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Generalized Inverses and Solutions to Equations in Rings with Involution

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Abstract. In this paper, we focus on partial isometry elements and strongly EP elements on a ring. We construct characterizing equations such that an element which is both group invertible and MP-invertible, is a partial isometry element, or is strongly EP, exactly when these equations have a solution in a given set. In particular, an element $a \in R^{\#} \cap R^{\dagger}$ is a partial isometry element if and only if the equation $x = x(a^{\dagger})^{*}a^{\dagger}$ has at least one solution in $\{a, a^{\#}, a^{\dagger}, a^{*}, (a^{\#})^{*}, (a^{\dagger})^{*}\}$. An element $a \in R^{\#} \cap R^{\dagger}$ is a strongly EP element if and only if the equation $(a^{\dagger})^{*}xa^{\dagger} = xa^{\dagger}a$ has at least one solution in $\{a, a^{\#}, a^{\dagger}, a^{*}, (a^{\#})^{*}, (a^{\dagger})^{*}\}$. These characterizations extend many well-known results.

1. Introduction

Throughout this paper, R denotes an associative ring with 1. We write $E(R)$ and $J(R)$ to denote the set of all idempotents and the Jacobson radical of R, respectively.

An element $a \in R$ is said to be group invertible if there exists an element $a^{\#} \in R$ such that

$$aa^{\#}a = a, \quad a^{\#}aa^{\#} = a^{\#}, \quad aa^{\#} = a^{\#}a.$$

The element $a^{\#}$ is called the group inverse of a, which is uniquely determined by the above equations [1, 9]. An involution $* : a \longmapsto a^{*}$ in a ring R is an anti-isomorphism of degree 2, that is,

$$(a^{*})^{*} = a, \quad (a + b)^{*} = a^{*} + b^{*}, \quad (ab)^{*} = b^{*}a^{*}.$$

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An element a in R is called normal if $aa^* = a^*a$. An element $a^\dagger$ in R is called the Moore-Penrose inverse (MP-inverse) of a [6, 10], if

$$aa^\dagger a = a, \quad a^\dagger aa^\dagger = a^\dagger, \quad (aa^\dagger)^* = aa^\dagger, \quad (a^\dagger a)^* = a^\dagger a.$$

If such $a^\dagger$ exists, then it is unique [6]. Denote by $R^\#$ and $R^\dagger$ the set of group invertible elements of R and the set of all MP-invertible elements of R, respectively. An element a is said to be EP if $a \in R^\# \cap R^\dagger$ and satisfies $a^\# = a^\dagger$ [3, 6]. We denote by R^{EP} the set of all EP elements of R. Note that if $a \in R^\dagger$ is normal, then $a \in R^{EP}$, see [6]. An element $a \in R$ is called normal EP if a is normal and $a \in R^\dagger$. Denote by R^{NEP} the set of all normal EP elements of R. An element a is called a partial isometry if $a^\dagger = a^*$ and a is called a strongly EP element if $a \in R^{EP}$ is a partial isometry. We denote the sets of all partial isometry elements and strongly EP elements of R by R^{PI} and R^{SEP}, respectively.

In [7], D. Mosić and D. S. Djordjević presented some characterizations of EP elements in rings with involution. In addition, some equivalent conditions for the element a in a ring with involution to be a partial isometry are given. Recent researches on EP elements in rings with involution have produced some interesting results, see [4, 9, 12]. The necessary and sufficient conditions for the existence of a common solution and the general common solution of the equation $axb = c$ (a, b are regular elements) were given for rings with involution in [2]. In [13, 15], a new kind of characterizations of generalized inverse elements has been studied by means of the solution of constructed equations recently.

Motivated by these articles above, this paper is intended to provide, by using certain equations admitting solutions in a definite set, further sufficient and necessary conditions for an element in a ring with involution to be an EP element, partial isometry, normal EP element, and strongly EP element. This is a new way to study generalized inverses in rings.

2. EP elements

Lemma 2.1.[5, 7] *Let $a \in R^\# \cap R^\dagger$. If $a^*aa^\#(1 - aa^\dagger) = 0$, then $a \in R^{EP}$.*

Proof. Pre-multiplying the equality $a^*aa^\#(1 - aa^\dagger) = 0$ by $(a^\dagger)^*$, we have $aa^\#(1 - aa^\dagger) = 0$. That is $aa^\# = aa^\dagger$. Hence $a \in R^{EP}$ by [7, Theorem 1.6] or [5]. □

Lemma 2.2. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{PI}$ if and only if $a^*a^\dagger = a^\dagger a^\dagger$.*

Proof. ⇒ The equality obviously holds since $a^* = a^\dagger$.

⇐ Post-multiplying $a^*a^\dagger = a^\dagger a^\dagger$ by a, one has $a^*a^\dagger a = a^\dagger a^\dagger a$. Applying the involution to the last equality, we have $a^\dagger a^2 = a^\dagger a(a^\dagger)^*$, it follows that $a^2 = a(a^\dagger)^*$. Post-multiplying the equality by a^*, we get $a^2a^* = a^2a^\dagger$. Pre-multiplying $a^2a^* = a^2a^\dagger$ by $a^\#$, one has $aa^* = aa^\dagger$. Thus $a \in R^{PI}$ by [7, Theorem 2.1]. □

In order to prove the theorems given in this paper more clearly, we briefly review the following existing conclusions:

Lemma 2.3.[11, Lemma 2.2] *Let $a \in R^{\#}$. Then $(a^{\#})^*R = a^*R$ and $R(a^{\#})^* = Ra^*$.*

Lemma 2.4.[11, Lemma 2.3] *Let $a \in R^{\dagger}$. Then*

(1) $aR = aa^{\dagger}R = aa^*R$ and $Ra = Ra^{\dagger}a = Ra^*a$.

(2) $a^*R = a^{\dagger}R = a^*aR = a^{\dagger}aR$ and $Ra^* = Ra^{\dagger} = Raa^* = Raa^{\dagger}$.

Lemma 2.5.[12, Theorem 3.9] *Let $a \in R$. Then the following are equivalent:*

(1) $a \in R^{EP}$;

(2) $a \in R^{\#}$ and $aR \subseteq a^*R$;

(3) $a \in R^{\#}$ and $Ra \subseteq Ra^*$;

(4) $a \in R^{\#}$ and $a^*R \subseteq aR$;

(5) $a \in R^{\#}$ and $Ra^* \subseteq Ra$.

Lemma 2.6.[14, Lemma 2.1] *Let $a \in R^{\#} \cap R^{\dagger}$. Then the following conditions are satisfied:*

(1) $a^*R = a^*a^2R = a^*aa^{\#}R = (a^{\#})^*R$;

(2) $Ra = Ra^{\#} = Raa^*a^{\#} = Ra^*a = Ra^*a^*a = Ra^{\dagger}a^*a$;

(3) $(a^{\#})^*aa^{\dagger}R = (a^{\#})^*a^{\#}a^{\dagger}R = (a^{\#})^*a^{\#}a^*R$;

(4) $a^{\#}R = aR$ and $Ra^* = Ra^{\dagger}$.

In [14, Theorem 2.4], the authors proved that an element $a \in R^{\#} \cap R^{\dagger}$ can be an EP element if and only if the equation $axa^{\#} + axa^* = xaa^{\dagger} + a^*ax$ has at least one solution in the set $\chi_a = \{a, a^{\#}, a^{\dagger}, a^*, (a^{\#})^*, (a^{\dagger})^*\}$.

