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A Characterization of Nonnil-Projective Modules

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Abstract. Recently, Zhao, Wang, and Pu introduced and studied new concepts of nonnil-commutative diagrams and nonnil-projective modules. They proved that an R-module that is nonnil-isomorphic to a projective module is nonnil-projective, and they proposed the following problem: Is every nonnil-projective module nonnil-isomorphic to some projective module? In this paper, we delve into some new properties of nonnil-commutative diagrams and answer this problem in the affirmative.

1. Introduction

In this paper, all rings are assumed to be commutative with non-zero identity and all modules are assumed to be unitary. For a ring R, we denote by $\mathrm{Nil}(R)$ and $Z(R)$ the ideal of all nilpotent elements of R and the set of all zero-divisors of R, respectively. A ring R is called a PN-ring if $\mathrm{Nil}(R)$ is a prime ideal of R and a ZN-ring if $Z(R) = \mathrm{Nil}(R)$. An ideal I of R is said to be nonnil if $I \nsubseteq \mathrm{Nil}(R)$.

Recall from [4] that a prime ideal P of R is said to be divided if it is comparable to every ideal of R. Let $\mathcal{H} := \{R \mid R$ be a commutative ring, and $\mathrm{Nil}(R)$ be a divided prime ideal of $R\}$. If $R \in \mathcal{H}$, then R is called a ϕ-ring. A ϕ-ring is called a strongly ϕ-ring if it is also a ZN-ring. Recall from [1] that for a ϕ-ring R with total quotient ring $T(R)$, the map $\phi : T(R) \to R_{\mathrm{Nil}(R)}$ such that $\phi\left(\frac{b}{a}\right) = \frac{b}{a}$ is a ring

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homomorphism, and the image of R, denoted by $\phi(R)$, is a strongly ϕ-ring. The classes of ϕ-rings and strongly ϕ-rings are good extensions of integral domains to commutative rings with zero-divisors. In 2002, Badawi [6] generalized the concept of Noetherian rings to that of nonnil-Noetherian rings in which all nonnil ideals are finitely generated. He showed that a ϕ-ring R is nonnil-Noetherian if and only if $\phi(R)$ is nonnil-Noetherian, if and only if $R/\operatorname{Nil}(R)$ is a Noetherian domain. Generalizations of Dedekind domains, Prüfer domains, Bézout domains, pseudo-valuation domains, Krull domains, valuation domains, Mori domains, piecewise Noetherian domains, and coherent domains to the context of rings that are in the class $\mathcal{H}$ are also introduced and studied. We recommend [2, 3, 5, 7, 8, 9, 10, 11] for studying the ring-theoretic characterizations on ϕ-rings.

To investigate module-theoretic characterizations on ϕ-rings, the authors [12, 14, 17, 18, 19, 21] introduce nonnil-injective modules, ϕ-projective, and ϕ-flat modules, and characterize nonnil-Noetherian rings, ϕ-von Neumann regular rings, nonnil-coherent rings, ϕ-coherent rings, ϕ-Dedekind rings, and ϕ-Prüfer rings. Let M be an R-module and set

$$\operatorname{Ntor}(M) := \{x \in M \mid sx = 0 \text{ for some } s \in R \setminus \operatorname{Nil}(R)\}.$$

If $\operatorname{Ntor}(M) = M$, then M is called a ϕ-torsion module, and if $\operatorname{Ntor}(M) = 0$, then M is called a ϕ-torsion-free module. Recall from [18] that an R-module F is said to be ϕ-flat if for every R-monomorphism $f : A \longrightarrow B$ with $\operatorname{Coker}(f)$ being a ϕ-torsion R-module, we have $1_F \otimes_R f : F \otimes_R A \longrightarrow F \otimes_R B$ is an R-monomorphism; equivalently, $\operatorname{Tor}_1^R(F, M) = 0$ for every ϕ-torsion R-module M (see for instance [15, 16, 18]). If R is a PN-ring, define $\phi : R \to R_{\operatorname{Nil}(R)}$ by $\phi(r) = \frac{r}{1}$ for every $r \in R$. Then $\phi(R)$ is a ZN-ring. In [17], Zhao defined the map $\psi : M \to M_{\operatorname{Nil}(R)}$ by $\psi(x) = \frac{x}{1}$ for every $x \in M$. This makes $\psi(M)$ a $\phi(R)$-module. If $f : M \to N$ is a homomorphism of R-modules, then f induces naturally a $\phi(R)$-homomorphism $\widetilde{f} : \psi(M) \to \psi(N)$ such that $\widetilde{f}\left(\frac{x}{1}\right) = \frac{f(x)}{1}$ for $x \in M$. A sequence of R-modules and homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is called ϕ-exact if the $\phi(R)$-sequence: $\psi(A) \xrightarrow{\widetilde{f}} \psi(B) \xrightarrow{\widetilde{g}} \psi(C)$ is exact, and an R-module P is said to be ϕ-projective (resp., ϕ-free) if $\psi(P)$ is projective (resp., free) as a $\phi(R)$-module. Let R be a PN-ring and let $f : A \to B$ be a homomorphism of R-modules. Set

