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Strong Convergence of a Bregman Projection Method for the Solution of Pseudomonotone Equilibrium Problems in Banach Spaces

Olawale Kazeem Oyewole*
The Technion – Israel Institute of Technology, 32000 Haifa, Israel
e-mail: oyewoleolawalekazeem@gmail.com, oyewoleok@campus.technion.ac.il

Lateef Olakunle Jolaoso
School of Mathematics, University of Southampton, SO171 BJ, United Kingdom
Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences, P. O. Box 94 Medunsa 0204, Ga-Rankuwa, South Africa
e-mail : jollatanu@yahoo.co.uk

Kazeem Olalekan Aremu
Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences, P. O. Box 94 Medunsa 0204, Ga-Rankuwa, South Africa
School of Mathematics, Usmanu Danfodiyo University Sokoto, P. M. B. 2346, Sokoto, Sokoto State, Nigeria
e-mail : aremu.kazeem@udusok.edu.ng, aremukazeemolalekan@gmail.com

Abstract. In this paper, we introduce an inertial self-adaptive projection method using Bregman distance techniques for solving pseudomonotone equilibrium problems in reflexive Banach spaces. The algorithm requires only one projection onto the feasible set without any Lipschitz-like condition on the bifunction. Using this method, a strong convergence theorem is proved under some mild conditions. Furthermore, we include numerical experiments to illustrate the behaviour of the new algorithm with respect to the Bregman function and other algorithms in the literature.

1. Introduction

Let C be a nonempty, closed and convex subset of a reflexive real Banach space E with dual space E^*. Throughout this paper, we shall denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the

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norm and the duality pairing between the elements of E and E^*, respectively. The Equilibrium Problem (EP) is formulated as:

$$\text{Find} \quad x^* \in C \quad \text{such that} \quad g(x^*, y) \geq 0, \quad \forall\, y \in C, \tag{1.1}$$

where $g : C \times C \to \mathbb{R}$ is a bifunction satisfying $g(x, x) = 0$ for all $x \in C$. The EP provides a unified framework for the study of various problems arising in pure and applied sciences such as complimentarity problems, fixed point problems, optimization problems, variational inequality problems and so on [4, 15, 31]. For instance, if $g(x, y) = \langle Fx, y - x\rangle$ for all $x, y \in C$ where $F : C \to E^*$ is a mapping, then the EP becomes the Variational Inequality Problem (VIP) (see [16, 35]) which consists of finding a point $x^* \in C$ such that

$$\langle F(x^*), x^* - y\rangle \geq 0, \quad \forall\, y \in C. \tag{1.2}$$

In the study of EP, research is split into existence results and the development of iterative algorithms for approximating solutions. For more on existence results, we refer the readers to the works of [4, 15] and the references therein. Iterative schemes for approximating solutions of equilibrium problems have been studied in both finite and infinite dimensional spaces (see [3, 12, 31]). In recent years, there have been several results about approximating the solution of equilibrium problems involving pseudomonotone and strongly pseudomonotone bifunctions (see [10, 12, 14, 19, 22, 32]). We note that in most of these results, there is the requirement that the bifunction g satisfies a certain Lipschitz-type condition on the set C. A bifunction $g : C \times C \to \mathbb{R}$ is said to satisfy Lipschitz-type condition, if there exist constants $c_1, c_2 > 0$ such that for all $x, y, z \in C$

$$g(x, y) + g(y, z) \geq g(x, z) - c_1\|x - y\|^2 - c_2\|y - z\|^2. \tag{1.3}$$

The Lipschitz condition (1.3) does not hold in general, and when it does, it is not always easy to find the Lipschitz constants c_1 and c_2. This may affect the efficiency of the method (see [13, 20, 23]). In addition to this, previous methods require solving two strongly convex programming problems; this is not efficient when the feasible set or the bifunction have complex structures. To reduce some of these drawbacks, Vinh and Gibali [17] recently introduced gradient projection type algorithms for solving the EP with a pseudomonotone bifunction in real Hilbert spaces as follows:

Algorithm 1.1. *Inertial Gradient projection method (IGPM) for EP*
Initialization: *Take* $\theta_n \in [0, 1)$ *and a positive sequence* $\{\beta_n\}_{n=0}^{\infty}$ *satisfying*

$$\sum_{n=0}^{\infty} \beta_n = +\infty, \quad \sum_{n=0}^{\infty} \beta_n^2 < +\infty. \tag{1.4}$$

Select initial points $x_0,\ x_1 \in C$ *and set* $n = 1$.

Iterative step: *Given x_{n-1} and x_n $(n \geq 1)$, choose α_n such that $0 \leq \theta_n \leq \bar{\theta}_n$ where*

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\beta_n^2}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise.} \end{cases}$$

Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

Take $g(x_n) \in \partial(x_n, \cdot)(x_n)(n \geq 1)$. Calculate

$$\eta_n = \max\{1, \|g(x_n)\|\}, \quad \lambda_n = \frac{\beta_n}{\eta_n}$$

and

$$x_{n+1} = P_C(w_n - \alpha_n g(x_n)).$$

Stopping criterion *If $x_{n+1} = w_n = x_n$ for some $n \geq 1$ then stop. Otherwise set $n := n + 1$ and return to* **Iterative step**.

Although, Algorithm 1.1 does not require a Lipschitz-like condition on the pseudomonotone bifunction, its convergence requires condition (1.4) which significantly affects the convergence rate of the algorithm. Rehman et al [32] proposed an explicit algorithm which does not requires condition (1.4) but rather the Lipschitz-like condition (1.3) in real Hilbert space as follows:

Algorithm 1.2. *Inertial explicit subgradient extragradient method (IESEM)*

Initialization: *Choose $x_{-1},\ y_{-1},\ x_0,\ y_0 \in H$, α_1, $\mu \in (0, h(\theta_n))$ and let $\alpha_n \in [0, \sqrt{5} - 2)$ be a nondecreasing sequence.*

Iterative step: *Given x_{n-1}, $\{y_{n-1}\}$, $\{x_n\}$, $\{y_n\}$ and $\alpha_n(n \geq 0)$.*

Step 1 *Construct a half space*

$$C_n = \{w \in H : \langle w_n - \alpha_n v_n - y_n, w - y_n\rangle \leq 0\},$$

where $v_n \in \partial f(y_{n-1}, y_n)$ and compute

$$x_{n+1} = \arg\min\{\alpha_n f(y_n, y) + \frac{1}{2}\|w_n - y\|^2 : y \in C_n\},$$

where $w_n = x_n + \theta_n(x_n - x_{n-1})$.

Step 2 *Set*

$$\alpha_{n+1} = \min\left\{\alpha_n, \frac{\mu(\|y_{n-1} - y_n\|^2 + \|x_{n+1} - y_n\|^2)}{2[f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1})]_+}\right\}$$

and compute

$$x_{n+1} = \arg\min\{\alpha_{n+1} f(y_n, y) + \frac{1}{2}\|w_n - y\|^2 : y \in C\},$$

where $w_{n+1} = x_{n+1} + \theta_{n+1}(x_{n+1} - x_n)$.

Stopping criterion *If* $x_{n+1} = w_n$ *and* $y_n = y_{n-1}$ *for some* $n \geq 0$ *then stop. Otherwise set* $n := n + 1$ *and return to* **Step 1**.

The authors proved that the sequences $\{x_n\}$ and $\{y_n\}$ generated by Algorithm 1.2 converges weakly to a solution of the EP.

Let us also mention that the inertial extrapolation term in Algorithms 1.1 and 1.2 is used as a means of speeding up the convergence properties of the algorithms. This method was first introduced by Polyak [30] and has been adopted by many other authors, see for instance [11, 19, 22, 27, 32].

Motivated by the above results, in this paper, we are concerned with finding an iterative method which does not involve the condition (1.4) and a Lipschitz-like condition for solving pseudomonotone EP in reflexive Banach space. We introduce a new inertial self-adaptive Bregman projection method which does not require the bifunction satisfying the Lipschitz-like condition and its convergence is proved without using condition (1.4). We also use the Bregman distance techniques which generalizes the Euclidean distance popularly used by other authors.

The rest of the paper is organized as follows. In Section 2, we collect some basic definitions and preliminary results required in our main results. In Section 3, we introduce our algorithm and prove a strong convergence result for the sequence generated by the algorithm. In Section 4, we give an application of our result to variational inequality problems. We provide some numerical reports and also compare the performance of our method with other methods in the literature in Section 5. We give some concluding remarks in Section 6.

