem 3.

Proof. Observe that since $\frac{\alpha}{\beta} > N > p$ and $q \geq \frac{N(p-1)+p}{N-p}$, then $\frac{N-p}{p-1} > \frac{p}{q+1-p}$ and $\frac{\alpha}{q\beta} > \frac{p}{q+1-p}$.

We consider the following logarithmic change

$$v(t) = r^{\frac{p}{q+1-p}} u(r) \text{ where } r > 0 \text{ and } t = \ln(r). \tag{5.3}$$

Since u is strictly positive, by (1.1) the function v satisfies

$$w'(t) + (p-1)\Big(\frac{N-p}{p-1} - \frac{p}{q+1-p}\Big)w(t) + \alpha v^q(t) + q\beta v^{q-1}(t)h(t) = 0, \tag{5.4}$$

where

$$w(t) = |h|^{p-2}h(t) \tag{5.5}$$

and

$$h(t) = v'(t) - \frac{p}{q+1-p}\,v(t) = r^{\frac{p}{q+1-p}+1}u'(r). \tag{5.6}$$

Define now the following energy function associated with equation (5.4).

$$F_1(t) = \frac{p-1}{p}\,|h(t)|^p + \frac{p}{q+1-p}w(t)v(t) + \frac{1}{q+1}\left(\alpha - \frac{q\beta p}{q+1-p}\right)v^{q+1}(t) - \frac{A}{p}\left(\frac{p}{q+1-p}\right)^{p-1} v^p(t), \tag{5.7}$$

where

$$A = \frac{q(N-p) - (N(p-1)+p)}{q+1-p}. \tag{5.8}$$

The proof of theorem will be done in three steps.

Step 1. The function $F_1(t)$ is converges when $t \to \infty$. By direct computation,

$$F_1'(t) = -AX(t) - q\beta v^{q-1}(t)\left(h(t) + \frac{p}{q+1-p}v(t)\right)^2. \tag{5.9}$$

Where

$$X(t)=\left(|h(t)|^{p-1}-\left(\frac{p}{q+1-p}\right)^{p-1}v^{p-1}(t)\right)\times \left(|h(t)|-\frac{p}{q+1-p}v(t)\right). \tag{5.10}$$

Since the function $s\to s^{p-1}$ is increasing, then $X(t)\geq 0$, moreover $A\geq 0$ (because $q\geq \frac{N(p-1)+p}{N-p}$), then $F_1'(t)\leq 0$ $\forall t\in(-\infty,\infty)$. This implies that $F_1(t)\leq 0$ $\forall t\in(-\infty,\infty)$. Indeed, we have

$$\lim_{r\to 0} r^{\frac{p}{q+1-p}}u(r)=0 \quad \text{and} \quad \lim_{r\to 0} r^{\frac{p}{q+1-p}+1}u'(r)=0. \tag{5.11}$$

This is equivalent to

$$\lim_{t\to-\infty} v(t)=\lim_{t\to-\infty} h(t)=0. \tag{5.12}$$

Therefore

$$\lim_{t\to-\infty} F_1(t)=0. \tag{5.13}$$

Since F_1 is decreasing, we have $F_1(t)\leq 0$ for any $t\in(-\infty,\infty)$. On the other hand, by (2.8), we have that v is bounded, and as $E_\sigma(r)>0$ $\forall r>0$ for $\sigma=\min\left(\frac{\alpha}{q\beta},\frac{N-p}{p-1}\right)$ by Lemma 2.4, we get for $t\in(-\infty,\infty)$,

$$0<|h(t)|<\sigma v(t). \tag{5.14}$$

Thus $h(t)$ is bounded for large t, and so $F_1(t)$ is bounded. We deduce that $F_1(t)$ converges and $\lim_{t\to\infty}F_1(t)=L_1<0$.