$$\operatorname{NKer}(f) := \{a \in A \mid sf(a) = 0 \text{ for some } s \in R \setminus \operatorname{Nil}(R)\} \quad \text{and}$$

$$\operatorname{NIm}(f) := \{b \in B \mid sb = sf(a) \text{ for some } a \in A \text{ and } s \in R \setminus \operatorname{Nil}(R)\}.$$

Because $\operatorname{Nil}(R)$ is prime, $\operatorname{NKer}(f)$ is a submodule of A, called the nonnil-kernel of f, and $\operatorname{NIm}(f)$ is a submodule of B, called the nonnil-image of f. We set $\operatorname{NCoker}(f) := B/\operatorname{NIm}(f)$. It is easy to verify that $\operatorname{Ker}(f) + \operatorname{Ntor}(A) \subseteq \operatorname{NKer}(f)$ and $\operatorname{Im}(f)+ \operatorname{Ntor}(B) = \operatorname{NIm}(f)$. Let A, B, C, D be R-modules and $f : A \to B, g : B \to D, h : A \to C, k : C \to D$ be homomorphisms of R-modules. Then the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{k} & D \end{array}$$

is said to be nonnil-commutative if $\mathrm{NIm}(gf - kh) = \mathrm{Ntor}(D)$; equivalently, $\mathrm{NKer}(gf - kh) = \mathrm{Ntor}(A)$. A sequence of R-modules and homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$ is called a nonnil-complex (resp., a nonnil-exact sequence) if it is ϕ-complex (resp., ϕ-exact); equivalently, $\mathrm{NIm}(f) \subseteq \mathrm{NKer}(g)$ (resp., $\mathrm{NIm}(f) = \mathrm{NKer}(g)$) according to [17, Theorem 2.6]. A homomorphism $f : A \to B$ of R-modules is called a nonnil-monomorphism if $\mathrm{NKer}(f) = \mathrm{Ntor}(A)$, equivalently $0 \to A \xrightarrow{f} B$ is a nonnil-exact sequence; f is called a nonnil-epimorphism if $\mathrm{NIm}(f) = B$ (i.e., $\mathrm{NCoker}(f) = 0$), equivalently $A \xrightarrow{f} B \to 0$ is a nonnil-exact sequence. Also f is called a nonnil-isomorphism if there exists a homomorphism $g : B \to A$ such that $\mathrm{NIm}\,(\mathbf{1}_A - gf) = \mathrm{Ntor}(A)$ and $\mathrm{NIm}\,(\mathbf{1}_B - fg) = \mathrm{Ntor}(B)$. If there exists a nonnil-isomorphism $f : A \to B$, we say that A and B are nonnil-isomorphic, denoted by $A \overset{N}{\simeq} B$. Note that if $f : A \to B$ is a nonnil-isomorphism, then f is both a nonnil-monomorphism and a nonnil-epimorphism. Interestingly, a homomorphism f of R-modules is both a nonnil-monomorphism and a nonnil-epimorphism without being a nonnil-isomorphism (see [20]). Following [20], an R-module P is said to be nonnil-projective if given any diagram of module homomorphisms

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle h}\swarrow & \downarrow{\scriptstyle f} & & \\ B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

with the bottom row nonnil-exact, there is a homomorphism $h : P \to B$ making this diagram nonnil-commutative. Also an R-module F_0 is said to be N-free if it is nonnil-isomorphic to a free module. Following [20, Theorem 3.7], an R-module is nonnil-projective if and only if it is a direct summand of an N-free module. If an R-module P is nonnil-isomorphic to a projective module, then P is nonnil-projective (cf. [20, Corollary 3.8]). Afterward, they proposed an interesting problem as follows.

Problem: Is every nonnil-projective module nonnil-isomorphic to some projective module?