Recall that an element a is said to be EP if $a \in R^{\#} \cap R^{\dagger}$ and satisfies $a^{\#} = a^{\dagger}$. Thus, we can modify the above existing theorem in [14] and construct the following equation, with the help of which we can explore a new kind of characterization of EP elements:

$$axa^{\dagger} + axa^* = xaa^{\#} + a^*ax. \tag{2.1}$$

Theorem 2.7. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if equation (2.1) has at least one solution in $\chi_a = \{a, a^{\#}, a^{\dagger}, a^*, (a^{\#})^*, (a^{\dagger})^*\}$.*

Proof. $\Rightarrow$ Obviously, $x = a^{\dagger}$ is a solution because $a^{\dagger} = a^{\#}$.

$\Leftarrow$ (1) If $x = a$ is a solution, then $a^2a^{\dagger} + a^2a^* = a^2a^{\#} + a^*a^2 = a + a^*a^2$. By Lemma 2.6, we have

$$a^*R = a^*a^2R = (a^2a^{\dagger} + a^2a^* - a)R \subseteq aR.$$

Therefore $a \in R^{EP}$ by Lemma 2.5.

(2) If $x = a^{\#}$ is a solution, then $aa^{\#}a^{\dagger}+aa^{\#}a^{*} = a^{\#}aa^{\#}+a^{*}aa^{\#} = a^{\#}+a^{*}aa^{\#}$. By Lemma 2.6, we obtain that

$$a^{*}R = a^{*}aa^{\#}R = (aa^{\#}a^{\dagger} + aa^{\#}a^{*} - a^{\#})R \subseteq aR.$$

The fact that $a \in R^{EP}$ follows from Lemma 2.5.

(3) If $x = a^{\dagger}$ is a solution, then $aa^{\dagger}a^{\dagger} + aa^{\dagger}a^{*} = a^{\dagger}aa^{\#} + a^{*}aa^{\dagger} = a^{\dagger}aa^{\#} + a^{*}$. Pre-multiplying it by $aa^{\#}$, we obtain $aa^{\dagger}a^{\dagger} + aa^{\dagger}a^{*} = a^{\#} + a^{\#}aa^{*}$. By Lemma 2.6, we have

$$Ra^{\#} = R(aa^{\dagger}a^{\dagger} + aa^{\dagger}a^{*} - a^{\#}aa^{*}) \subseteq Ra^{\dagger} + Ra^{*} = Ra^{\dagger}.$$

Since $Ra^{\#} = Ra, Ra^{\dagger} = Ra^{*}$ by Lemma 2.6, we get $Ra \subseteq Ra^{*}$. From Lemma 2.5, $a \in R^{EP}$.

(4) If $x = a^{*}$ is a solution, then $aa^{*}a^{\dagger}+aa^{*}a^{*} = a^{*}aa^{\#}+a^{*}aa^{*}$. Post-multiplying it by $(1 - aa^{\dagger})$, we have $a^{*}aa^{\#}(1 - aa^{\dagger}) = 0$. By Lemma 2.1, we get $a \in R^{EP}$.

(5) If $x = (a^{\#})^{*}$ is a solution, then $a(a^{\#})^{*}a^{\dagger}+a(a^{\#})^{*}a^{*} = (a^{\#})^{*}aa^{\#}+a^{*}a(a^{\#})^{*}$. Post-multiplying it by $(1-aa^{\dagger})$, we get $(a^{\#})^{*}aa^{\#}(1-aa^{\dagger}) = 0$. Pre-multiplying the last equation by $(a^{2})^{*}$, we get $a^{*}aa^{\#}(1 - aa^{\dagger}) = 0$. Therefore $a \in R^{EP}$ by Lemma 2.1.

(6) If $x = (a^{\dagger})^{*}$ is a solution, then $a(a^{\dagger})^{*}a^{\dagger} + a(a^{\dagger})^{*}a^{*} = (a^{\dagger})^{*}aa^{\#} + a^{*}a(a^{\dagger})^{*}$. Taking involution of the above equality, we obtain that

$$(a^{\dagger})^{*}a^{\dagger}a^{*} + aa^{\dagger}a^{*} = (a^{\#})^{*}a^{*}a^{\dagger} + a^{\dagger}a^{*}a.$$

Lemma 2.6 now leads to

$$Ra = Ra^{\dagger}a^{*}a = R((a^{\dagger})^{*}a^{\dagger}a^{*} + aa^{\dagger}a^{*} - (a^{\#})^{*}a^{*}a^{\dagger}) \subseteq Ra^{*} + Ra^{\dagger} = Ra^{\dagger} = Ra^{*}.$$

From Lemma 2.5, $a \in R^{EP}$. □

Multipying the equation (2.1) on the right by a, we obtain:

$$axa^{\dagger}a + axa^{*}a = ax + a^{*}axa. \tag{2.2}$$

Theorem 2.8. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if the equation* (2.2) *has at least one solution in χ_a.*

Proof. $\Rightarrow$Obviously, $x = a^{\dagger}$ is a solution.

$\Leftarrow$ (1) If $x = a$ is a solution, then $a^{2}a^{\dagger}a + a^{2}a^{*}a = a^{2} + a^{*}a^{3}$. It is immediate that $a^{2}a^{*}a = a^{*}a^{3}$. By Lemma 2.4, we obtain that

$$a^{*}R = a^{*}a^{3}R = a^{2}a^{*}aR \subseteq aR.$$

Hence $a \in R^{EP}$ by Lemma 2.5.

(2) If $x = a^{\#}$ is a solution, then $aa^{\#}a^{\dagger}a + aa^{\#}a^{*}a = aa^{\#} + a^{*}aa^{\#}a$. That is $aa^{\#}a^{*}a = a^{*}a$. Post-multiplying it by a, we obtain that $aa^{\#}a^{*}a^{2} = a^{*}a^{2}$. By Lemma 2.6, we get

$$a^{*}R = a^{*}a^{2}R = aa^{\#}a^{*}a^{2}R \subseteq aR.$$

Therefore, $a \in R^{EP}$ by Lemma 2.5.

(3) If $x = a^\dagger$ is a solution, then $aa^\dagger a^\dagger a + aa^\dagger a^* a = aa^\dagger + a^* aa^\dagger a = aa^\dagger + a^* a$. Post-multiplying it by a, we have $aa^\dagger a^\dagger a^2 + aa^\dagger a^* a^2 = a + a^* a^2$. We thus get

$$a^* R = a^* a^2 R = (aa^\dagger a^\dagger a^2 + aa^\dagger a^* a^2 - a)R \subseteq aR$$

by Lemma 2.6. And then it follows from Lemma 2.5 that $a \in R^{EP}$.

(4) If $x = a^*$ is a solution, then $aa^* a^\dagger a + aa^* a^* a = aa^* + a^* aa^* a$. We conclude from Lemma 2.4 that

$$Ra^* = Raa^* = R(aa^* a^\dagger a + aa^* a^* a - a^* aa^* a) \subseteq Ra.$$

Hence $a \in R^{EP}$ by Lemma 2.5.

