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A Study on Two Subclasses of Analytic and Univalent Functions with Negative Coefficients Involving the Poisson Distribution Series

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Abstract. This paper introduces two new subclasses of analytical functions with negative coefficients and derives coefficient estimates for these novel subclasses. Further, inclusion relations and necessary and sufficient conditions for the Poisson distribution series to belong to these subclasses are established.

1. Introduction

Let $\mathcal{A}$ be the class of all functions ξ with Taylor series expansion of the form

$$\xi(z) = z + \sum_{k=2}^{\infty} a_k z^k \ , \tag{1.1}$$

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that are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Further, let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of normalized and univalent functions in U. Let $S^*(\alpha)$ and $K(\alpha)$ denote, respectively, the two well-known subclasses of S that are starlike and convex functions of order α, $0 \le \alpha < 1$, which were introduced by Robertson [17]. It is well-known that

$$S^*(\alpha) = \left\{ \xi \in \mathcal{A} : \Re\left(\frac{z\xi^{'}(z)}{\xi(z)} \right) > \alpha,\ z \in U \right\}, \tag{1.2}$$

and

$$K(\alpha) = \left\{ \xi\ \in \mathcal{A} : \Re\left(1 + \frac{z\xi^{''}(z)}{\xi^{'}(z)} \right) > \alpha,\ z \in U \right\}. \tag{1.3}$$

Definition 1.1. Let $\mathcal{A}\,(\beta, \alpha)$ denote a class of function $\xi \in \mathcal{A}$ which satisfy the following condition

$$\left| \beta\left(\frac{z\xi^{'}(z)}{\xi(z)} - 1 \right) + (1-\beta)\,\frac{z^2\xi^{\ ''}(z)}{\xi(z)} \right| < 1 - \alpha, \tag{1.4}$$

where $0 \le \beta,\ \alpha < 1$.

Fukui [6], introduced the function class $\mathcal{A}\,(\beta, \alpha)$ and he proved that $\mathcal{A}\,(\beta, \alpha) \subset S^*(\alpha)$. We note that $\mathcal{A}\,(1, \alpha) \equiv S^*(\alpha)$. In [19] Singh and Singh introduced the class of functions $\xi \in \mathcal{A}$ which satisfy the following condition

$$\Re\left(\beta\left(1 + \frac{z\xi^{''}(z)}{\xi^{'}(z)} \right) + (1-\beta)\,\frac{1}{\xi^{\ '}(z)} \right) < \frac{1+2a}{2}, \tag{1.5}$$

where $a > \frac{1}{2}$ and $z \in U$ and they gave some criteria for univalence expresing by $\Re\{\xi^{'}(z)\} > 0$.

Definition 1.2. Let $\mathcal{F}(\beta, \alpha)$ denote a class of functions $\xi \in \mathcal{A}$ which satisfy

$$\Re\left(\beta\left(1 + \frac{z\xi^{''}(z)}{\xi^{'}(z)} \right) + (1-\beta)\,\frac{1}{\xi^{\ '}(z)} \right) > \alpha, \tag{1.6}$$

where $0 \le \beta,\ \alpha < 1$.

We note that
1. $\mathcal{F}(1, \alpha) \equiv K(\alpha)$.
2. $\mathcal{F}(0, \alpha)$ is the class of univalent close to convex functions that satisfies

$$\Re\{\xi^{\ '}(z)\} > 0.$$

Let $\mathcal{T}$ be the subclass of $\mathcal{S}$ consisting of functions with negative coefficients of the form

$$\xi(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0. \tag{1.7}$$

Further, we define the class $\mathcal{T}(\beta, \alpha)$ by

$$\mathcal{T}(\beta, \alpha) = \mathcal{A}(\beta, \alpha) \cap \mathcal{T}. \tag{1.8}$$

Also, we define the class $\mathcal{R}(\beta, \alpha)$ by

$$\mathcal{R}(\beta, \alpha) = \mathcal{F}(\beta, \alpha) \cap \mathcal{T}. \tag{1.9}$$

The Poisson distribution was created in 1837 by Siméon Denis Poisson. This distribution expresses the probability of a given number of events occurring in a fixed interval of time or space. Recently, Poisson, Pascal, Logarithmic, and Binomial distributions have been partially studied in Geometric Function Theory (see [1, 3, 5, 7, 12, 15]). A variable $\varkappa$ is said to have Poisson distribution if it takes the values $0, 1, 2, 3, ...$ with probabilities

$$e^{-m}, \frac{me^{-m}}{1!}, \frac{m^2 e^{-m}}{2!}, \frac{m^3 e^{-m}}{3!}, ..., \tag{1.10}$$

respectively, where m is called the parameter. Thus

$$P(\varkappa = k) = \frac{m^k e^{-m}}{k!}, \quad k = 0, 1, 2, 3, ... \tag{1.11}$$