One of the main aims of this paper is to answer this problem. Section 2 studies some new properties of nonnil-commutative diagrams and nonnil-exact sequences. In the last section, we solved the previous problem in the affirmative: An R-module is nonnil-projective if and only if it is nonnil-isomorphic to a projective module (Theorem 3.1 and Remark 3.6). In this paper, R **always denotes a PN-ring.**

2. On Nonnil-Commutative Diagrams

We start this section by providing a nonnil-analog of Five Lemma.

Theorem 2.1. *Consider the following nonnil-commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} D & \xrightarrow{h} & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{k} & E \\ {\scriptstyle\delta}\downarrow & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\mu}\downarrow \\ D' & \xrightarrow{h'} & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{k'} & E' \end{array}$$

(1) *If α and γ are nonnil-monomorphisms and δ is a nonnil-epimorphism, then β is a nonnil-monomorphism.*

(2) *If α and γ are nonnil-epimorphisms and μ is a nonnil-monomorphism, then β is a nonnil-epimorphism.*

Proof. (1) Let $b \in \mathrm{NKer}(\beta)$. Then there exists $t_1 \in R \setminus \mathrm{Nil}(R)$ such that $t_1\beta(b) = 0$. On the other hand, there exists $t_2 \in R \setminus \mathrm{Nil}(R)$ such that $t_2\gamma \circ g(b) = t_2 g' \circ \beta(b)$. Hence $t_1t_2\gamma \circ g(b) = t_2g'(t_1\beta(b)) = 0$. Therefore, $g(b) \in \mathrm{NKer}(\gamma)$. Since γ is a nonnil-monomorphism, there exists $t_3 \in R \setminus \mathrm{Nil}(R)$ such that $t_3g(b) = 0$, and so $b \in \mathrm{NKer}(g) = \mathrm{NIm}(f)$. Then $t_4b = t_4f(a)$ for some $a \in A$ and $t_4 \in R \setminus \mathrm{Nil}(R)$. Hence $t_4(\beta \circ f(a) - f' \circ \alpha(a)) = t_4(\beta(b) - f'(\alpha(a)))$. Since $a \in A$, it follows that $t_5(f' \circ \alpha(a) - \beta \circ f(a)) = 0$ for some $t_5 \in R \setminus \mathrm{Nil}(R)$. Therefore

$$\begin{aligned} 0 &= t_1t_4t_5(f' \circ \alpha(a) - \beta \circ f(a)) \\ &= -t_1t_5t_4(\beta(b) + f' \circ \alpha(a)) \\ &= -t_5t_4\beta(t_1b) + t_1t_4t_5f' \circ \alpha(a) \\ &= t_1t_4t_5f' \circ \alpha(a). \end{aligned}$$

Hence $\alpha(a) \in \mathrm{NKer}(f') = \mathrm{NIm}(h)$, and so $t_6\alpha(a) = t_6h'(x')$ for some $t_6 \in R\setminus\mathrm{Nil}(R)$ and $x' \in D'$. Since δ is a nonnil-epimorphism, there exist some $x \in D$ and $t_7 \in R \setminus \mathrm{Nil}(R)$ such that $t_7\delta(x) = t_7x'$. Hence

$$\begin{aligned} t_6t_7\alpha(a) &= t_7t_6h'(x') \\ &= t_6h'(t_7x') \\ &= t_6h'(t_7\delta(x)) \\ &= t_6t_7h' \circ \delta(x). \end{aligned}$$

On the other hand, since $x \in D$, it follows that $t_8h'\delta(x) = t_8\alpha h(x)$ for some $t_8 \in R\setminus\mathrm{Nil}(R)$. So $t_6t_7t_8\alpha(a) = t_6t_7t_8h'\circ\delta(x) = t_6t_7t_8\alpha\circ h(x)$, and hence $t_6t_7t_8\alpha(a-h(x)) = 0$. Therefore, $a - h(x) \in \mathrm{NKer}(\alpha) = \mathrm{Ntor}(A)$, and hence there exists $t_9 \in R \setminus \mathrm{Nil}(R)$ such that $t_9a = t_9h(x)$. Since $h(x) \in Im(h) \subseteq \mathrm{NIm}(h) = \mathrm{NKer}(f)$, we get $t_{10}f \circ h(x) = 0$ for some $t_{10} \in R \setminus \mathrm{Nil}(R)$. Then

$$\begin{aligned} t_4t_9t_{10}b &= t_9t_{10}t_4f(a) \\ &= t_{10}t_4f(t_9h(a)) \\ &= t_4t_9t_{10}f \circ h(a) = 0. \end{aligned}$$

Therefore, $tb = 0$ with $t := t_4t_9t_{10} \in R \setminus \mathrm{Nil}(R)$, and so $b \in \mathrm{Ntor}(B)$. Thus β is a nonnil-monomorphism.