2. Preliminaries

In this section, we give some definitions and preliminary results which will be used in our convergence analysis. Let C be a nonempty, closed and convex subset of a real Banach space E with the norm $\|\cdot\|$ and dual space E^*. We denote the weak and strong convergence of a sequence $\{x_n\} \subset E$ to a point $x \in E$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively.

A function $f : E \to (-\infty, +\infty]$ is said to be

(i) proper, if $\mathrm{dom}(f) = \{x \in E : f(x) < \infty\} \neq \emptyset$;

(ii) f is strongly convex with strongly convexity constant $\rho > 0$, i.e

$$f(x) \geq f(y) - \langle \nabla f(y), x - y\rangle + \frac{\rho}{2}\|x - y\|^2, \quad \forall\ x, y \in E;$$

(iii) Gâteaux differentiable at $x \in E$, if there exists an element in E denoted by $f'(x)$ or $\nabla f(x)$ such that

$$\lim_{t\to 0} \frac{f(x + ty) - f(x)}{t} = \langle y, f'(x)\rangle, \quad \forall y \in E,$$

where $f'(x)$ or $\nabla f(x)$ is called the Gâteaux differential or gradient of f at x. We note that $\nabla f(\nabla f^*(x^*)) = x^*$ for all $x^* \in E^*$;

(iv) Fréchet differentiable at x, if the limit in (iii) above exists uniformly on the unit sphere of E.

Lemma 2.1. ([33]) *Let $f : E \to \mathbb{R}$ be uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from strong topology of E to the strong topology of E^*.*

The subdifferential set f at a point x denoted by ∂f is defined by

$$\partial f(x) := \{x^* \in E^* : f(x) - f(y) \leq \langle y - x, x^* \rangle, \quad y \ \in E\}.$$

Every element $x^* \in \partial f(x)$ is called a subgradient of f at x. If f is continuously differentiable, then $\partial f(x) = \{\nabla f(x)\}$, which is the gradient of f at x. The Fénchel conjugate of f is the convex functional $f^* : E^* \to \mathbb{R} \cup \{+\infty\}$ defined $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}$. Let E be a reflexive Banach space, the function f is said to be Legendre if and if only it satisfies the following two conditions (see [1]):

(L1) int dom$(f) \neq \emptyset$ and ∂f is single-valued on its domain ;

(L2) int dom$(f) \neq \emptyset$ and ∂f^* is single-valued on its domain.

Let f be a strictly convex and Gâteaux differentiable function. The function D_f : dom $(f)\times$int dom (f)$\to [0, \infty)$ defined by

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y)\rangle,$$

is called the Bregman distance with respect to the function f. It is worthy of mentioning that the bifunction D_f is not a metric in the usual sense because it does not satisfy the symmetry and triangle inequality properties. However, it posses the following important property called the three points identity:

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle, \tag{2.1}$$

for $x \in$ dom(f) and $y, z \in$ int dom(f). Also, from the strong convexity of f and the definition of the Bregman distance, we have that

$$D_f(x, y) \geq \frac{\rho}{2}\|x - y\|^2. \tag{2.2}$$

The Bregman distance function has been widely used by many authors in the literature (see [1, 7, 8] and the references therein).

Remark 2.2. Practical important examples of Bregman distance functions can be found in [2]. For example, if $f(x) = \frac{1}{2}\|\cdot\|$, then $D_f(x, y) = \frac{1}{2}\|x - y\|^2$ which is the Euclidean distance. Also, if $f(x) = -\sum\limits_{j=1}^{m} x_j \log(x_j)$ which is the Shannon's entropy for the non-negative orthant $\mathbb{R}^m_{++} := \{x \in \mathbb{R}^m : x_j > 0\}$, we obtain the Kullback-Leibler cross entropy defined by

$$D_f(x,y) = \sum_{j=1}^{m}\left(x_j \log\left(\frac{x_j}{y_j}\right) - 1\right) + \sum_{j=1}^{m} y_j. \tag{2.3}$$

Definition 2.3. ([5]) Let C be a nonempty, closed and convex subset of a reflexive real Banach space E. A Bregman projection of $x \in \operatorname{int} \operatorname{dom}(f)$ onto $C \subset \operatorname{int} \operatorname{dom}(f)$ is the unique vector $\Pi_C x \in C$ which satisfies

$$D_f(\Pi_C x, x) = \inf\{D_f(y,x) : y \in C\}.$$

Lemma 2.4. ([9]) *Let C be a nonempty, closed and convex subset of E and $x \in E$. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Then*

(i) $q = \Pi_C x$ if and only if $\langle \nabla f(x) - \nabla f(q), y - q\rangle \leq 0$, for all $y \in C$;

(ii) $D_f(y, \Pi_C x) + D_f(\Pi_C(x), x) \leq D_f(y,x)$, for all $y \in C$.

Definition 2.5. ([6, 9]) The bifunction $v_f : \operatorname{int} \operatorname{dom}(f) \times [0, +\infty)$ defined by

$$v_f(x,t) := \inf\{D_f(y,x) : y \in dom(f), \|y - x\| = t\}$$

is called the modulus of total convexity at x. The function f is called totally convex at $x \in \operatorname{int} \operatorname{dom}(f)$ if $v_f(x,t)$ is positive for any $t > 0$. The modulus of total convexity of f on C is the bifunction $v_f : intdom(f) \times [0, +\infty)$, defined by

$$v_f(C,t) := \inf\{v_f(x,t) : x \in C \cap intdom(f)\}.$$

he function f is called totally convex on bounded subsets if $v_f(C,t) > 0$ for any nonempty and bounded subset C and any $t > 0$. Also, f is said to be coercive, if $\lim\limits_{\|x\| \to +\infty} \left|\frac{f(x)}{\|x\|}\right| = +\infty$.

Proposition 2.6. ([6]) *If $x \in int\ dom(f)$, then the following are equivalent:*

(i) the function f is totally convex at x,

(ii) f is sequentially consistent, i.e., for any sequence $\{y_n\} \subset dom(f)$,

$$\lim_{n\to\infty} D_f(y_n, x) = 0 \Rightarrow \lim_{n\to\infty} \|y_n - x\| = 0.$$

We also recall (see [6]) that the function f is called sequentially consistent, if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded,

$$\lim_{n\to\infty} D_f(x_n, y_n) = 0 \Rightarrow \lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{2.4}$$

Proposition 2.7. ([6]) *If $dom(f)$ contains at least two points, then the function f is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Proposition 2.8. ([34]) *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $\bar{x} \in E$ and the sequence $\{D_f(x_n, \bar{x})\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 2.9. ([29]) *If $f : E \to (-\infty, +\infty]$ is a proper lower semicontinuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semicontinuous and convex function. Thus for all $y \in E$, we have*

$$D_f\left(y, \nabla f^*\left(\sum_{i=1}^{N} \lambda_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^{N} \lambda_i D_f(y, x_i)$$

where $\{x_i\}_i^N \subset E$ and $\{\lambda_i\}_i^N \subset (0,1)$ with $\sum\limits_{i=1}^{N} \lambda_i = 1$.

A Banach space is said to satisfy the Opial property [28], if for any sequence $\{x_n\} \subset E$ such that $x_n \rightharpoonup x$ for some $x \in E$, we have

$$\lim_{n\to\infty} \|x_n - x\| < \lim_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$. We note that all Hilbert spaces, all finite dimensional Banach spaces and the Banach space ℓ^p $(1 \leq p < \infty)$ satisfy the Opial property. It is also worthy of mentioning that not every Banach space satisfy the Opial property (see [18]). In order to extend this property to cover all Banach spaces, Huang et al. [25] established the following lemma.

Lemma 2.10. ([25]) *Let E be a Banach space and $f : E \to (-\infty, +\infty]$ be a proper strictly convex function that is Gâteaux differentiable and $\{x_n\}$ is a sequence in E such that $x_n \rightharpoonup x$ for some $x \in E$. Then*

$$\lim_{n\to\infty} D_f(x, x_n) < \lim_{n\to\infty} D_f(y, x_n)$$

for all $y \in domf$ with $y \neq x$.

Lemma 2.11. ([17]) *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative real sequences such that*

$$a_{n+1} \leq a_n - b_n.$$

Then, $\{a_n\}$ is bounded and $\sum\limits_{n=1}^{\infty} b_n < \infty$.