(5) If $x = (a^\#)^*$ is a solution, then $a(a^\#)^* a^\dagger a + a(a^\#)^* a^* a = a(a^\#)^* + a^* a(a^\#)^* a$. Pre-multiplying it by $a^\dagger$, we get $(a^\#)^* a^\dagger a + (a^\#)^* a^* a = (a^\#)^* + a^\dagger a^* a(a^\#)^* a$. Then from Lemma 2.3, we obtain that

$$Ra^* = R(a^\#)^* = R((a^\#)^* a^\dagger a + (a^\#)^* a^* a - a^\dagger a^* a(a^\#)^* a) \subseteq Ra,$$

which yields $a \in R^{EP}$ by Lemma 2.5.

(6) If $x = (a^\dagger)^*$ is a solution, then $a(a^\dagger)^* a^\dagger a + a(a^\dagger)^* a^* a = a(a^\dagger)^* + a^* a(a^\dagger)^* a$. That is $a^2 = a^* a(a^\dagger)^* a$. Post-multiplying it by $a^\#$, we obtain that $a = a^* a(a^\dagger)^* aa^\#$. Then

$$aR = a^* a(a^\dagger)^* aa^\# R \subseteq a^* R.$$

Therefore, $a \in R^{EP}$ by Lemma 2.5. □

Further, we revised the equation (2.2) as follows:

$$axa^\dagger a + xaa^* a = ax + a^* axa. \tag{2.3}$$

Theorem 2.9. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{EP}$ if and only if the equation* (2.3) *has at least one solution in χ_a.*

Proof. $\Rightarrow$ $x = a^\dagger$ is a solution since $aa^\dagger = a^\dagger a$.

$\Leftarrow$ (1) If $x = a$ is a solution, then $a^2 a^\dagger a + a^2 a^* a = a^2 + a^* a^3$. It is immediate from the proof of Theorem 2.8(1) that $a \in R^{EP}$.

(2) If $x = a^\#$ is a solution, then $aa^\# a^\dagger a + a^\# aa^* a = aa^\# + a^* aa^\# a$. Then $a \in R^{EP}$ by the proof of Theorem 2.8(2) since $aa^\# = a^\# a$.

(3) If $x = a^\dagger$ is a solution, then $aa^\dagger a^\dagger a + a^\dagger aa^* a = aa^\dagger + a^* aa^\dagger a = aa^\dagger + a^* a$. That is $aa^\dagger a^\dagger a = aa^\dagger$. Applying the involution, one has $aa^\dagger = a^\dagger a^2 a^\dagger$. By Lemma 2.4 and Lemma 2.5, we have $a \in R^{EP}$.

(4) If $x = a^*$ is a solution, then $aa^* a^\dagger a + a^* aa^* a = aa^* + a^* aa^* a$. That is $aa^* a^\dagger a = aa^*$. From Lemma 2.4, we obtain that

$$Ra^* = Raa^* = Raa^* a^\dagger a \subseteq Ra.$$

By Lemma 2.5, $a \in R^{EP}$.

(5) If $x = (a^\#)^*$ is a solution, then $a(a^\#)^* a^\dagger a + (a^\#)^* a a^* a = a(a^\#)^* + a^* a(a^\#)^* a$. Pre-multiplying it by $a^\dagger$, we have $(a^\#)^* a^\dagger a + a^\dagger (a^\#)^* a a^* a = (a^\#)^* + a^\dagger a^* a (a^\#)^* a$. By Lemma 2.3, we have

$$Ra^* = R(a^\#)^* = R((a^\#)^* a^\dagger a + a^\dagger (a^\#)^* a a^* a - a^\dagger a^* a (a^\#)^* a) \subseteq Ra,$$

which gives $a \in R^{EP}$ by Lemma 2.5.

(6) If $x = (a^\dagger)^*$ is a solution, then $a(a^\dagger)^* a^\dagger a + (a^\dagger)^* a a^* a = a(a^\dagger)^* + a^* a (a^\dagger)^* a$. That is $(a^\dagger)^* a a^* a = a^* a (a^\dagger)^* a$. Pre-multiplying it by $(1 - aa^\dagger)$, we have

$$(1 - aa^\dagger) a^* a (a^\dagger)^* a = 0.$$

Hence $0 = (1 - aa^\dagger) a^* a (a^\dagger a a^\dagger)^* a = (1 - aa^\dagger) a^* a (a^\dagger)^* a^\dagger a^2$. Multiplying the last equality by $a^\#$ on the right, we obtain that $0 = (1 - aa^\dagger) a^* a (a^\dagger)^* a^\dagger a = (1 - aa^\dagger) a^* a (a^\dagger)^*$. Post-multiply it by a^* and then we have

$$(1 - aa^\dagger) a^* a^2 a^\dagger = 0.$$

Post-multiplying it by $aa^\# a^\dagger$, we get $(1 - aa^\dagger) a^* = 0$, which implies $a = a^2 a^\dagger$. Consequently, $a \in R^{EP}$. □

Theorem 2.10. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{EP}$ if and only if the equality $a^\dagger x a = a^*$ has at least one solution.*

Proof. ⇒ Since $a \in R^{EP}$, we have $aa^\dagger = a^\dagger a$. Hence $x = aa^* a^\dagger$ is a solution of the equation $a^\dagger x a = a^*$.

⇐ Assume that $a^\dagger x a = a^*$ have a solution x_0. Then $Ra^* = Ra^\dagger x_0 a \subseteq Ra$, it follows that $a \in R^{EP}$ by Lemma 2.5. □

Theorem 2.11. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{EP}$ if and only if the equation $a^\dagger x a = aa^\# - aa^\dagger$ has at least one solution.*

Proof. ⇒ Since $a \in R^{EP}$, we have $aa^\# - aa^\dagger = 0$. Then $x = aa^\# - aa^\dagger$ is a solution of the equation $a^\dagger x a = aa^\# - aa^\dagger$.

⇐ Assume that $a^\dagger x a = aa^\# - aa^\dagger$ has a solution. Then, by [2, Theorem 2.1], we get

$$aa^\# - aa^\dagger = a^\dagger a (aa^\# - aa^\dagger) a^\dagger a. \tag{2.4}$$

Pre-multiplying (2.4) by a, we obtain that

$$a - a^2 a^\dagger = a(aa^\# - aa^\dagger) a^\dagger a = a - a^2 a^\dagger a^\dagger a.$$

That is $a^2 a^\dagger = a^2 a^\dagger a^\dagger a$.

On the other hand, post-multiply (2.4) by a, we have

$$a^\dagger a (aa^\# - aa^\dagger) a^\dagger a^2 = 0. \tag{2.5}$$

Pre-multiplying (2.5) by a and then post-multiplying the last equation by $a^{\#}$, we obtain that

$$a(aa^{\#} - aa^{\dagger})a^{\dagger}a = 0.$$

That is $a = a^2a^{\dagger}a^{\dagger}a$. Obviously, we can deduce that $a = a^2a^{\dagger}$. Consequently, $a \in R^{EP}$. □

Let $a \in R$. Write $a^0 = \{x \in R | ax = 0\}$. Clearly, a^0 is a right ideal of R, which is called the right annihilator of a. Similarly, we can define ${}^0a$. Then, we have the following theorem.