(2) Let $b' \in B'$. Since γ is a nonnil-epimorphism, there exist $c \in C$ and $t_1 \in R \setminus \mathrm{Nil}(R)$ such that $t_1\gamma(c) = t_1g'(b')$. Nonnil-commutativity of the right square gives $t_2\mu \circ k(c) = t_2jk' \circ \gamma(c)$ for some $t_2 \in R \setminus \mathrm{Nil}(R)$. Then

$$\begin{aligned} t_1t_2\mu \circ k(c) &= t_2k'(t_1\gamma(c)) \\ &= t_2k'(t_1g'(b')) \\ &= t_1t_2k' \circ g'(b'). \end{aligned}$$

Since $g'(b') \in Im(g') \subseteq \mathrm{NIm}(g') = \mathrm{NKer}(k')$, there exists $t_3 \in R \setminus \mathrm{Nil}(R)$ such that $t_3k' \circ g'(b') = 0$, and so $t_1t_2t_3\mu \circ k(c) = 0$. Therefore, $k(c) \in \mathrm{NKer}(\gamma) = \mathrm{Ntor}(E)$. Consequently there exists $t_4 \in R \setminus \mathrm{Nil}(R)$ such that $t_4k(c) = 0$, and hence $c \in \mathrm{NKer}(k) = \mathrm{NIm}(g)$, that is, $t_5c = t_5g(b)$ for some $t_5 \in R \setminus \mathrm{Nil}(R)$ and $b \in B$. On the other hand, since $b \in B$, there exists $t_6 \in R \setminus \mathrm{Nil}(R)$ such that $t_6\gamma \circ g(b) = t_6g' \circ \beta(b)$. Then

$$\begin{aligned} t_1t_5t_6g'(b') &= t_1t_5t_6\gamma(c) \\ &= t_1t_6\gamma(t_5g(b) \\ &= t_1t_5t_6g' \circ \beta(b). \end{aligned}$$

Thus $t_1t_5t_6g'(b' - \beta(b)) = 0$, and so $b' - \beta(b) \in \mathrm{NKer}\, g' = \mathrm{NIm}(f')$. Hence there exist $t_7 \in R \setminus \mathrm{Nil}(R)$ and $a' \in A'$ such that $t_7(b' - \beta(b)) = t_7f'(a')$. Since α is a nonnil-epimorphism, there exist some $a \in A$ and $t_8 \in R \setminus \mathrm{Nil}(R)$ such that $t_8\alpha(a) = t_8a'$. Hence

$$t_8t_7(b' - \beta(b)) = t_8t_7f'(a') = t_7t_8f' \circ \alpha(a).$$

Since $a \in A$, there exists $t_9 \in R \setminus \mathrm{Nil}(R)$ such that $t_9f' \circ \alpha(a) = t_9\beta \circ f(a)$, and so $t_9t_8t_7(b' - \beta(b)) = t_7t_8t_9\beta \circ f(a)$. Thus $tb' = t\beta(b + f(a)$ with $t := t_7t_8t_9 \in R \setminus \mathrm{Nil}(R)$. Consequently β is a nonnil-epimorphism. □

Let M be an R-module. Define $\psi : M \to M_{\mathrm{Nil}(R)}$ such that $\psi(x) = \frac{x}{1}$ for every $x \in M$.

Proposition 2.2. *Let $f : A \to B$ be an R-module homomorphism. Then* $A/\mathrm{NKer}(f) \cong \psi(\mathrm{Im}(f))$.

Proof. Let $x, y \in A$. Then we have:

$$\begin{aligned} \frac{f(x)}{1} = \frac{f(y)}{1} \in \psi(\mathrm{Im}\,(f)) &\iff \exists s \in (R \setminus \mathrm{Nil}(R)) : sf(x) = sf(y) \\ &\iff \exists s \in (R \setminus \mathrm{Nil}(R)) : sf(x-y) = 0 \\ &\iff x - y \in \mathrm{NKer}(f) \\ &\iff \bar{x} = \bar{y} \in A/\,\mathrm{NKer}(f). \end{aligned}$$

Hence the homomorphism:

$$\begin{aligned} g : A/\,\mathrm{NKer}(f) &\to \psi(\mathrm{Im}\,(f)) \\ \bar{x} &\mapsto g(\bar{x}) = \frac{f(x)}{1} \end{aligned}$$

is an isomorphism. □

A nonempty subset S of R is said to be a multiplicative subset if $1 \in S$, $0 \notin S$, and for each $a, b \in S$, we have $ab \in S$. Note that if there exists $s \in S \cap \mathrm{Nil}(R)$, then there exists a positive integer n such that $0 = s^n \in S$, a contradiction. Hence **we always assume that** $S \cap \mathrm{Nil}(R) = \emptyset$**.**

It is well known that if $M' \xrightarrow{f} M \xrightarrow{g} M''$ is an exact sequence of R-modules, then $M'_S \xrightarrow{f_S} M_S \xrightarrow{g_S} M''_S$ is also exact. The following theorem gives the nonnil-version of this result.