Definition 2.12. Let C be a nonempty, closed and convex subset of a Banach space E and $g : C \times C \to \mathbb{R}$ be a bifunction, g is said to be:

(i) strongly γ-monotone on C, if there exists $\gamma > 0$ such that

$$g(x,y) + g(y,x) \leq -\gamma\|x-y\|^2, \quad \forall\ x,y \in C;$$

(ii) monotone on C, if

$$g(x,y) + g(y,x) \leq 0, \quad \forall\ x,y \in C;$$

(iii) strongly γ-pseudomonotone on C, if there exists $\gamma > 0$ such that for any $x,y \in C$

$$g(x,y) \geq 0 \Rightarrow g(y,x) \leq -\gamma\|x-y\|^2;$$

(iv) pseudomonotone on C, if

$$g(x,y) \geq 0 \Rightarrow g(y,x) \leq 0, \quad \forall x,y \in C.$$

It is easy to see that $(i) \Rightarrow (ii) \Rightarrow (iv)$ and $(i) \Rightarrow (iii) \Rightarrow (iv)$ but the converse implications are not always true, see, for instance [17, 20, 23].

3. Main Result

In this section, we give a concise and precise statement of our algorithm and discuss some of its elementary properties convergence analysis. Let C be a nonempty, closed and convex subset of a reflexive Banach space E and $f : E \to \mathbb{R} \cup \{+\infty\}$ be a bounded Legendre function which is uniformly Fréchet differentiable, strongly coercive, strongly convex and totally convex on bounded subsets of E. Let $g : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

(A1) $g(x,\cdot)$ is convex and lower semicontinuous for every $x \in E$;

(A2) g is pseudomonotone on C;

(A3) $EP(g,C) \neq \emptyset$;

(A4) If $\{x_n\}_{n=0}^{\infty} \subset E$ is bounded, then the sequence $\{h(x_n) \in \partial g(x_n,\cdot)(x_n)\}_{n=0}^{\infty}$ is bounded.

(A5) For $L > 0$, we assume $h(x) \in \partial g(x,\cdot)(x)$ is L-Lipschitz continuous. However, the knowledge of L is not required in execution and practice of our method.

Remark 3.1. We note that condition (A4) is a standard assumption and holds when $g(x,\cdot)$ is bounded on bounded sets (see, for instance [9, Proposition 1.1.11]).

Next, we present our algorithm as follows:

Algorithm 3.2. *Modified Bregman Popov Extragradient Method for EP* **Initialization:** *Choose* x_0, x_1, y_0 *and* $y_1 \in C$ *and* $\alpha_1 > 0$, $\mu \in (0, \rho(\sqrt{2}-1))$, $\theta \in (0,1)$.

Iterative step: *Having* $x_{n-1}, x_n, y_{n-1}, y_n$ *and* α_n. *Calculate* x_{n+1}, y_{n+1} *and* α_{n+1} *for each* $n \geq 1$ *as follows*

(3.1)
$$\begin{cases} w_n = \nabla f^*\big(\nabla f(x_n) + \theta(\nabla f(x_{n-1}) - \nabla f(x_n)\big), \\ x_{n+1} = \Pi_{T_n} \nabla f^*\big((\nabla f(w_n) - \alpha_n h(y_n))\big), \\ \alpha_{n+1} = \begin{cases} \min\left\{\alpha_n, \dfrac{\mu\|y_n - y_{n-1}\|}{\|h(y_n) - h(y_{n-1})\|}\right\}, & \text{if } \|h(y_n) - h(y_{n-1})\| > 0, \\ \alpha_n, & \text{otherwise}, \end{cases} \\ y_{n+1} = \Pi_C \nabla f^*\big((\nabla f(x_{n+1}) - \alpha_{n+1} h(y_n))\big), \end{cases}$$

where
$$T_n = \{y \in E : \langle \nabla f(w_n) - \alpha_n h(y_{n-1}) - \nabla f(y_n), y - y_n\rangle \leq 0\}$$
and $h(x) \in \partial g(x, \cdot)(x)$ *for each* $x \in C$.

Stopping criterion *If* $x_{n+1} = w_n$ *and* $y_{n+1} = y_n = y_{n-1}$ *for some* $n \geq 1$ *then stop. Otherwise set* $n := n + 1$ *and return to* **Iterative step**.

Remark 3.3. The sequence $\{\alpha_n\}$ given in (3.1) is nonincreasing and
$$\lim_{n\to\infty} \alpha_n = \alpha \geq \min\left\{\alpha_1, \frac{\mu}{L}\right\}.$$

Proof: It follows from the definition of $\{\alpha_n\}$ that $\alpha_{n+1} \leq \alpha_n$. Thus, $\{\alpha_n\}$ is nonincreasing. Now, since $\|h(y_n) - h(y_{n-1})\| \leq L\|y_n - y_{n-1}\|$ for $L > 0$, we get that
$$\frac{\mu\|y_n - y_{n-1}\|}{\|h(y_n) - h(y_{n-1})\|} \geq \frac{\mu}{L}, \quad \text{if } \|h(y_n) - h(y_{n-1})\| > 0.$$
This together with (3.1) implies
$$\alpha_n \geq \min\left\{\alpha_1, \frac{\mu}{L}\right\}.$$
Thus, the sequence $\{\alpha_n\}$ is lower bounded. The conclusion follows. The following inequality is satisfied for $\{x_n\}$ and $\{y_n\}$.

Lemma 3.4. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be defined by Algorithm 3.2, then for every* $x^* \in EP(g, C)$ *the following inequality holds*

$$\begin{aligned} D_f(x^*, x_{n+1}) &\leq D_f(x^*, x_n) - \left(1 - \sqrt{2}\frac{\mu\alpha_n}{\rho\alpha_{n+1}}\right) D_f(x_{n+1}, y_n) \\ &\quad - \left(1 - (1+\sqrt{2})\frac{\mu\alpha_n}{\rho\alpha_{n+1}}\right) D_f(y_n, w_n) + \theta\Big(D_f(x^*, x_{n-1}) - D_f(x^*, x_n)\Big). \end{aligned} \tag{3.2}$$

Proof: Since $x^* \in \mathrm{EP}(g, C)$ and $x_{n+1} = \Pi_{T_n} \nabla f^*(\nabla f(w_n) - \alpha_n h(y_n))$, we have by Lemma 2.4 (i), that
$$\langle \nabla f(w_n) - \alpha_n h(y_n) - \nabla f(x_{n+1}), x^* - x_{n+1}\rangle \leq 0,$$

this implies

$$\langle \nabla f(w_n) - \nabla f(x_{n+1}), x^* - x_{n+1}\rangle \leq \alpha_n \langle h(y_n), x^* - x_{n+1}\rangle. \tag{3.3}$$

Using the three points identity (2.1) and subdifferential of g in the second variable in (3.3), we obtain

$$\begin{aligned}
D_f(x^*, x_{n+1}) &= D_f(x^*, w_n) - D_f(x_{n+1}, w_n) + \langle \nabla f(w_n) - \nabla f(x_{n+1}), x^* - x_{n+1}\rangle \\
&\leq D_f(x^*, w_n) - D_f(x_{n+1}, w_n) + \alpha_n \langle h(y_n), x^* - x_{n+1}\rangle \\
&= D_f(x^*, w_n) - D_f(x_{n+1}, w_n) + \alpha_n \langle h(y_n), y_n - x_{n+1}\rangle \\
&\quad + \alpha_n \langle h(y_n), x^* - y_n\rangle \qquad (3.4)\\
&\leq D_f(x^*, w_n) - D_f(x_{n+1}, w_n) + \alpha_n \langle h(y_n), y_n - x_{n+1}\rangle + \alpha_n g(y_n, x^*) \\
&= D_f(x^*, w_n) - D_f(x_{n+1}, w_n) + \alpha_n \langle h(y_{n-1}), y_n - x_{n+1}\rangle \\
&\quad + \alpha_n \langle h(y_n) - h(y_{n-1}), y_n - x_{n+1}\rangle + \alpha_n g(y_n, x^*). \qquad (3.5)
\end{aligned}$$

Note that

$$\begin{aligned}
\alpha_n \langle h(y_{n-1}), y_n - x_{n+1}\rangle &= \langle \nabla f(w_n) - \alpha_n h(y_{n-1}) - \nabla f(y_n), x_{n+1} - y_n\rangle \\
&\quad + \langle \nabla f(y_n) - \nabla f(w_n), x_{n+1} - y_n\rangle.
\end{aligned}$$

Since $x_{n+1} \in T_n$, $\langle \nabla f(w_n) - \alpha_n h(y_{n-1}) - \nabla f(y_n), x_{n+1} - y_n\rangle \leq 0$ implies that

$$\alpha_n \langle h(y_{n-1}), y_n - x_{n+1}\rangle \leq \langle \nabla f(y_n) - \nabla f(w_n), x_{n+1} - y_n\rangle. \tag{3.6}$$