Theorem 2.12. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if $a^0 = (a^{\dagger})^0$.*

Proof. $\Rightarrow$ Since $a \in R^{EP}$, we have $a^{\#} = a^{\dagger}$. Therefore $(a^{\#})^o = (a^{\dagger})^0$. Note that $a^0 = (a^{\#})^0$. Then $a^0 = (a^{\dagger})^0$.

$\Leftarrow$ Assume that $a^0 = (a^{\dagger})^0$. Note that $1 - a^{\dagger}a \in (a^{\#})^0$. Then $1 - a^{\dagger}a \in (a^{\dagger})^0$, which implies $a^{\dagger} = a^{\dagger}a^{\dagger}a$. Hence $Ra^{\dagger} \subseteq Ra$, one obtains $a \in R^{EP}$ by Lemma 2.4 and Lemma 2.5. □

3. Partial Isometry Elements

Recall that an element $c \in R$ is semi-idempotent if $c - c^2 \in J(R)$. Using the semi-idempotent elements of R, we have the following theorem.

Theorem 3.1. *Let $a \in R^{\dagger}$. Then $a \in R^{PI}$ if and only if the following two conditions hold:*
(1) aa^* is a semi-idempotent;
(2) $a^{\dagger} - a^* \in R^{\#}$.

Proof. The equality $a^* = a^{\dagger}$ implies that

$$aa^* - aa^*aa^* = 0 \in J(R) \quad \textit{and} \quad a^{\dagger} - a^* = 0 \in E(R) \subseteq R^{\#}.$$

On the contrary, assume that aa^* is semi-idempotent and $a^{\dagger} - a^* \in R^{\#}$. Take $aa^* - aa^*aa^* = x \in J(R)$. Then, by the proof of [14, Theorem 2.6], one has $a^{\dagger} - a^* = a^{\dagger}(a^{\dagger})^*a^{\dagger}x \in J(R)$. Set $z = (a^{\dagger} - a^*)^{\#}$ because $a^{\dagger} - a^* \in R^{\#}$. Then $a^{\dagger} - a^* = (a^{\dagger} - a^*)z(a^{\dagger} - a^*)$. Thus, we get

$$(a^{\dagger} - a^*)(1 - z(a^{\dagger} - a^*)) = (a^{\dagger} - a^*) - (a^{\dagger} - a^*)z(a^{\dagger} - a^*) = 0.$$

Since $z(a^{\dagger} - a^*) \in J(R)$, we obtain that $1 - z(a^{\dagger} - a^*)$ is invertible. Hence $a^{\dagger} - a^* = 0$, so $a \in R^{PI}$. □

Theorem 3.2. *Let $a \in R^{\dagger}$. Then the following conditions are equivalent:*
(1) $a^{\dagger}(a^{\dagger})^* \in E(R)$;
(2) $(a^{\dagger})^*a^{\dagger} \in E(R)$;
(3) $a \in R^{PI}$;
(4) $a^{\dagger}(a^{\dagger})^*$ is a semi-idempotent and $a^* - a^{\dagger} \in E(R)$;

(5) $a^\dagger a - a^\dagger (a^\dagger)^* \in E(R)$*;*
(6) $aa^\dagger - (a^\dagger)^* a^\dagger \in E(R)$*;*
(7) $(a^\dagger)^* a^\dagger$ *is a semi-idempotent and* $a^* - a^\dagger \in E(R)$*.*

Proof. (1)⇒(2) From the assumption, we know that $a^\dagger (a^\dagger)^* = a^\dagger (a^\dagger)^* a^\dagger (a^\dagger)^*$. Pre-multiplying it by a and then post-multiplying the last equality by $a^\dagger$, we get $(a^\dagger)^* a^\dagger = (a^\dagger)^* a^\dagger (a^\dagger)^* a^\dagger$.

(2)⇒(3) From (2), we obtain that $(a^\dagger)^* a^\dagger = (a^\dagger)^* a^\dagger (a^\dagger)^* a^\dagger$. Post-multiplying it by aa^*, we get $aa^\dagger = (a^\dagger)^* a^\dagger$. Multiply the equality by a on the right and then we get $a = (a^\dagger)^*$. Applying involution to $a = (a^\dagger)^*$, we get $a^* = a^\dagger$. Consequently, $a \in R^{PI}$.

(3)⇒(4) Since $a \in R^{PI}$, $a^* = a^\dagger$. Then we know that $a^* - a^\dagger = 0 \in E(R)$ and $a^\dagger (a^\dagger)^* - a^\dagger (a^\dagger)^* a^\dagger (a^\dagger)^* = 0 \in J(R)$. It is immediate that $a^\dagger (a^\dagger)^*$ is a semi-idempotent and $a^* - a^\dagger \in E(R)$.

(4)⇒(5) Write $x = a^\dagger a - a^\dagger (a^\dagger)^*$. Then $x - x^2 = a^\dagger (a^\dagger)^* - a^\dagger (a^\dagger)^* a^\dagger (a^\dagger)^* \in J(R)$ by hypothesis. Clearly, $a(x - x^2)a^* a = a - (a^\dagger)^*$. Note that $a^* - a^\dagger \in E(R)$. Then $a - (a^\dagger)^* \in E(R)$, this gives $a(x - x^2)a^* a \in J(R) \cap E(R)$, so $a(x - x^2)a^* a = 0$. It follows that $a = (a^\dagger)^*$. Hence $a^\dagger a - a^\dagger (a^\dagger)^* \in E(R)$.

(5)⇒(6) From (5), we know that $a^\dagger (a^\dagger)^* = a^\dagger (a^\dagger)^* a^\dagger (a^\dagger)^*$. Pre-multiplying it by a and then multiplying the last equality by $a^\dagger$ on the right, we obtain that

$$(a^\dagger)^* a^\dagger = (a^\dagger)^* a^\dagger (a^\dagger)^* a^\dagger.$$

Hence $aa^\dagger - (a^\dagger)^* a^\dagger - (aa^\dagger - (a^\dagger)^* a^\dagger)(aa^\dagger - (a^\dagger)^* a^\dagger) = (a^\dagger)^* a^\dagger - (a^\dagger)^* a^\dagger (a^\dagger)^* a^\dagger = 0$. Consequently, $aa^\dagger - (a^\dagger)^* a^\dagger \in E(R)$.

(6)⇒(7) From (6), we obtain that $(a^\dagger)^* a^\dagger = (a^\dagger)^* a^\dagger (a^\dagger)^* a^\dagger$. So $(a^\dagger)^* a^\dagger$ is an idempotent. By (2)⇒(3), we get $a^* = a^\dagger$, which gives $a^* - a^\dagger = 0 \in E(R)$.