Step 2. $\lim_{t\to\infty}v'(t)=0$. First we show that $\liminf_{t\to\infty}v(t)>0$. Suppose, towards contradiction, that $\liminf_{t\to\infty}v(t)=0$. There exists a sequence $s_i\to\infty$ such that s_i is local minima of v and $\lim_{i\to\infty}v(s_i)=0$. By (5.14) $\lim_{i\to\infty}h(s_i)=0$ and so $\lim_{i\to\infty}F_1(s_i)=0$. But this contradicts the fact that F_1 is negative and decreasing and $\lim_{t\to\infty}F_1(t)=L_1<0$. We deduce that $\liminf_{t\to\infty}v(t)>0$ and so that there exists $C_3>0$ such that $v(t)>C_3$ for large t. On the other hand by inequality (2.13), we have

$$|u'(r)|^{p-1}>\beta r u^q(r) \qquad \forall\, r>0. \tag{5.15}$$

This gives,

$$|w(t)|>\beta v^q(t) \qquad \forall t\in(-\infty,\infty), \tag{5.16}$$

consequently,

$$|h(t)| > C \quad \text{for large } t, \quad \text{where } C = \beta^{\frac{1}{p-1}} C_3^{\frac{q}{p-1}}. \tag{5.17}$$

Now we distinguish two cases.

Case 1. $A > 0$. We show that $\lim_{t\to\infty} X(t) = 0$. We will apply the idea of the Lemma 5.1. For this, we write $X(t)$ in the following form:

$$X(t) = |h(t)|^p + \frac{p}{q+1-p} v\,w + \left(\frac{p}{q+1-p}\right)^{p-1} v^{p-1}\,h + \left(\frac{p}{q+1-p}\right)^{p} v^p. \tag{5.18}$$

From whence

$$\begin{aligned} X'(t) &= \frac{p}{p-1} h\,w' + \frac{p}{q+1-p} v'\,w + \frac{p}{q+1-p} v\,w' + \\ &(p-1)\left(\frac{p}{q+1-p}\right)^{p-1} v^{p-2}\,v'\,h + p\left(\frac{p}{q+1-p}\right)^{p} v^{p-1}\,v' + \\ &\left(\frac{p}{q+1-p}\right)^{p-1} v^{p-1}\,h'. \end{aligned} \tag{5.19}$$

Since $v(t)$ is bounded for large t, by (5.14) and (5.5) we have $h(t)$ and $w(t)$ are bounded for large t, and therefore by (5.4), (5.6) and (5.17), we get $w'(t)$, $v'(t)$ and $h'(t) = \frac{1}{p-1} w'(t)|h(t)|^{2-p}$ are bounded for large t (h' exists because $u'(r) < 0$). Consequently, $X'(t)$ is bounded for large t.

Now we show that $\int\limits_{t_0}^{\infty} X(s)\,ds < \infty$. By (5.9) and (5.6), we obtain

$$AX(t) = -F_1'(t) - q\beta v^{q-1}(t) v'^2(t) \le -F_1'(t). \tag{5.20}$$

Then,

$$0 \le \int\limits_{t_0}^{t} X(s)\,ds \le \frac{-1}{A} \int\limits_{t_0}^{t} F_1'(s)\,ds = \frac{-1}{A} F_1(t) + \frac{1}{A} F_1(t_0). \tag{5.21}$$

Since F_1 converges, then $\int\limits_{t_0}^{t} X(s)\,ds$ is bounded. Moreover this integral is increasing, therefore $\lim_{t\to\infty} \int\limits_{t_0}^{t} X(s)\,ds$ exists and is finite, consequently, by Lemma 5.1, we obtain $\lim_{t\to\infty} X(t) = 0$, this yields $\lim_{t\to\infty} v'(t) = 0$.

Case 2. $A = 0$. By (5.9), we have $F_1'(t) = -q\beta v^{q-1}(t) v'^2(t)$, hence the integral $\int\limits_{t_0}^{t} v^{q-1}(s) v'^2(s)\,ds$ converges as $t \to \infty$ (because its is increasing and bounded).