Using the three points identity (2.1), we have

$$\langle \nabla f(y_n) - \nabla f(w_n), x_{n+1} - y_n\rangle = D_f(x_{n+1}, w_n) - D_f(x_{n+1}, y_n) - D_f(y_n, w_n).$$

Substituting this into (3.6), we get

$$\alpha_n \langle h(y_{n-1}), y_n - x_{n+1}\rangle \leq D_f(x_{n+1}, w_n) - D_f(x_{n+1}, y_n) - D_f(y_n, w_n). \tag{3.7}$$

From (3.7) and (3.5), we obtain that

$$\begin{aligned}
D_f(x^*, x_{n+1}) &\leq D_f(x^*, w_n) - D_f(x_{n+1}, y_n) - D_f(y_n, w_n) + \alpha_n g(y_n, x^*) \\
&\quad + \alpha_n \langle h(y_{n-1}) - h(y_n), x_{n+1} - y_n\rangle. \qquad (3.8)
\end{aligned}$$

Observe from the definition of α_n, that

$$\begin{aligned}\alpha_n\langle h(y_{n-1})-h(y_n), x_{n+1}-y_n\rangle &\leq \alpha_n\|h(y_{n-1})-h(y_n)\|\|x_{n+1}-y_n\| \\ &\leq \frac{\mu\alpha_n}{\alpha_{n+1}}\|y_{n-1}-y_n\|\|x_{n+1}-y_n\| \\ &\leq \frac{\mu\alpha_n}{\alpha_{n+1}}\left(\frac{1}{2\sqrt{2}}\|y_{n-1}-y_n\|^2+\frac{1}{\sqrt{2}}\|x_{n+1}-y_n\|^2\right) \\ &\leq \frac{\mu\alpha_n}{2\sqrt{2}\alpha_{n+1}}\left((2+\sqrt{2})\|y_n-w_n\|^2+\sqrt{2}\|w_n-y_{n-1}\|^2\right) \\ &\quad+\frac{\mu\alpha_n}{\sqrt{2}\alpha_{n+1}}\|x_{n+1}-y_n\|^2 \\ &\leq (1+\sqrt{2})\frac{\mu\alpha_n}{\rho\alpha_{n+1}}D_f(y_n,w_n)+\frac{\mu\alpha_n}{\rho\alpha_{n+1}}D_f(w_n,y_{n-1}) \\ &\quad+\sqrt{2}\frac{\mu\alpha_n}{\rho\alpha_{n+1}}D_f(x_{n+1},y_n), \end{aligned} \tag{3.9}$$

where we have used $2ab \leq \frac{1}{\sqrt{2}}a^2+\sqrt{2}b^2$ and $(a+b)^2 \leq \sqrt{2}a^2+(2+\sqrt{2})b^2$ in separate steps and the strong convexity of f.
By substituting (3.9) into (3.8), we obtain

$$\begin{aligned}D_f(x^*,x_{n+1}) \leq D_f(x^*,w_n) &- \left(1-\sqrt{2}\frac{\mu\alpha_n}{\rho\alpha_{n+1}}\right)D_f(x_{n+1},y_n) \\ &-\left(1-(1+\sqrt{2})\frac{\mu\alpha_n}{\rho\alpha_{n+1}}\right)D_f(y_n,w_n)+\alpha_n g(y_n,x^*).\end{aligned} \tag{3.10}$$

Since $x^* \in \mathrm{EP}(g,C)$ and $y_n \in C$, we have that $g(x^*,y_n) \geq 0$, it follows from the fact that g is pseudomonotone that $g(y_n,x^*) \leq 0$. Therefore, we obtain from (3.10), that

$$\begin{aligned}D_f(x^*,x_{n+1}) \quad\leq\quad & D_f(x^*,w_n)-\left(1-\sqrt{2}\frac{\mu\alpha_n}{\rho\alpha_{n+1}}\right)D_f(x_{n+1},y_n) \\ &-\left(1-(1+\sqrt{2})\frac{\mu\alpha_n}{\rho\alpha_{n+1}}\right)D_f(y_n,w_n).\end{aligned} \tag{3.11}$$

Observe from (3.1) and Lemma 2.9, that

$$\begin{aligned}D_f(x^*,w_n) &= D_f(x^*,\nabla f^*((1-\theta)\nabla f(x_n)+\theta\nabla f(x_{n-1}))) \\ &\leq (1-\theta)D_f(x^*,x_n)+\theta D_f(x^*,x_{n-1}).\end{aligned} \tag{3.12}$$

It therefore follows from (3.11) and (3.12), that

$$\begin{aligned}D_f(x^*,x_{n+1}) \leq D_f(x^*,x_n) &- \left(1-\sqrt{2}\frac{\mu\alpha_n}{\rho\alpha_{n+1}}\right)D_f(x_{n+1},y_n) \\ -\left(1-(1+\sqrt{2})\frac{\mu\alpha_n}{\rho\alpha_{n+1}}\right)&D_f(y_n,w_n)+\theta\Big(D(x^*,x_{n-1})-D_f(x^*,x_n)\Big).\end{aligned} \tag{3.13}$$

The prove is complete.
Now, we present the weak convergence theorem for Algorithm 3.2.

Theorem 3.5. *Assume that the conditions A1-A4 are satisfied and* $\theta D_f(x_n, x_{n-1}) < \infty$, *then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *given by Algorithm 3.2 converge weakly to a solution* $x^* \in EP(g, C)$.

Proof. Let $x^* \in \mathrm{EP}(g, C)$. First, we show that $\{x_n\}$ is bounded. Indeed, we have from Lemma 3.4 that (3.13) satisfies $a_{n+1} \le a_n - b_n$ where

$$a_n = D_f(x^*, x_n) + \frac{\mu\alpha_n}{\rho\alpha_{n+1}} D_f(x_n, y_{n-1}) + \theta D_f(x^*, x_{n-1})$$

and

$$b_n = \left(1 - (1+\sqrt{2})\frac{\mu\alpha_n}{\rho\alpha_{n+1}}\right) D_f(y_n, w_n) + \left(1 - (1+\sqrt{2})\frac{\mu\alpha_n}{\rho\alpha_{n+1}} + \frac{\mu\alpha_{n+1}}{\rho\alpha_{n+2}}\right) D_f(x_{n+1}, y_n). \tag{3.14}$$

Then, by Lemma 2.11, $\{a_n\}$ is bounded, which implies that $\{x_n\}$ is bounded, $\lim_{n\to\infty} D_f(x_{n+1}, y_n) = 0$ and $\lim_{n\to\infty} D_f(y_n, w_n) = 0$. Consequently, $\|y_n - w_n\| \to 0$ and $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. Again, we see from (3.1) and Lemma 2.9, that

$$\begin{aligned} D_f(x_n, w_n) &\le (1-\theta) D_f(x_n, x_n) + \theta D_f(x_n, x_{n-1}) \\ &\le \theta D_f(x_n, x_{n-1}). \end{aligned} \tag{3.15}$$

Thus, we have that $D_f(x_n, w_n) \to 0$ as $n \to \infty$, which implies by (2.4), that $\|x_n - w_n\| \to 0$ as $n \to \infty$. Consequently, one gets that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Furthermore

$$\|y_{n+1} - y_n\| \le \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - y_n\| \to 0, \quad \text{as} \quad n \to \infty.$$

Since f is uniformly Fréchet differentiable, we have that $\|\nabla f(y_{n+1}) - \nabla f(x_{n+1})\| \to 0$ as $n \to \infty$.
From the boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$. Consequently, $\{w_{n_k}\}$ and $\{y_{n_k}\}$ converge both converge weakly to $\bar{x}$. It follows easily that $\bar{x} \in C$. From the definition of y_{n+1} and Lemma 2.4 (i), we have

$$\langle \nabla f(x_{n+1}) - \alpha_n h(y_n) - \nabla f(y_{n+1}), y - y_{n+1} \rangle \le 0, \quad \forall\, y \in C, \tag{3.16}$$

that is

$$\langle \nabla f(x_{n+1}) - \nabla f(y_{n+1}), y - y_{n+1} \rangle - \alpha_n \langle h(y_n), y - y_{n+1} \rangle \le 0, \quad \forall\, y \in C$$

and

$$\left\langle \frac{\nabla f(x_{n+1}) - \nabla f(y_{n+1})}{\alpha_n}, y - y_{n+1} \right\rangle + \langle h(y_n), y_{n+1} - y_n \rangle \le \langle h(y_n), y - y_n \rangle, \quad \forall\, y \in C.$$