(7)⇒(1) Similar to (4)⇒(5), we obtain that $a^* = a^\dagger$. Then

$$a^\dagger (a^\dagger)^* a^\dagger (a^\dagger)^* = a^\dagger a a^\dagger (a^\dagger)^* = a^\dagger (a^\dagger)^*.$$

Therefore, $a^\dagger (a^\dagger)^* \in E(R)$. □

Lemma 3.3. *Let* $a \in R^\dagger$*. Then* $a \in R^{PI}$ *if and only if* $a^\dagger = a^\dagger (a^\dagger)^* a^\dagger$*.*

Proof. ⇒ Since $a \in R^{PI}$, $a^* = a^\dagger$. And then we can easily get

$$a^\dagger (a^\dagger)^* a^\dagger = a^\dagger a a^\dagger = a^\dagger.$$

⇐ From the assumption, we know that $a^\dagger = a^\dagger (a^\dagger)^* a^\dagger$. Pre-multiplying it by a and then post-multiplying the last equality by a, we get $a = (a^\dagger)^*$. Taking involution of the above equality, we obtain $a^* = a^\dagger$. So $a \in R^{PI}$. □

Lemma 3.4. *Let* $a \in R^\# \cap R^\dagger$*. Then* $a \in R^{PI}$ *if and only if* $a^2 = a(a^\dagger)^*$*.*

Proof. $\Rightarrow$ We know that $a^* = a^\dagger$ according to the assumption. Then we can get the following equality.

$$a(a^\dagger)^* = a(a^*)^* = a^2.$$

$\Leftarrow$ From the assumption, we know that $a^2 = a(a^\dagger)^*$. Post-multiplying it by a^*, we have $a^2a^* = a^2a^\dagger$. Hence $a \in R^{PI}$ by [8, Theorem 2.1]. □

Lemma 3.5. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{PI}$ if and only if the following equation has at least one solution in χ_a:*

$$x = x(a^\dagger)^*a^\dagger. \tag{2.1}$$

Proof. $\Rightarrow$ Obviously, a^* is a solution.

$\Leftarrow$ (1) If $x = a$ is a solution, then $a = a(a^\dagger)^*a^\dagger$. Post-multiplying it by a, we have $a^2 = a(a^\dagger)^*$. By Lemma 3.4, $a \in R^{PI}$.

(2) If $x = a^\#$ is a solution, then $a^\# = a^\#(a^\dagger)^*a^\dagger$ is a solution. Pre-multiplying it by a^2, we get $a = a(a^\dagger)^*a^\dagger$. By the proof of (1), we know $a \in R^{PI}$.

(3) If $x = a^\dagger$ is a solution, then $a^\dagger = a^\dagger(a^\dagger)^*a^\dagger$. Hence $a \in R^{PI}$ by Lemma 3.3.

(4) If $x = a^*$ is a solution, then $a^* = a^*(a^\dagger)^*a^\dagger = a^\dagger$. It is immediate that $a \in R^{PI}$.

(5) If $x = (a^\#)^*$ is a solution, then $(a^\#)^* = (a^\#)^*(a^\dagger)^*a^\dagger$. Post-multiplying it by a, we get $(a^\#)^*a = (a^\#)^*(a^\dagger)^*$. Applying involution to the above equality, we have $a^*a^\# = a^\dagger a^\#$. Then we deduce that $a \in R^{PI}$ by [8, Theorem 2.1].

(6) If $x = (a^\dagger)^*$ is a solution, then $(a^\dagger)^* = (a^\dagger)^*(a^\dagger)^*a^\dagger$. Firstly, multiply the equality on the right by a, apply involution to the latest equation, and then we get $a^*a^\dagger = a^\dagger a^\dagger$. By Lemma 2.2, $a \in R^{PI}$. □

Similarly, we have the following theorem.

Theorem 3.6. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{PI}$ if and only if the following equation has at least one solution in χ_a:*

$$x = (a^\dagger)^*a^\dagger x. \tag{2.2}$$

Using the symmetricity, we have the following corollary.

Corollary 3.7. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{PI}$ if and only if the following equation has at least one solution in χ_a:*

$$x = xa^\dagger(a^\dagger)^*. \tag{2.3}$$

4. Normal EP Elements

Lemma 4.1. *Let $a \in R^{\dagger}$ and $x \in R$.*
(1) If $a^{\dagger}(a^{\dagger})^{*}a^{\dagger}x = 0$, then $a^{\dagger}x = 0$;
(2) If $xa^{\dagger}(a^{\dagger})^{*}a^{\dagger} = 0$, then $xa^{\dagger} = 0$.

Proof. (1) Pre-multiplying the equality $a^{\dagger}(a^{\dagger})^{*}a^{\dagger}x = 0$ by $a^{*}a$, we immediately have $a^{\dagger}x = 0$.

(2) Similarly, we can prove (2). □

Lemma 4.2. *Let $a \in R^{\dagger} \cap R^{\#}$ and $x \in R$. If $a(a^{\dagger})^{*}a^{\dagger}x = 0$, then $a^{\dagger}x = 0$.*

Proof. Since $a(a^{\dagger})^{*} = a^{2}a^{\dagger}(a^{\dagger})^{*}$, pre-multiplying the equality $a(a^{\dagger})^{*}a^{\dagger}x = 0$ by $a^{\#}$, one has $(a^{\dagger})^{*}a^{\dagger}x = 0$. Pre-multiplying the last equality by a^{*}, one obtains $a^{\dagger}x = 0$. □

Lemma 4.3. *Let $a \in R^{\dagger} \cap R^{\#}$ and $x \in R$. If $x(a^{\#})^{*}(a^{\dagger})^{*}a^{\dagger} = 0$, then $x(a^{\#})^{*} = 0$.*
Proof. Post-multiplying $x(a^{\#})^{*}(a^{\dagger})^{*}a^{\dagger} = 0$ by aa^{*}, we have $x(a^{\#})^{*}aa^{\dagger} = 0$. Note that $(a^{\#})^{*}aa^{\dagger} = (a^{\#})^{*}$. Thus, $x(a^{\#})^{*} = 0$. □

Lemma 4.4. *Let $a \in R^{\dagger} \cap R^{\#}$ and $x \in R$. If $(a^{\dagger})^{*}(a^{\dagger})^{*}a^{\dagger}x = 0$, then $a^{\dagger}x = 0$.*
Proof. Pre-multiplying $(a^{\dagger})^{*}(a^{\dagger})^{*}a^{\dagger}x = 0$ by $a^{\#}a^{*}$, we have $a^{\#}(a^{\dagger})^{*}a^{\dagger}x = 0$. Pre-multiplying the last equation by a^{2}, we obtain that $a(a^{\dagger})^{*}a^{\dagger}x = 0$. By Lemma 4.2, $a^{\dagger}x = 0$. □

Lemma 4.5.[14, Lemma 2.3] *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{EP}$ if and only if one of the following conditions holds:*
(1)$Ra^{\dagger} \subseteq Ra$;
(2)$Ra \subseteq Ra^{\dagger}$;
(4)$aR \subseteq a^{\dagger}R$;
(6)$a^{\dagger}R \subseteq aR$;
(3)$Ra^{\#} \subseteq Ra^{*}$;
(5)$Ra^{\#} \subseteq Ra^{\dagger}$.

Lemma 4.6.[14, Lemma 2.11] *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{NEP}$ if and only if $(a^{\dagger})^{*}a^{\dagger} = a^{\dagger}(a^{\dagger})^{*}$.*

Theorem 4.7. *Let $a \in R^{\#} \cap R^{\dagger}$. Then $a \in R^{NEP}$ if and only if the following equation has at least one solution in χ_a:*

$$xa^{\dagger}(a^{\dagger})^{*} = x(a^{\dagger})^{*}a^{\dagger}. \tag{2.1}$$

Proof. ⇒ By [11, Corollary 2.8], we know that $x = a$ is a solution.