On the other hand,

$$\left(v^{q-1}(t)v'^2(t)\right)' = (q-1)v^{q-2}(t)v'^3(t) + 2v'(t)v''(t)v^{q-1}(t). \tag{5.22}$$

As $v(t)$, $v'(t)$ and $v''(t) = h'(t) + \frac{p}{q+1-p}v'(t)$ are bounded for large t, we obtain $\left(v^{q-1}(t)v'^2(t)\right)'$ is bounded for large t. Therefore by Lemma 5.1, we have $\lim_{t\to\infty} v^{q-1}(t)v'^2(t) = 0$. Since $v^{q-1}(t)v'^2(t) > C_3^{q-1}v'^2(t) \geq 0$ for large t, we obtain $\lim_{t\to\infty} v'(t) = 0$.

Step 3. The function $v(t)$ converges when $t \to \infty$. Recall that v is bounded. We argue by contradiction. Suppose that there exist two sequences $s_i \to \infty$ and $k_i \to \infty$ such that $s_i < k_i < s_{i+1}$, and s_i and k_i are local minima and local maxima of v, respectively, satisfying

$$0 \leq \liminf_{t\to\infty} v(t) = \lim_{i\to\infty} v(s_i) = m < \limsup_{t\to\infty} v(t) = \lim_{i\to\infty} v(k_i) = M < \infty. \tag{5.23}$$

We have $\lim_{i\to\infty} F_1(s_i) = \phi(m)$ and $\lim_{i\to\infty} F_1(k_i) = \phi(M)$, where for $s \geq 0$,

$$\begin{aligned}\phi(s) =& \frac{1}{q+1}\left(\alpha - \frac{q\beta p}{q+1-p}\right)s^{q+1} - \\ & \frac{p-1}{p}\left(\frac{N-p}{p-1} - \frac{p}{q+1-p}\right)\left(\frac{p}{q+1-p}\right)^{p-1} s^p.\end{aligned} \tag{5.24}$$

Since $\lim_{t\to\infty} F_1(t) = L_1 < 0$, we have

$$\phi(m) = \phi(M) = L_1 < 0. \tag{5.25}$$

Therefore, there exists $\gamma \in (m, M)$ and $t_i \in (s_i, k_i)$ such that $v(t_i) = \gamma$, $\phi'(\gamma) = 0$ and $\phi(\gamma) \neq L_1$. On the other hand, since $\lim_{t\to\infty} v'(t) = 0$, then in particular $\lim_{i\to\infty} v'(t_i) = 0$, then by (5.6), we get $\lim_{i\to\infty} h(t_i) = -\frac{p}{q+1-p}\gamma$. Therefore, $\lim_{i\to\infty} F_1(t_i) = \phi(\gamma) = L_1$; which is a contradiction. Consequently, v converges and since $\liminf_{t\to\infty} v(t) > 0$, necessarily we have $\lim_{t\to\infty} v(t) = d > 0$. Then using Step 2, (5.6) and (5.5), we obtain $\lim_{t\to\infty} h(t) = -\frac{p}{q+1-p}d$ and $\lim_{t\to\infty} w(t) = -\left(\frac{p}{q+1-p}\right)^{p-1} d^{p-1}$. Therefore by (5.4), necessarily $\lim_{t\to\infty} w'(t) = 0$ and letting $t \to \infty$ in the same equation, we obtain

$$d = \left(\frac{(p-1)\left(\frac{N-p}{p-1} - \frac{p}{q+1-p}\right)\left(\frac{p}{q+1-p}\right)^{p-1}}{\alpha - q\beta\frac{p}{q+1-p}}\right)^{\frac{1}{q+1-p}}.$$

Which ends the proof of theorem. □

6. Conclusion

By adding a convection term to Emden-Fowler's equation, we obtained in the case $\frac{\alpha}{\beta} > N$ an improved result concerning the asymptotic behavior of solutions and their derivatives. We showed that under certain conditions, the problem (P) admits solutions which change sign, and under other conditions admits strictly positive solutions. For the latter, we gave explicit desriptions of the asymptotic behavior. Our study of the problem (P) is for the case that $\alpha > 0$ and $\beta > 0$. The cases where $\alpha \leq 0$ or $\beta \leq 0$ are not yet studied, they will be the subject of a future research work.

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