In 2014, Porwal [13] introduced a power series such that its coefficients are probabilities of the Poisson distribution

$$N(m, z) = z + \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} e^{-m} z^k, \quad m > 0, \ z \in U. \tag{1.12}$$

Also, he introduced the series

$$\begin{aligned} R(m, z) &= 2z - N(m, z), \\ &= z - \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} e^{-m} z^k, \quad m > 0, \ z \in U. \end{aligned} \tag{1.13}$$

Based on earlier results using hypergeometric functions associated with various subclasses of analytic and univalent functions [4, 8, 18, 20], Porwal's recent research [14, 16], and the results using generalized Bessel functions to connect various subclasses of analytic and univalent functions [2, 9, 10, 11], this paper examines some properties and characteristics of the two classes $\mathcal{T}(\beta, \alpha)$ and $\mathcal{R}(\beta, \alpha)$. We also provide necessary and sufficient conditions for the Poisson distribution to belong to these classes.

2. Main Results

Firstly, we determine the sufficient and necessary conditions for the function $\xi \in \mathcal{T}$ to belong to the class $\mathcal{T}(\beta,\alpha)$.

Theorem 2.1. *Let the function* ξ *be defined by* (1.7). *Then* $\xi \in \mathcal{T}(\beta,\alpha)$ *if and only if*

$$\sum_{k=2}^{\infty}\left((k-1)(k(1-\beta)+\beta)+(1-\alpha)\right)a_k < 1-\alpha. \tag{2.1}$$

Proof. Assume that the inequality (2.1) holds true, then we need to show that

$$\left|\beta\left(\frac{z\xi^{'}(z)}{\xi(z)}-1\right)+(1-\beta)\frac{z^2\xi^{''}(z)}{\xi(z)}\right| < 1-\alpha.$$

Since

$$\begin{aligned}\left|\beta\left(\frac{z\xi^{'}(z)}{\xi(z)}-1\right)+(1-\beta)\frac{z^2\xi^{''}(z)}{\xi(z)}\right| &= \left|\frac{\sum\limits_{k=2}^{\infty}(k-1)(k(1-\beta)+\beta)a_kz^{k-1}}{1-\sum\limits_{k=2}^{\infty}a_kz^{k-1}}\right| \\ &\leq \frac{\sum\limits_{k=2}^{\infty}(k-1)(k(1-\beta)+\beta)a_k}{1-\sum\limits_{k=2}^{\infty}a_k}.\end{aligned}$$

Then

$$\left|\beta\left(\frac{z\xi^{'}(z)}{\xi(z)}-1\right)+(1-\beta)\frac{z^2\xi^{''}(z)}{\xi(z)}\right| < 1-\alpha,$$

if

$$\sum_{k=2}^{\infty}(k-1)(k(1-\beta)+\beta)a_k < (1-\alpha)\left(1-\sum_{k=2}^{\infty}a_k\right),$$

or equivalently

$$\sum_{k=2}^{\infty}((k-1)(k(1-\beta)+\beta)+(1-\alpha))a_k < 1-\alpha.$$

Conversely, assume that the function $\xi \in \mathcal{T}$ is in the class $\mathcal{T}(\beta,\alpha)$. Since $\Re\{z\} \leq |z|$, then we have

$$\Re\left(\frac{\sum\limits_{k=2}^{\infty}(k-1)(k(1-\beta)+\beta)a_kz^{k-1}}{1-\sum\limits_{k=2}^{\infty}a_kz^{k-1}}\right) < 1-\alpha.$$

Choosing z on the real axis, then

$$\left(\frac{\sum\limits_{k=2}^{\infty}(k-1)(k(1-\beta)+\beta)a_k z^{k-1}}{1-\sum\limits_{k=2}^{\infty}a_k z^{k-1}}\right)$$

is real. Let $z \to 1^-$ through real values, we get

$$\sum_{k=2}^{\infty}((k-1)(k(1-\beta)+\beta)+(1-\alpha))a_k < 1-\alpha$$

which is equivalent to (2.1). And this ends the proof of the theorem. $\square$

Corollary 2.2. *Let $\xi \in \mathfrak{T}$,$0 \le \beta_1 \le \beta_2 < 1$ and $0 \le \alpha \le 1$. If $\xi(z) \in \mathfrak{T}(\beta_1, \alpha)$, then $\xi(z) \in \mathfrak{T}(\beta_2, \alpha)$ .i.e $\mathfrak{T}(\beta_1, \alpha) \subset \mathfrak{T}(\beta_2, \alpha)$.*