⇐ (1) If $x = a$ is a solution, then $aa^{\dagger}(a^{\dagger})^{*} = a(a^{\dagger})^{*}a^{\dagger}$. That is $(a^{\dagger})^{*} = a(a^{\dagger})^{*}a^{\dagger}$. This infers that $Ra = R(a^{\dagger})^{*} = Ra(a^{\dagger})^{*}a^{\dagger} \subseteq Ra^{\dagger} = Ra^{*}$ by [11, Lemma 2.1]. It follows from Lemma 2.5 that $a \in R^{EP}$. Moreover, post-multiplying $(a^{\dagger})^{*} = a(a^{\dagger})^{*}a^{\dagger}$

by a, we get $(a^\dagger)^*a = a(a^\dagger)^*$. Take involution of the last equation. It follows $a^*a^\dagger = a^\dagger a^*$. By [11, Lemma 2.7], a is normal. Hence $a \in R^{NEP}$.

(2) If $x = a^\#$ is a solution, then $a^\# a^\dagger (a^\dagger)^* = a^\# (a^\dagger)^* a^\dagger$. Note that $(a^\#)^0 = a^0$. Then we get $aa^\dagger (a^\dagger)^* = a(a^\dagger)^* a^\dagger$. Hence $a \in R^{NEP}$ by (1).

(3) If $x = a^\dagger$ is a solution, then $a^\dagger a^\dagger (a^\dagger)^* = a^\dagger (a^\dagger)^* a^\dagger$. Note that $(a^\dagger)^* = (a^\dagger)^* a^\dagger a$. Then $a^\dagger (a^\dagger)^* a^\dagger (1 - a^\dagger a) = 0$. By Lemma 4.1, we have $a^\dagger (1 - a^\dagger a) = 0$. Then, $Ra^\dagger = Ra^\dagger a^\dagger a \subseteq Ra$. Thus, by Lemma 4.5, we obtain $a \in R^{EP}$ and $aa^\dagger = a^\dagger a$. On the other hand,

$$a^\dagger (a^\dagger)^* = a^\dagger a a^\dagger (a^\dagger)^* = aa^\dagger a^\dagger (a^\dagger)^* = aa^\dagger (a^\dagger)^* a^\dagger = (a^\dagger)^* a^\dagger,$$

which shows $a \in R^{NEP}$ by Lemma 4.6.

(4) If $x = a^*$ is a solution, then $a^* a^\dagger (a^\dagger)^* = a^* (a^\dagger)^* a^\dagger = a^\dagger a a^\dagger = a^\dagger$. Similar to the proof of (1), $a \in R^{NEP}$.

(5) If $x = (a^\#)^*$ is a solution, then $(a^\#)^* a^\dagger (a^\dagger)^* = (a^\#)^* (a^\dagger)^* a^\dagger$. Applying involution to it, then $a^\dagger (a^\dagger)^* a^\# = (a^\dagger)^* a^\dagger a^\#$. We get $a^\dagger (a^\dagger)^* a = (a^\dagger)^*$ because ${}^0a ={}^0 (a^\#)$. Similar to the proof of (1), $a \in R^{NEP}$.

(6) If $x = (a^\dagger)^*$ is a solution, then $(a^\dagger)^* a^\dagger (a^\dagger)^* = (a^\dagger)^* (a^\dagger)^* a^\dagger$. Taking involution of the equality, we obtain that

$$a^\dagger (a^\dagger)^* a^\dagger = (a^\dagger)^* a^\dagger a^\dagger.$$

Similar to the proof of (3), $a \in R^{NEP}$. □

Theorem 4.8. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{NEP}$ if and only if the following equation has at least one solution in χ_a:*

$$x(a^\dagger)^* a^\dagger = (a^\dagger)^* a^\dagger x. \tag{2.2}$$

Proof. $\Rightarrow$ Since $a \in R^{NEP}$, $x = a$ is a solution.

$\Leftarrow$ (1) If $x = a$ is a solution, then $a(a^\dagger)^* a^\dagger = (a^\dagger)^* a^\dagger a$. Post-multiplying it by $(1 - a^\dagger a)$, we have $a(a^\dagger)^* a^\dagger (1 - a^\dagger a) = 0$. By Lemma 4.2, $a^\dagger (1 - a^\dagger a) = 0$. Hence $a \in R^{EP}$. It follows that

$$(a^\dagger)^* a^\dagger = aa^\dagger (a^\dagger)^* a^\dagger = a^\dagger a (a^\dagger)^* a^\dagger = a^\dagger (a^\dagger)^* a^\dagger a = a^\dagger (a^\dagger)^*.$$

Therefore, $a \in R^{NEP}$ according to Lemma 4.6.

(2) If $x = a^\#$ is a solution, then $a^\# (a^\dagger)^* a^\dagger = (a^\dagger)^* a^\dagger a^\#$. Post-multiplying it by $(1 - a^\dagger a)$, we get

$$a^\# (a^\dagger)^* a^\dagger (1 - a^\dagger a) = 0.$$

By Lemma 4.2 and the proof of (1), $a \in R^{EP}$. Post-multiplying $a^\# (a^\dagger)^* a^\dagger = (a^\dagger)^* a^\dagger a^\#$ by a, we have $a^\dagger (a^\dagger)^* = a^\# (a^\dagger)^* = (a^\dagger)^* a^\dagger$. Hence $a \in R^{NEP}$ by Lemma 4.6.

(3) If $x = a^\dagger$ is a solution, then $a^\dagger (a^\dagger)^* a^\dagger = (a^\dagger)^* a^\dagger a^\dagger$. Pre-multiplying it by $(1 - aa^\dagger)$, we have

$$(1 - aa^\dagger) a^\dagger (a^\dagger)^* a^\dagger = 0.$$

By Lemma 4.1, $(1-aa^\dagger)a^\dagger=0$. Hence $a\in R^{EP}$. Then $x=a^{\#}$ is a solution. By (2), $a\in R^{NEP}$.

(4) If $x=a^*$ is a solution, then $a^*(a^\dagger)^*a^\dagger=(a^\dagger)^*a^\dagger a^*$. That is $a^\dagger=(a^\dagger)^*a^\dagger a^*$. Similar to the proof of (1) in Theorem 4.7, we have $a\in R^{NEP}$.

(5) If $x=(a^{\#})^*$ is a solution, then $(a^{\#})^*(a^\dagger)^*a^\dagger=(a^\dagger)^*a^\dagger(a^{\#})^*$. Pre-multiplying it by $(1-aa^\dagger)$, we get

$$(1-aa^\dagger)(a^{\#})^*(a^\dagger)^*a^\dagger=0.$$

By Lemma 4.3, $(1-aa^\dagger)(a^{\#})^*=0$. This gives $a^{\#}=a^{\#}aa^\dagger$. Therefore, $a\in R^{EP}$. Multiplying $(a^{\#})^*(a^\dagger)^*a^\dagger=(a^\dagger)^*a^\dagger(a^{\#})^*$ on the left by a^*, we obtain that

$$(a^\dagger)^*a^\dagger=a^\dagger(a^{\#})^*=a^\dagger(a^\dagger)^*,$$

which implies $a\in R^{NEP}$ by Lemma 4.6.