Hence, we have by the definition of subdifferential, that forall $y \in C$,

(3.17)
$$\left\langle \frac{\nabla f(x_{n_k+1}) - \nabla f(y_{n_k+1})}{\alpha_{n_k}}, y - y_{n_k+1} \right\rangle + \langle h(y_{n_k}), y_{n_k+1} - y_{n_k} \rangle \leq g(y_{n_k}, y).$$

Observe that $\dfrac{\nabla f(x_{n_k+1}) - \nabla f(y_{n_k+1})}{\alpha_{n_k}} \to 0$ as $k \to \infty$, since $\alpha_{n_k} \geq \alpha > 0$. Hence, by passing limit over (3.17), we obtain that $g(\bar{x}, y) \geq 0, \ \forall\, y \in C$, thus $\bar{x} \in \mathrm{EP}(g, C)$. Finally, we show that $\bar{x}$ is unique. Assume the contrary, then there exists a subsequence x_{n_i} such that $x_{n_i} \rightharpoonup \hat{x}$. Following similar arguments as above, we have that $\hat{x} \in \mathrm{EP}(g, C)$. It follows from the Bregman Opial-like property of E (Lemma 2.10), that

$$\begin{aligned} \lim_{n\to\infty} D_f(\bar{x}, x_n) &= \lim_{k\to\infty} D_f(\bar{x}, x_{n_k}) = \liminf_{k\to\infty} D_f(\bar{x}, x_{n_k}) \\ &< \liminf_{k\to\infty} D_f(\hat{x}, x_{n_k}) = \lim_{k\to\infty} D_f(\hat{x}, x_{n_k}) \\ &= \lim_{n\to\infty} D_f(\hat{x}, x_n). \end{aligned}$$

Thus, we arrive at a contradiction. Therefore is $\bar{x} = \hat{x}$. $\square$

We are now in position to establish the strong convergence of Algorithm 3.2, however we replace condition $A2$ of Assumption A by $A2^*$ that the bifunction $g : C \times C \to \mathbb{R} \cup \{+\infty\}$ is strongly pseudomonotone.

Strong convergence theorem for Algorithm 3.2.

Theorem 3.6. *Assume that conditions $A1$, $A2^*$ and $A3$-$A4$ are satisfied.The sequences $\{x_n\}$ and $\{y_n\}$ of Algorithm 3.2 converge strongly to a unique solution of EP(g, C).*

Proof: Let $x^* \in \mathrm{EP}(g, C)$. By the definition of x_{n+1} and Lemma 2.4, we have

$$\langle \nabla f(w_n) - \alpha_n h(y_n) - \nabla f(x_{n+1}), x^* - x_{n+1} \rangle \leq 0,$$

alternatively

$$\langle \nabla f(w_n) - \nabla f(x_{n+1}), x^* - x_{n+1} \rangle \leq \alpha_n \langle h(y_n), x^* - x_{n+1} \rangle.$$

By using the three points identity 2.1, we have

$$D_f(x^*, x_{n+1}) + D_f(x_{n+1}, w_n) - D_f(x^*, w_n) \leq \alpha_n \langle h(y_n), x^* - x_{n+1} \rangle,$$

that is

$$D_f(x^*, x_{n+1}) \leq \quad D_f(x^*, w_n) - D_f(x_{n+1}, w_n) + \alpha_n \langle h(y_n), x^* - x_{n+1} \rangle. \tag{3.18}$$

Since $x^* \in \mathrm{EP}(g, C)$, we have $g(x^*, x) \geq 0$ for all $x \in C$. Using the fact that g is strongly pseudomonotone, we obtain $g(x, x^*) \leq -\gamma\|x - x^*\|^2$. Taking $x = y_n \in C$, we get $g(y_n, x^*) \leq -\gamma\|y_n - x^*\|^2$. Now, using the definition of subdifferential of g at y_n, we have

$$\begin{aligned}\langle h(y_n), x^* - x_{n+1}\rangle &\leq \langle h(y_n), x^* - y_n\rangle + \langle h(y_n), y_n - x_{n+1}\rangle \\ &\leq g(y_n, x^*) + \langle h(y_n), y_n - x_{n+1}\rangle \\ &\leq -\gamma\|y_n - x^*\|^2 + \langle h(y_n), y_n - x_{n+1}\rangle. \end{aligned} \tag{3.19}$$

Substituting (3.18) into (3.19), we get

$$\begin{aligned} D_f(x^*, x_{n+1}) &\leq D_f(x^*, w_n) - D_f(x_{n+1}, w_n) - \gamma\alpha_n\|y_n - x^*\|^2 \\ &+ \alpha_n\langle h(y_n), y_n - x_{n+1}\rangle, \end{aligned} \tag{3.20}$$

again by using (2.1), we get

$$\begin{aligned} D_f(x^*, x_{n+1}) \leq\ & D_f(x^*, w_n) - D_f(w_n, y_n) - D_f(x_{n+1}, y_n) \\ & - \langle \nabla f(y_n) - \nabla f(w_n), x_{n+1} - y_n\rangle \\ & - \gamma\alpha_n\|y_n - x^*\|^2 + \alpha_n\langle h(y_n), y_n - x_{n+1}\rangle \\ =\ & D_f(x^*, w_n) - D_f(w_n, y_n) - D_f(x_{n+1}, y_n) \\ & + \langle \nabla f(w_n) - \alpha_n h(y_n) - \nabla f(y_n), x_{n+1} - y_n\rangle \\ & - \gamma\alpha_n\|y_n - x^*\|^2. \end{aligned} \tag{3.21}$$

Observe that

$$\begin{aligned} \langle \nabla f(w_n) &- \alpha_n h(y_n) - \nabla f(y_n), x_{n+1} - y_n\rangle \\ &= \langle \nabla f(w_n) - \alpha_n h(y_{n-1}) - \nabla f(y_n), x_{n+1} - y_n\rangle \\ &\quad + \alpha_n\langle h(y_{n-1}) - h(y_n), x_{n+1} - y_n\rangle, \end{aligned} \tag{3.22}$$

and $\langle \nabla f(w_n) - \alpha_n h(y_{n-1}) - \nabla f(y_n), x_{n+1} - y_n\rangle \leq 0$, since $x_{n+1} \in T_n$. Hence

$$\begin{aligned} \langle \nabla f(w_n) - \alpha_n h(y_n) - \nabla f(y_n), x_{n+1} - y_n\rangle &\leq \alpha_n\langle h(y_{n-1}) - h(y_n), x_{n+1} - y_n\rangle \\ &\leq \alpha_n\|h(y_{n-1}) - h(y_n)\|\|x_{n+1} - y_n\| \\ &\leq \frac{\mu\alpha_n}{\alpha_{n+1}}\|y_{n-1} - y_n\|\|x_{n+1} - y_n\| \\ &\leq \frac{\mu\alpha_n}{\alpha_{n+1}}\{\frac{1}{2\sqrt{2}}\|y_{n-1} - y_n\|^2 + \frac{1}{\sqrt{2}}\|x_{n+1} - y_n\|^2\} \\ &\leq \frac{\mu\alpha_n}{2\sqrt{2}\alpha_n}\Big((2+\sqrt{2})\|y_n - w_n\|^2 + \sqrt{2}\|w_n - y_{n-1}\|^2\Big) \\ &\quad + \frac{\mu\alpha_n}{\sqrt{2}\alpha_{n+1}}\|x_{n+1} - y_n\|^2 \\ &\leq (1+\sqrt{2})\frac{\mu\alpha_n}{\rho\alpha_{n+1}}D_f(y_n, w_n) + \frac{\mu\alpha_n}{\rho\alpha_{n+1}}D_f(w_n, y_{n-1}) \\ &\quad + \sqrt{2}\frac{\mu\alpha_n}{\rho\alpha_{n+1}}D_f(x_{n+1}, y_n). \end{aligned} \tag{3.23}$$

Using (3.23) in (3.21), we get

$$D_f(x^*,x_{n+1}) \leq D_f(x^*,w_n) - \left(1-\sqrt{2}\frac{\mu\alpha_n}{\alpha_{n+1}}\right)D_f(x_{n+1},y_n)$$

(3.24)

$$-\left(1-(1+\sqrt{2})\frac{\mu\alpha_n}{\alpha_{n+1}}\right)D_f(y_n,w_n)+\frac{\mu\alpha_n}{\alpha_{n+1}}D_f(w_n,y_{n-1})-\gamma\alpha_n\|y_n-x^*\|^2.$$

by using (3.12), we obtain

$$\begin{aligned} D_f(x^*,x_{n+1}) &\leq D_f(x^*,x_n)-\left(1-\sqrt{2}\frac{\mu\alpha_n}{\alpha_{n+1}}\right)D_f(x_{n+1},y_n) \\ &\quad -\left(1-(1+\sqrt{2})\frac{\mu\alpha_n}{\alpha_{n+1}}\right)D_f(y_n,w_n)+\frac{\mu\alpha_n}{\alpha_{n+1}}D_f(w_n,y_{n-1}) \\ &\quad +\theta\Big(D_f(x^*,x_{n-1})-D_f(x^*,x_n)\Big)-\gamma\alpha_n\|y_n-x^*\|^2. \end{aligned}$$