Proof. Let the function $\xi(z) \in \mathfrak{T}(\beta_1, \alpha)$. Then, by Theorem 2.1, we have

$$\sum_{k=2}^{\infty}((k-1)(k(1-\beta_1)+\beta_1)+(1-\alpha))a_k < 1-\alpha.$$

Since $\beta_1 \le \beta_2$, we get

$$\begin{aligned} & \sum_{k=2}^{\infty}((k-1)(k-(k-1)\beta_2)+(1-\alpha))a_k \\ \le\ & \sum_{k=2}^{\infty}((k-1)(k-(k-1)\beta_1)+(1-\alpha))a_k \\ <\ & 1-\alpha, \end{aligned}$$

which implies that $\xi(z) \in \mathfrak{T}(\beta_2, \alpha)$. $\square$

Corollary 2.3. *All functions in the class $\mathfrak{T}(\beta, \alpha)$ are starlike.*

Proof. Since $\beta < 1$. An application of Corollary 2.2 gives $\mathfrak{T}(\beta, \alpha) \subset \mathfrak{T}(1, \alpha)$. $\square$

We apply the Poisson distribution function to the class $\mathfrak{T}(\beta, \alpha)$ in Theorem 2.4 below.

Theorem 2.4. *Let $m > 0$ and $R(m, z)$ be defined by (1.13), then $R(m, z) \in \mathfrak{T}(\beta, \alpha)$ if and only if*

$$(1-\beta)m^2+(2-\beta)m \le (1-\alpha)\, e^{-m}. \tag{2.2}$$

Proof. According to Theorem 2.1, it is sufficient to show that condition (2.2) is equivalent to

$$\sum_{k=2}^{\infty}((k-1)(k(1-\beta)+\beta)+(1-\alpha))\frac{m^{k-1}}{(k-1)!}e^{-m} \le 1-\alpha.$$

Thus,

$$\begin{aligned} &\sum_{k=2}^{\infty}((k-1)(k(1-\beta)+\beta)+(1-\alpha))\frac{m^{k-1}}{(k-1)!}e^{-m} \\ = \; & e^{-m}\sum_{k=2}^{\infty}((1-\beta)(k-1)(k-2)+(k-1)(2-\beta)+(1-\alpha))\frac{m^{k-1}}{(k-1)!} \\ = \; & e^{-m}\left((1-\beta)m^2\sum_{k=3}^{\infty}\frac{m^{k-3}}{(k-3)!}+(2-\beta)m\sum_{k=2}^{\infty}\frac{m^{k-2}}{(k-2)!}+(1-\alpha)\sum_{k=2}^{\infty}\frac{m^{k-1}}{(k-1)!}\right) \\ = \; & (1-\beta)m^2+(2-\beta)m+(1-\alpha)(1-e^{-m}) \le 1-\alpha. \end{aligned}$$

Which ends the proof. □

Secondly, we determine the sufficient and necessary conditions for the function $\xi \in \mathcal{T}$ to belong to the class $\mathcal{R}(\beta,\alpha)$.

Theorem 2.5. *Let the function* $\xi \in \mathcal{T}$ *be defined by* (1.7) *and let* $\beta \ge \frac{1}{2}$. *Then* $\xi \in \mathcal{R}(\beta,\alpha)$ *if and only if*

$$\sum_{k=2}^{\infty}(k(\beta k-\alpha))a_k \le 1-\alpha. \tag{2.3}$$

Proof. Let us assume inequality (2.3) holds. The goal is to prove that

$$\left|\beta\left(1+\frac{z\xi''(z)}{\xi'(z)}\right)+(1-\beta)\frac{1}{\xi'(z)}-1\right| < 1-\alpha. \tag{2.4}$$

Since

$$\begin{aligned} &\left|\beta\left(1+\frac{z\xi''(z)}{\xi'(z)}\right)+(1-\beta)\frac{1}{\xi'(z)}-1\right| \\ = \; & \left|\frac{\sum_{k=2}^{\infty}(k(k\beta-1))a_k z^{k-1}}{1-\sum_{k=2}^{\infty}ka_k z^{k-1}}\right| \\ \le \; & \frac{\sum_{k=2}^{\infty}(k(k\beta-1))a_k}{1-\sum_{k=2}^{\infty}ka_k} \le 1-\alpha. \end{aligned}$$