(6) If $x=(a^\dagger)^*$ is a solution, then $(a^\dagger)^*(a^\dagger)^*a^\dagger=(a^\dagger)^*a^\dagger(a^\dagger)^*$. Post-multiplying it by $(1-a^\dagger a)$, we have $(a^\dagger)^*(a^\dagger)^*a^\dagger(1-a^\dagger a)=0$. By Lemma 4.4, we obtain that $a^\dagger(1-a^\dagger a)=0$, which yields $a\in R^{EP}$. Therefore, $x=(a^{\#})^*$ is a solution. By (5), $a\in R^{NEP}$. □

5. Strongly EP elements

Theorem 5.1. *Let $a\in R^{\#}\cap R^\dagger$. Then $a\in R^{SEP}$ if and only if the following equation has at least one solution in χ_a:*

$$x=(a^{\#})^*xa^{\#}. \tag{2.1}$$

Proof. $\Rightarrow$ Note that $a^{\#}=a^\dagger=a^*$ since $a\in R^{SEP}$. Hence $x=a^*$ is a solution.

$\Leftarrow$ (1) If $x=a$ is a solution, then $a=(a^{\#})^*aa^{\#}$. Post-multiplying the equality by a, one gets $a^2=(a^{\#})^*a$. Hence

$$a^2a^\dagger=(a^{\#})^*aa^\dagger=(a^{\#})^*,$$

which leads to $a^{\#}=aa^\dagger a^*$. Then we have $Ra^{\#}=Raa^\dagger a^*\subseteq Ra^*$. Thus, $a\in R^{EP}$ by Lemma 4.5. Then, we get $a^\dagger=a^*$, which implies $a\in R^{PI}$. Hence $a\in R^{SEP}$.

(2) If $x=a^{\#}$ is a solution, then $a^{\#}=(a^{\#})^*a^{\#}a^{\#}$. Multiplying the equality by a^2 from the right, one obtains $a=(a^{\#})^*aa^{\#}$. By the proof of (1), we get $a\in R^{SEP}$.

(3) If $x=a^\dagger$ is a solution, then $a^\dagger=(a^{\#})^*a^\dagger a^{\#}$. Multiplying the equality by $a^\dagger a$ from the right, we have

$$a^\dagger a^\dagger a=(a^{\#})^*a^\dagger a^{\#}a^\dagger a=(a^{\#})^*a^\dagger a^{\#}=a^\dagger.$$

Then, $Ra^\dagger=Ra^\dagger a^\dagger a\subseteq Ra$. Thus, by Lemma 4.5, we obtain $a\in R^{EP}$, which gives $x=a^\dagger=a^{\#}$. By (2), $a\in R^{SEP}$.

(4) If $x = a^*$ is a solution, then $a^* = (a^\#)^* a^* a^\#$. We can deduce that

$$a^* a^\dagger a = (a^\#)^* a^* a^\# a^\dagger a = (a^\#)^* a^* a^\# = a^*.$$

Applying the involution to the equality, one has $a = a^\dagger a^2$. Thus, we get $aR = a^\dagger a^2 R \subseteq a^\dagger R$, which implies $a \in R^{EP}$ by Lemma 4.5. Then, we find that

$$a^* = (a^\#)^* a^* a^\# = (a^\dagger)^* a^* a^\# = a a^\dagger a^\# = a^\#,$$

which gives $a \in R^{SEP}$.

(5) If $x = (a^\#)^*$ is a solution, then $(a^\#)^* = (a^\#)^*(a^\#)^* a^\#$. Hence we deduce that

$$a^* = a^* a^* (a^\#)^* = a^* a^* (a^\#)^* (a^\#)^* a^\# = (a^\#)^* a^* a^\#.$$

By (4), we get $a \in R^{SEP}$.

(6) If $x = (a^\dagger)^*$ is a solution, then $(a^\dagger)^* = (a^\#)^*(a^\dagger)^* a^\#$. Applying involution to the equality, we have $a^\dagger = (a^\#)^* a^\dagger a^\#$. By (3), $a \in R^{SEP}$. □

Theorem 5.2. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{SEP}$ if and only if the following equation has at least one solution in χ_a:*

$$x a^\dagger a = x (a^\dagger)^* a^\dagger. \tag{2.2}$$

Proof. $\Rightarrow$ Obviously $x = a$ is a solution since $a^* = a^\dagger = a^\#$.

$\Leftarrow$ (1) If $x = a$ is a solution, then $a = a a^\dagger a = a (a^\dagger)^* a^\dagger$. Hence

$$Ra = Ra(a^\dagger)^* a^\dagger \subseteq Ra^\dagger.$$

By Lemma 4.5, $a \in R^{EP}$. Post-multiplying $a = a(a^\dagger)^* a^\dagger$ by a, we get $a^2 = a(a^\dagger)^*$. Thus $a^* = a^\dagger$ by Lemma 3.4, which implies $a \in R^{SEP}$.

(2) If $x = a^\#$ is a solution, then $a^\# = a^\# a^\dagger a = a^\# (a^\dagger)^* a^\dagger$. Pre-multiplying the equality by a^2, we have $a = a(a^\dagger)^* a^\dagger$. By (1), $a \in R^{SEP}$.

(3) If $x = a^\dagger$ is a solution, then $a^\dagger a^\dagger a = a^\dagger (a^\dagger)^* a^\dagger$. Note that $(a^\dagger)^0 = (a^*)^0$. Then we get $a^* a^\dagger a = a^\dagger$. Therefore $a \in R^{SEP}$ by [7, Theorem 2.3].

(4) If $x = a^*$ is a solution, then $a^* a^\dagger a = a^* (a^\dagger)^* a^\dagger = a^\dagger$. This gives $a^\dagger (1 - a^\dagger a) = 0$, so $a \in R^{EP}$. Post-multiplying $a^* a^\dagger a = a^\dagger$ by $a^\dagger$, we get $a^* a^\dagger = a^\dagger a^\dagger$. Then we obtain that $a \in R^{SEP}$ by Lemma 2.2.

(5) If $x = (a^\#)^*$ is a solution, then $(a^\#)^* a^\dagger a = (a^\#)^* (a^\dagger)^* a^\dagger$. Taking involution of the equality, we deduce that

$$a^\dagger a a^\# = (a^\dagger)^* a^\dagger a^\#.$$

This implies $a^\dagger a a = (a^\dagger)^* a^\dagger a = (a^\dagger)^*$ because ${}^0(a^\#) = {}^0 a$. Hence $a^* a^\dagger a = a^\dagger$, which infers $a \in R^{SEP}$ by [7, Theorem 2.3].

(6) If $x = (a^\dagger)^*$ is a solution, then $(a^\dagger)^* = (a^\dagger)^* a^\dagger a = (a^\dagger)^* (a^\dagger)^* a^\dagger$. Post-multiplying the equality $(a^\dagger)^* = (a^\dagger)^* (a^\dagger)^* a^\dagger$ by $(1 - a a^\dagger)$, we have $(a^\dagger)^* (1 - a a^\dagger) = 0$

which implies $a^\dagger = aa^\dagger a^\dagger$. Hence $a \in R^{EP}$. Post-multiplying $(a^\dagger)^* = (a^\dagger)^*(a^\dagger)^*a^\dagger$ by a, we get $(a^\dagger)^*a = (a^\dagger)^*(a^\dagger)^*$. Applying involution to the equality, we obtain that $a^*a^\dagger = a^\dagger a^\dagger$. Therefore, $a \in R^{SEP}$ according to Lemma 2.2. □

Theorem 5.3. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{SEP}$ if and only if the following equation has at least one solution in χ_a:*

$$(a^\dagger)^*xa^\dagger = xa^\dagger a. \tag{2.3}$$

Proof. $\Rightarrow$ $x = a$ is a solution since $a^\dagger = a^* = a^\#$.