Following similar argument as in Theorem 3.5, we obtain that $\{x_n\}$ is bounded. It follows also that $\|w_n-x_n\|\to 0$, $\|y_n-x_n\|\to 0$, $\|x_{n+1}-y_n\|\to 0$ and $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$. Since f is continuous on bounded sets, coercive and uniformly Fréchet differentiable, we get by Lemma 2.1, that $\|f(x_{n+1})-f(w_n)\|\to 0$, $\|f(x_n)-f(x_{n-1})\|\to 0$ and $\|\nabla f(x_{n+1})-\nabla f(x_n)\|\to 0$ as $n\to\infty$. Therefore,

(3.25)

$$\begin{aligned} &D_f(x^*,w_n)-D_f(x^*,x_{n+1}) \\ &\quad = f(x^*)-f(w_n)-\langle\nabla f(w_n),x^*-w_n\rangle - f(x^*)+f(x_{n+1}) \\ &\quad + \langle\nabla f(x_{n+1}),x^*-x_{n+1}\rangle \\ &\quad = f(x_{n+1})-f(w_n)+\langle\nabla f(x_{n+1}),x^*-x_{n+1}\rangle-\langle\nabla f(w_n),x^*-w_n\rangle \\ &\quad = f(x_{n+1})-f(w_n)+\langle\nabla f(x_{n+1})-\nabla f(w_n),x^*-w_n\rangle \\ &\quad + \langle\nabla f(x_{n+1}),w_n-x_{n+1}\rangle. \end{aligned} \tag{3.26}$$

Thus, passing limits over (3.26), we get

$$\lim_{n\to\infty} D_f(x^*,w_n)-D_f(x^*,x_{n+1})=0.$$

Also,

$$D_f(w_n,y_{n-1})=f(w_n)-f(y_{n-1})-\langle\nabla f(y_{n-1}),w_n-y_{n-1}\rangle\to 0 \quad \text{as } n\to\infty.$$

It follows from (3.24), that

$$\gamma\alpha_n\|y_n-x^*\|^2 \leq D_f(x^*,w_n)-D_f(x^*,x_{n+1})+\frac{\mu\alpha_n}{\alpha_{n+1}}D_f(w_n,y_{n-1})\to 0.$$

Thus, since $\gamma\alpha_n > 0$, we get that

$$\lim_{n\to\infty} \|y_n - x^*\| = 0.$$

This implies $\{y_n\}$ converges strongly to x^*. Consequently, $\{x_n\}$ converges strongly to $x^* \in \mathrm{EP}(g, C)$.

4. Application

In this section, we give an application of our main result to the Market Equilibrium Model. First, we give an adaptation of our method to the Variational Inequality Problem (VIP). Suppose the function $g : C \times C \to \mathbb{R}$ in (1.1) is given by

$$g(x,y) := \begin{cases} \langle F(x), y - x\rangle \geq 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise}, \end{cases} \tag{4.1}$$

where $F : C \to E^*$, then the EP (1.1) reduces to the classical variational inequality problem (1.2). That is finding a point $x^* \in C$ such that

$$\langle F(x^*), x - x^*\rangle \geq 0, \quad \forall x \in C.$$

Denote by VIP(C,F) the solution set of VIP (1.2). Variational inequalities play an important role in studying a wide class of unilateral, obstacle and equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework (see [17, 21]) and the references therein. For this and more there have been extensive studies of this problem by several authors (see [24, 25, 26]) for more.

The mapping $F : C \to E^*$ is said to be strongly γ-pseudomonotone if there exists $\gamma > 0$ such that for any $x, y \in C$

$$\langle Fx, y - x\rangle \geq 0, \quad \Longrightarrow \langle Fy, y - x\rangle \geq \gamma\|x - y\|^2. \tag{4.2}$$

By this adaptation, Algorithm 3.2 provides a new method for variational inequalities. In fact, we have the following Popov subgradient extragradient method for VIP.

We obtain the following for solving variational inequality problem.

Algorithm 4.1. *Modified Bregman Popov Extragradient Method for VIP* **Initialization:** *Choose x_0, x_1, y_0 and $y_1 \in C$ and $\alpha_1 > 0$, $\mu \in (0, \rho(\sqrt{2} - 1))$, $\theta \in (0,1)$.*
Iterative step: *Having x_{n-1}, x_n, y_n and α_n, calculate x_{n+1}, y_{n+1} and α_{n+1} for each $n \geq 1$ as follows*

$$\begin{cases} w_n = \nabla f^*(\nabla f(x_n) + \theta(\nabla f(x_{n-1}) - \nabla f(x_n))), \\ x_{n+1} = \Pi_{T_n} \nabla f^*(\nabla f(w_n) - \alpha_n F y_n), \\ \alpha_{n+1} = \begin{cases} \min\left\{\alpha_n, \dfrac{\mu\|y_n - y_{n-1}\|}{\|h(y_n) - h(y_{n-1})\|}\right\}, & \text{if } \|h(y_n) - h(y_{n-1})\| > 0, \\ \alpha_n, & \text{otherwise}, \end{cases} \\ y_{n+1} = \Pi_C \nabla f^*(\nabla f(x_{n+1}) - \alpha_{n+1} F y_n), \end{cases}$$

where

$$T_n = \{y \in E : \langle \nabla f(x_n) - \alpha_n F(y_{n-1}) - \nabla f(y_n), y - y_n \rangle \leq 0\}.$$

Stopping criterion *If $x_{n+1} = w_n = y_n$ for some $n \geq 1$ then stop. Otherwise set $n := n + 1$ and return to* **Iterative step**.

Theorem 4.2. *Let C be a nonempty, closed and convex subset of a reflexive real Banach E with dual E^*. Assume $F : C \to E^*$ be a strongly γ-pseudomonotone operator which is bounded on bounded sets such that $VIP(C, F) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by Algorithm 4.1, then $\{x_n\}$ converges strongly to an element in $VIP(C, F)$.*

Proof. For each $x, y \in C$, let the bifunction $g : C \times C \to \mathbb{R}$ be given by (4.1). It follows by hypothesis that the assumptions (A1)-(A4) are satisfied. Following the conclusion of Theorem 3.5 that $x^* \in EP(g, C)$, we have that $x^* \in VIP(C, F)$. □

5. Numerical Examples

In this section, we report some numerical experiments to illustrate the performance of our method for some known Bregman distances. We first list some of functions with their corresponding distances.

(i) Squared Euclidean distance (SED) with $dom f = \mathbb{R}^n$,

$$f(x) = \frac{1}{2}x^T x, \qquad \nabla f(x) = x, \qquad D_f(x, y) = \frac{1}{2}\|x - y\|_2^2.$$

m

(ii) General quadratic kernel (GQK) with $dom f = \mathbb{R}^n$,

$$f(x) = \frac{1}{2}x^T Ax, \qquad \nabla f(x) = Ax, \qquad D_f(x, y) = \frac{1}{2}(x - y)^T A(x - y),$$

where

- A is symmetric positive definite;
- in some applications, A is positive semidefinite, but not positive definite.

(iii) Relative entropy (RE) with itemize $dom f = \mathbb{R}^n_+$,

$$f(x) = \sum_{i=1}^{n} x_i \log x_i, \qquad \nabla f(x) = \begin{bmatrix} \log x_1 + 1 \\ \vdots \\ \log x_i + 1 \end{bmatrix},$$

$$D_f(x, y) = \sum_{i=1}^{n} \left(x_i \log \frac{x_i}{y_i} - x_i + y_i \right).$$

$D_f(x, y)$ is called the Kullback–Leibler distance.

Table 1: Computational result for Example 5.1.