Hence ξ satisfies the condition (2.4). Conversely, assume that the function $\xi \in \mathcal{T}$ is in the class $\mathcal{R}(\beta,\alpha)$. Then we have

$$\begin{aligned}
&\Re\left(\beta\left(1+\frac{z\xi^{''}(z)}{\xi'(z)}\right)+(1-\beta)\frac{1}{\xi\,'(z)}\right)\\
=\;&\Re\left(\frac{1-\sum\limits_{k=2}^{\infty}\beta k^2 a_k z^{k-1}}{1-\sum\limits_{k=2}^{\infty} k a_k z^{k-1}}\right)>\alpha.
\end{aligned}$$

If z is chosen on the real axis, then

$$\left(\frac{1-\sum\limits_{k=2}^{\infty}\beta k^2 a_k z^{k-1}}{1-\sum\limits_{k=2}^{\infty} k a_k z^{k-1}}\right)$$

is real. Letting $z \to 1^-$ through real values, we get

$$\sum_{k=2}^{\infty}(k(k\beta-\alpha))a_k < 1-\alpha.$$

Which is equivalent to (2.3) and this completes the proof. $\square$

Corollary 2.6. *Let $\xi \in \mathcal{T}$, $\frac{1}{2} \le \beta_1 \le \beta_2 < 1$ and $0 \le \alpha \le 1$, if $\xi \in \mathcal{R}(\beta_2,\alpha)$, then $\xi \in \mathcal{R}(\beta_1,\alpha)$. i.e $\mathcal{R}(\beta_2,\alpha) \subset \mathcal{R}(\beta_1,\alpha)$.*

Proof. Let the function $\xi \in \mathcal{R}(\beta_2,\alpha)$. Then, by Theorem 2.5, we have

$$\sum_{k=2}^{\infty}(k(\beta_2 k-\alpha))a_k \le 1-\alpha. \tag{2.5}$$

Since $\beta_1 \le \beta_2$, we get

$$\begin{aligned}
\sum_{k=2}^{\infty}(k(\beta_1 k-\alpha))a_k &\le \sum_{k=2}^{\infty}(k(\beta_2 k-\alpha))a_k,\\
&\le 1-\alpha,
\end{aligned}$$

which implies that $\xi(z) \in \mathcal{R}(\beta_1,\alpha)$. $\square$

We apply the Poisson distribution function to the class $\mathcal{R}(\beta,\alpha)$ in Theorem 2.7 below.

Theorem 2.7. *Let $m > 0$ and $R(m,z)$ be defined by (1.13), then $R(m,z) \in \mathcal{R}(\beta,\alpha)$, if and only if*

$$\beta m^2+(3\beta-\alpha)m+(\beta-\alpha)(1-e^{-m}) \le 1-\alpha. \tag{2.6}$$

Proof. According to Theorem 2.5, we need to show that (2.6) is equivalent to the inequality

$$\sum_{k=2}^{\infty}(k(\beta k-\alpha))\frac{m^{k-1}}{(k-1)!}e^{-m}\le 1-\alpha.$$

Thus,

$$\begin{aligned}
&\sum_{k=2}^{\infty}(k^2\beta-k\alpha)\frac{m^{k-1}}{(k-1)!}e^{-m}\\
= \quad & e^{-m}\sum_{k=2}^{\infty}(\beta(k-1)(k-2)+(3\beta-\alpha)(k-1)+(\beta-\alpha))\frac{m^{k-1}}{(k-1)!}\\
= \quad & e^{-m}\left(\beta m^2\sum_{k=3}^{\infty}\frac{m^{k-3}}{(k-3)!}+(3\beta-\alpha)m\sum_{k=2}^{\infty}\frac{m^{k-2}}{(k-2)!}+(\beta-\alpha)\sum_{k=2}^{\infty}\frac{m^{k-1}}{(k-1)!}\right)\\
= \quad & \beta m^2+(3\beta-\alpha)m+(\beta-\alpha)(1-e^{-m})\le 1-\alpha.
\end{aligned}$$

Which completes the proof. □

3. Conclusion

Our paper introduces two novel classes of analytic and univalent functions with negative coefficients and derives coefficient inequalities and inclusion relations for these two classes. Further, using a similar method to Porwal's [13], necessary and sufficient conditions for the Poisson distribution series to belong to these classes are also discussed.

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