$\Leftarrow$ (1) If $x = a$ is a solution, then $(a^\dagger)^*aa^\dagger = aa^\dagger a = a$. Hence $Ra = R(a^\dagger)^*aa^\dagger \subseteq Ra^\dagger$. By Lemma 4.5, $a \in R^{EP}$. Post-multiplying $(a^\dagger)^*aa^\dagger = a$ by a, we obtain that $(a^\dagger)^*a = a^2$. Per-multiplying the equation by a^*, we get $a^\dagger a^2 = a^*a^2$. Post-multiplying the last equation by $a^\# a^\dagger$, we obtain $a^\dagger = a^*$. Therefore $a \in R^{SEP}$.

(2) If $x = a^\#$ is a solution, then $(a^\dagger)^*a^\# a^\dagger = a^\# a^\dagger a = a^\#$. Observe that

$$Ra^\# = R(a^\dagger)^*a^\# a^\dagger \subseteq Ra^\dagger.$$

This implies that $a \in R^{EP}$ by Lemma 4.5. Then, we can obtain $(a^\dagger)^*a^\dagger a^\# = a^\#$. Since ${}^0(a) = {}^0(a^\#)$, we get $(a^\dagger)^*a^\dagger a = a$. That is $(a^\dagger)^* = a$. Therefore, $a \in R^{SEP}$.

(3) If $x = a^\dagger$ is a solution, then $(a^\dagger)^*a^\dagger a^\dagger = a^\dagger a^\dagger a$. Taking involution of the equality, we have $(a^\dagger)^*(a^\dagger)^*a^\dagger = a^\dagger a(a^\dagger)^*$. Pre-multiplying the last equality by $(1 - a^\dagger a)$, we have

$$(1 - a^\dagger a)(a^\dagger)^*(a^\dagger)^*a^\dagger = 0.$$

Post-multiplying by aa^*, we get $(1 - a^\dagger a)(a^\dagger)^*aa^\dagger = 0$, it is immediate that

$$(1-a^\dagger a)(a^\dagger)^* = (1-a^\dagger a)(a^\dagger)^*a^\dagger a = (1-a^\dagger a)(a^\dagger)^*a^\dagger a^2a^\dagger a^\# a = (1-a^\dagger a)(a^\dagger)^*aa^\dagger a^\# a = 0.$$

Hence $a \in R^{EP}$. On the other hand, pre-multiply $(a^\dagger)^*a^\dagger a^\dagger = a^\dagger a^\dagger a$ by a^*, and we obtain that $a^\dagger a^\dagger = a^*a^\dagger$, which implies $a \in R^{SEP}$ by Lemma 2.2.

(4) If $x = a^*$ is a solution, then $(a^\dagger)^*a^*a^\dagger = a^*a^\dagger a$. Taking involution of the equality, we get $(a^\dagger)^*aa^\dagger = a^\dagger a^2$. Hence we obtain that

$$Ra = Ra^\# a^2 = Ra^\# aa^\dagger a^2 \subseteq Ra^\dagger a^2 = R(a^\dagger)^*aa^\dagger \subseteq Ra^\dagger.$$

Therefore $a \in R^{EP}$ by Lemma 4.5. Post-multiplying $(a^\dagger)^*aa^\dagger = a^\dagger a^2$ by a, we have $(a^\dagger)^*a = a^2$. Then we deduce that $a^* = a^\dagger$ by the proof of (1). Therefore $a \in R^{SEP}$.

(5) If $x = (a^\#)^*$ is a solution, then $(a^\dagger)^*(a^\#)^*a^\dagger = (a^\#)^*a^\dagger a$. Applying involution to it, we get $(a^\dagger)^*a^\# a^\dagger = a^\dagger aa^\#$. Post-multiplying the last equality by $aa^\dagger$, we obtain that $(a^\dagger)^*a^\# a^\dagger = a^\dagger$. Then, by [11, Lemma 2.1] we know that

$$a^\dagger R = (a^\dagger)^*a^\# a^\dagger R \subseteq (a^\dagger)^*R = aR.$$

By Lemma 4.5, $a \in R^{EP}$. Post-multiplying $(a^\dagger)^*a^\# a^\dagger = a^\dagger$ by a^3, we get $(a^\dagger)^*a = a^2$. From Lemma 3.4, we deduce that $a^* = a^\dagger$, which implies $a \in R^{SEP}$.

(6) If $x = (a^\dagger)^*$ is a solution, then $(a^\dagger)^*(a^\dagger)^*a^\dagger = (a^\dagger)^*a^\dagger a$. By Theorem 5.2 (6), $a \in R^{SEP}$. □

Theorem 5.4. *Let $a \in R^\# \cap R^\dagger$. Then $a \in R^{SEP}$ if and only if the following equation has at least one solution in χ_a:*

$$a^\dagger x(a^\dagger)^* = xaa^\dagger. \tag{2.4}$$

Proof. $\Rightarrow$ $x = a^*$ is a solution since $a^\dagger = a^* = a^\#$.

$\Leftarrow$ (1) If $x = a$ is a solution, then $a^\dagger a(a^\dagger)^* = a^2a^\dagger$. Taking involution of the equality, we have $a^\dagger a^\dagger a = aa^\dagger a^*$. By [14, Theorem 2.15] (3), we deduce that $a \in R^{SEP}$.

(2) If $x = a^\#$ is a solution, then $a^\dagger a^\#(a^\dagger)^* = a^\# aa^\dagger$. Pre-multiplying the equality by a, we get $a^\#(a^\dagger)^* = aa^\dagger$. Taking involution of the last equality and then post-multiplying the obtained equality by a, we obtain that $a^\dagger(a^\#)^*a = a$. It is evident that

$$aR = a^\dagger(a^\#)^*aR \subseteq a^\dagger R,$$

which shows $a \in R^{EP}$ by Lemma 4.5. Furthermore, post-multiply $a^\#(a^\dagger)^* = aa^\dagger$ by a^* and thus we get $a^\dagger = a^*$. Hence $a \in R^{SEP}$.

(3) If $x = a^\dagger$ is a solution, then $a^\dagger a^\dagger(a^\dagger)^* = a^\dagger aa^\dagger = a^\dagger$. Similar to the proof of Theorem 5.1 (3), we deduce that $a \in R^{SEP}$.

(4) If $x = a^*$ is a solution, then $a^\dagger a^*(a^\dagger)^* = a^*aa^\dagger = a^*$. Similar to the proof of Theorem 5.3 (1), we get $a \in R^{SEP}$.

(5) If $x = (a^\#)^*$ is a solution, then $a^\dagger(a^\#)^*(a^\dagger)^* = (a^\#)^*aa^\dagger$. Taking involution of the equality, we get $a^\dagger a^\#(a^\dagger)^* = aa^\dagger a^\# = a^\#$. Similar to the proof of Theorem 5.1 (2), we have $a \in R^{SEP}$.

(6) If $x = (a^\dagger)^*$ is a solution, then $a^\dagger(a^\dagger)^*(a^\dagger)^* = (a^\dagger)^*aa^\dagger$. Similar to the proof of Theorem 5.3 (3), we know that $a \in R^{SEP}$. □

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