		SED	MD	KLD	ISD
$M = 10$	Iter.	17	21	21	15
	Time (sec)	0.0521	0.1353	0.0709	0.0393
$M = 30$	Iter.	16	10	22	14
	Time (sec)	0.1658	0.2733	0.2486	0.1423
$M = 50$	No of Iter.	16	17	23	14
	Time (sec)	0.3395	0.9916	0.4868	0.3484
$M = 100$	No of Iter.	20	9	24	13
	Time (sec)	0.9158	1.2521	1.4161	0.6622

(iv) Logarithmic barrier (LB) with $dom f = \mathbb{R}^n_{++}$,

$$f(x) = -\sum_{i=1}^{n} \log x_i,\ \nabla f(x) = \begin{bmatrix} -\frac{1}{x_1} \\ \vdots \\ -\frac{1}{x_n} \end{bmatrix},\ D_f(x,y) = \sum_{i=1}^{n} \left(\frac{x_i}{y_i} - \log \frac{x_i}{y_i} - 1 \right),$$

$D_f(x,y)$ in this example is called Itakura-Saito divergence.

Example 5.1. We consider the EP (1.1) with the bifunction $g : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$g(x,y) = \langle Px + Qy + q, y - x \rangle$$

where q is a vector in $\mathbb{R}^N$, P and Q are $N \times N$ matrices such that P is symmetric and positive semidefinite and $Q - P$ is negative semidefinite. The feasible set C is defined by

$$C = \{x = (x_1, x_2, \dots, x_N)^T \in \mathbb{R}^N : \|x\| \le 1 \quad \text{and} \quad x_i > 0, i = 1, 2, \dots, N\}.$$

Note that g is monotone (hence, pseudomonotone) and the unique solution of the EP is $\bar{x} = (0, 0, \dots, 0)^T$ (see [31]). The entries of the matrices P and Q are generated randomly, while q is generated randomly and uniformly distributed in $[-2, 2]$. We compare the performance of Algorithm 3.2 for various kind of the convex function f listed above. We test the algorithm for $N = 10, 30, 50, 100$, $\mu = 0.35$, $\alpha_1 = 0.24$ and the initial points x_0, x_1 are generated randomly in $\mathbb{R}^N$. The projection onto the feasible set is calculated explicitly and we study the convergence of the sequence generated by Algorithm 3.2 using $Error = \|x_{n+1} - w_n\|^2 + \|y_n - x_n\|^2 < 10^{-4}$ as stopping criterion. The numerical results are shown in Table 1 and Figure 1.

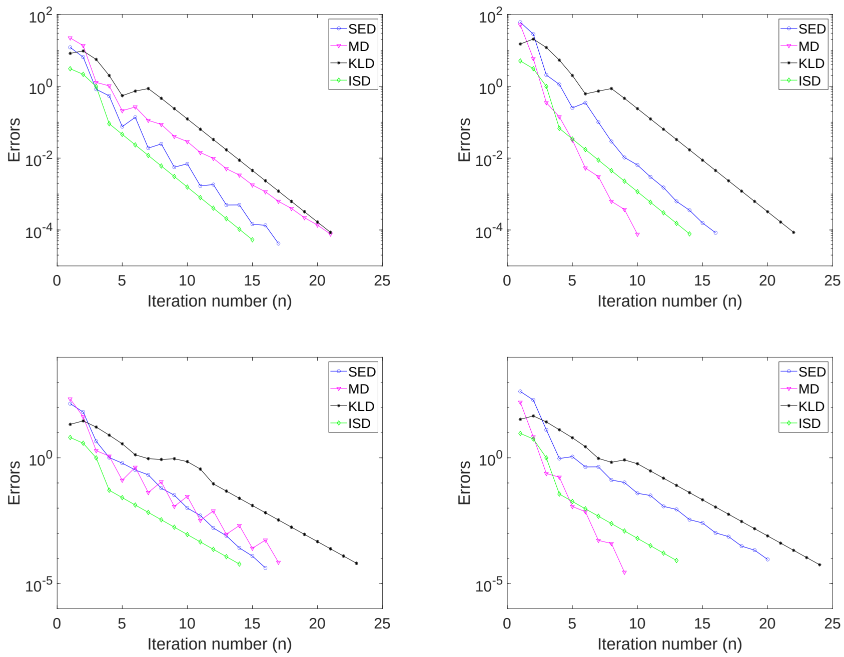

Figure 1: Example 5.1, From Top – Bottom: $M = 10, 30, 50, 100$.

Example 5.2. In this example, we consider the EP (1.1) with the bifunction $g : C \times C \to \mathbb{R}$ defined by

$$g(x, y) = \sum_{i=1}^{N} \Big[(x_i + 1 + y_i)(y_i - x_i)\Big] \quad \text{and} \quad C = \left\{ x \in \mathbb{R}_+ : \sum_{i=1}^{N} x_i = 1 \right\}.$$

We compare the performance of Algorithm 3.2 using the convex functions as given above. The initial values x_0, x_1, y_0, y_1 are generated randomly in $\mathbb{R}^N$ where $N = 10, 30, 50$ and 100. We choose $\mu = 0.63$, $\alpha_1 = 0.5$. and study the convergence of the algorithm using $Error = \|x_{n+1} - w_n\|^2 + \|y_n - x_n\|^2 < 10^{-4}$ as stopping criterion.

Table 2: Computational result for Example 5.2.

		SED	MD	KLD	ISD
$M = 10$	Iter.	22	5	25	18
	Time (sec)	0.0108	0.0479	0.3417	0.0344
$M = 30$	Iter.	23	19	67	19
	Time (sec)	0.0132	0.4386	0.0646	0.0196
$M = 50$	No of Iter.	23	24	56	20
	Time (sec)	0.0068	1.3560	0.0507	0.0089
$M = 100$	No of Iter.	27	31	26	22
	Time (sec)	0.0115	2.9870	0.0164	0.0131

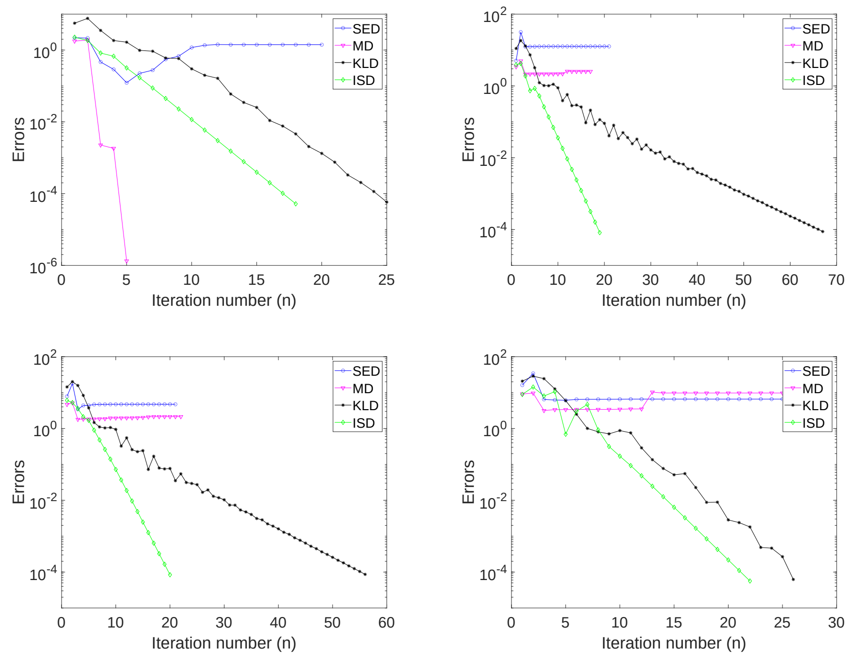

Figure 2: Example 5.1, From Top – Bottom: $M = 10, 30, 50, 100$.

Example 5.3. Let $E = \ell_2(\mathbb{R})$ be the linear spaces whose elements are all 2-summable sequences $\{x_i\}_{i=1}^{\infty}$ of scalars in $\mathbb{R}$, that is

$$\ell_2(\mathbb{R}) := \left\{ x = (x_1, x_2 \cdots, x_i \cdots), \quad x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\},$$

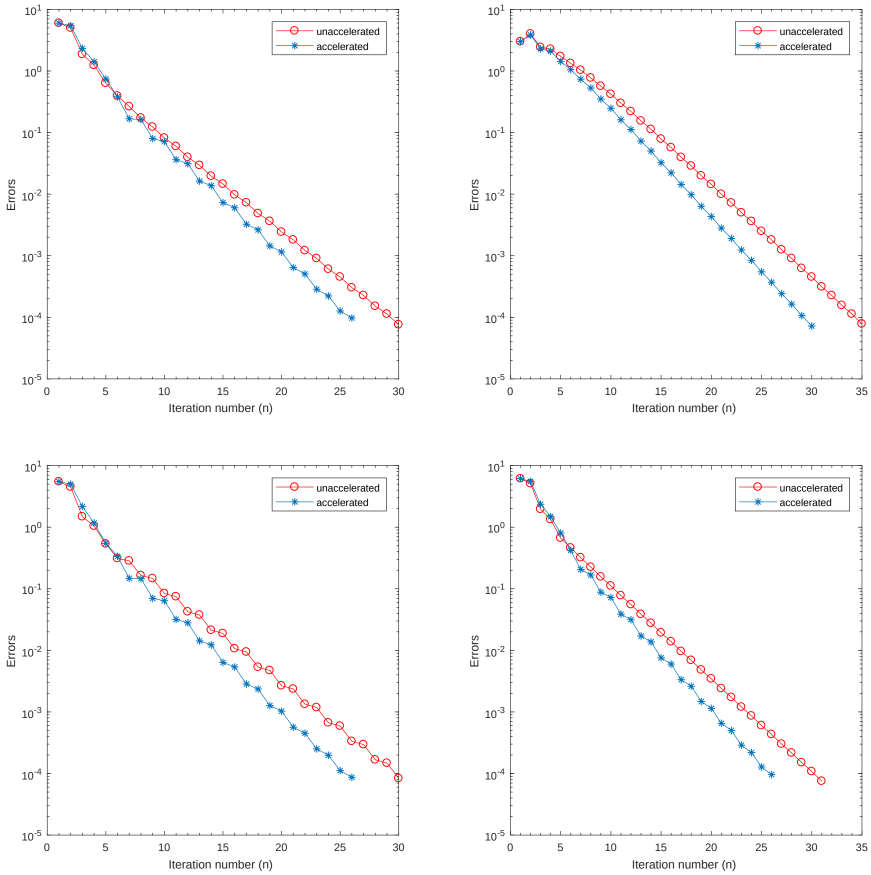

Figure 3: Example 5.3, Top left: **Case 1**, Top right: **Case 2**, Bottom left: **Case 3**, Bottom right **Case 4** .

with the inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle x, y \rangle := \sum_{i=1}^{\infty} x_i y_i$ and the norm $\| \cdot \| : \ell_2 \to \mathbb{R}$ by $\|x\| := \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$, where $x = \{x_i\}_{i=1}^{\infty}$, $y = \{y_i\}_{i=1}^{\infty}$. Suppose $f : \ell_2 \to \ell_2$ be given by $\frac{1}{2}\|x\|^2$ for all $x \in \ell_2$ then, $\nabla f(x) = \|x\|$. Let $C = \{x \in E : \|x\| \leq 3\}$, and let $h(x) \in \partial g(x, \cdot)(x)$ be given by $x(5 - \|x\|)$. The projection onto C is easily computed as

$$P_C(x) = \begin{cases} x \quad \text{if} \quad \|x\| \leq 3 \\ \dfrac{3x}{\|x\|} \quad \text{otherwise.} \end{cases}$$

In this experiment for Algorithm 3.2, we choose $\mu = 0.63$, $\alpha_1 = 0.5$. $\theta = \frac{1}{7}$. We also compare the algorithm in case $\theta \neq 0$ and $\theta = 0$ (without inertial term) for varying values of x_0 and x_1 as follows:

Case 1 $x_0 = (1, 0, 0, \ldots, 0, \ldots)'$ and $x_1 = (-2, 0, 0, \ldots, 0, \ldots)'$
Case 2 $x_0 = (2, 0, 0, \ldots, 0, \ldots)'$ and $x_1 = (1, 0, 0, \ldots, 0, \ldots)'$
Case 3 $x_0 = (2, 0, 0, \ldots, 0, \ldots)'$ and $x_1 = (-1.5, 0, 0, \ldots, 0, \ldots)'$
Case 4 $x_0 = (1, 0, 1, \ldots, 0, \ldots)'$ and $x_1 = (-2, 0, 1, \ldots, 0, \ldots)'$

For Example 5.3, we chose Error= $\|x_{n+1} - w_n\|^2 + \|y_n - x_n\|^2$ and $Error = \|x_{n+1} - x_n\|^2 + \|y_n - x_n\|^2$ respectively for the accelerated and unaccelerated algorithm. The comparisons are demonstrated in Figure 3.

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Example 5.4. In this example we make a comaprison of Algorithm 3.2 and Algorithm 1.1. Let $E = \mathbb{R}$ and $g : C \times C \to \mathbb{R}$ be given by $g(x, y) = (2.5 - \|x\|)(y - x)$. Let $C = \{x \in E : \|x\| \leq 3\}$. Then the projection onto C is easily computed as

$$P_C(x) = \begin{cases} x \quad \text{if} \quad \|x\| \leq 3, \\ \dfrac{3x}{\|x\|} \quad \text{otherwise.} \end{cases}$$

For Algorithm 3.2 choose the sequences $\theta = \frac{1}{3}$, $\alpha_1 = 2.5$ and $\beta_n = \frac{1}{2n+3}$ for Algorithm 1.1. The execution of this example is terminated at $E_n = \|x_{n+1} - w_n\| = 10^{-4}$. The result of this example is reported in Figure 4 for various values of the initial points x_0, y_0, x_1 and y_1.

Case 1 $x_0 = 0.56$, $x_1 = 0.76$, $y_0 = 0.98$, and $y_1 = 0.65$.
Case 2 $x_0 = 0.91$, $x_1 = 0.75$, $y_0 = 0.98$, and $y_1 = 0.12$.
Case 3 $x_0 = 1.01$, $x_1 = -1.76$, $y_0 = 1.12$, and $y_1 = 0.65$.
Case 4 $x_0 = 1.56$, $x_1 = 2.06$, $y_0 = -0.98$, and $y_1 = -0.65$.

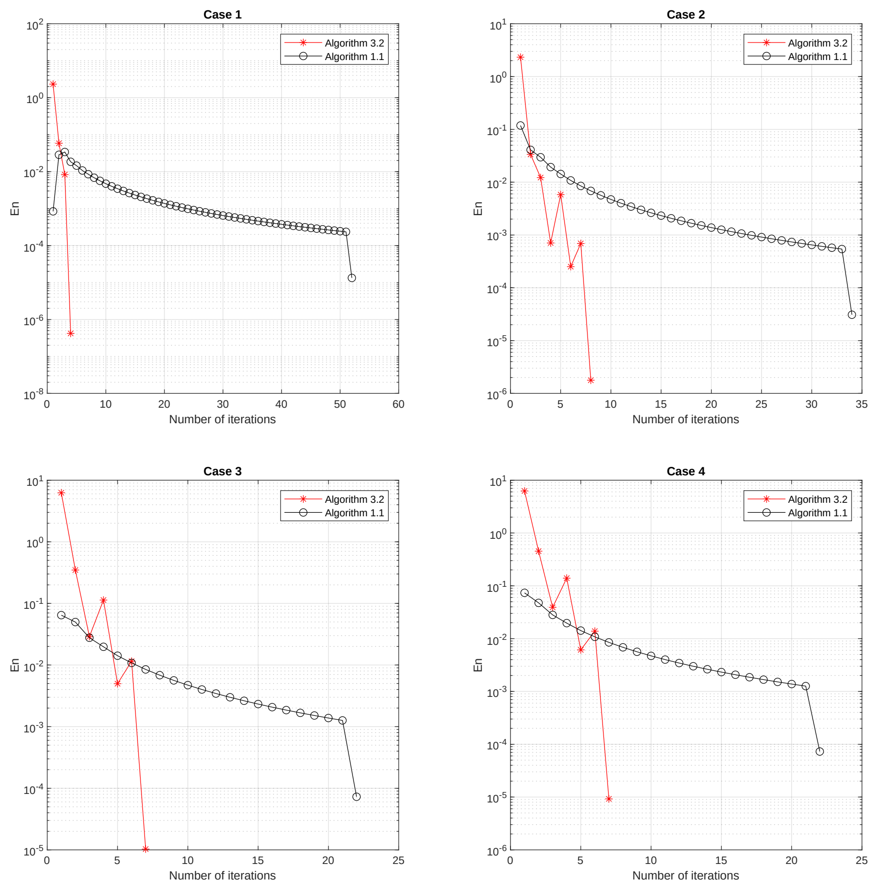

Figure 4: Example 5.4, Top left: **Case 1**, Top right: **Case 2**, Bottom left: **Case 3**, Bottom right **Case 4** .

6. Conclusion

This paper presented a Popov inspired subgradient extragradient algorithm from obtaining the solutions of an equilibrium problem. The method uses a step size which is carefully selected for easy computation and does not depend on a Lipschitz-type condition. Based on this method, we state and prove weak and strong convergence theorems under some certain monotonicity and standard assumptions. By numerical illustrations, we displayed the efficiency of this method compared to other previous obtained results in this direction.

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