Theorem 2.3. *Let R be a ring, S be a multiplicative subset of R, and $M' \xrightarrow{f} M \xrightarrow{g} M''$ be a nonnil-exact sequence of R-modules. Then $M'_S \xrightarrow{f_S} M_S \xrightarrow{g_S} M''_S$ is a nonnil-exact sequence.*

Proof. Let $\frac{y}{s} \in \mathrm{NIm}(f_S)$. Then there exist $\frac{t}{s_1} \in R_S \setminus \mathrm{Nil}(R_S)$ and $\frac{x'}{s'} \in M_S$ such that $\frac{t}{s_1}\frac{y}{s} = \frac{t}{s_1} f_S(\frac{x'}{s'}) = \frac{tf(x')}{s_1 s'}$. Thus there exists $s_2 \in S$ such that $s_2 t s' s_1 y = s_2 s_1 s t f(x') = s_2 t f(s_1 s x')$. Hence $s_1 s y \in \mathrm{NIm}(f) = \mathrm{NKer}(g)$ since $s_2 t \in R \setminus \mathrm{Nil}(R)$, and so $t' g(s_1 s y) = 0$ for some $t' \in R \setminus \mathrm{Nil}(R)$. Therefore, $\frac{t' g(y)}{s} = 0$, whence $\frac{t'}{1} g_S(\frac{y}{s}) = 0$ and $\frac{t'}{1} \in R_S \setminus \mathrm{Nil}(R_S)$. Thus $\frac{y}{s} \in \mathrm{NKer}(g_S)$.

Conversely, let $\frac{x}{s} \in \mathrm{NKer}(g_S)$. Then $\frac{t}{s_1}\frac{g(x)}{s} = 0$ for some $\frac{t}{s_1} \in R_S \setminus \mathrm{Nil}(R_S)$. Thus there exists $s_2 \in S$ such that $t s_2 f(x) = 0$, whence $s_2 x \in \mathrm{NKer}(f) = \mathrm{NIm}(g)$ since $t \in R \setminus \mathrm{Nil}(R)$, that is, $t_1 s_2 x = t_1 f(x')$ for some $x' \in M'$ and $t_1 \in R \setminus \mathrm{Nil}(R)$. Then

$$\frac{t_1 s_2}{1}\frac{x}{s} = \frac{t_1 f(x')}{s} = \frac{t_1 s_2 f(x')}{s_2 s} = \frac{t_1 s_2}{1} f_S(\frac{x'}{s_2 s}).$$

Thus $\frac{x}{s} \in \mathrm{NIm}(f_S)$ since $\frac{t_1 s_2}{1} \in R_S \setminus \mathrm{Nil}(R_S)$. □

Remark 2.4. If $S := R \setminus \mathrm{Nil}(R)$, then $M' \xrightarrow{f} M \xrightarrow{g} M''$ is a nonnil-exact sequence if and only if $M'_S \xrightarrow{f_S} M_S \xrightarrow{g_S} M''_S$ is exact.

Note that a nonnil-monomorphism is not always a monomorphism (see [17]). But if we consider K as a field and M as a K-vector space, and let $R = K \propto M$ be the trivial extension. Then the homomorphism $g : M \to R$ defined by $g(x) = (0, x)$ is not a nonnil-epimorphism; in fact, $(1, 0) \notin \mathrm{NIm}(g)$. Now we give an example of a nonnil-epimorphism which is not an epimorphism.

Example 2.5. Let $R = \mathbb{Z} \propto \mathbb{Z}/2\mathbb{Z}$ and consider $g : \mathbb{Z} \to R$ defined by $g(a) = (a, 0)$. Since $2(0 \propto \mathbb{Z}/2\mathbb{Z}) = 0$, it follows that $(0 \propto \mathbb{Z}/2\mathbb{Z}) \subseteq \mathrm{Ntor}(R)$. Then $\mathrm{NIm}(g) = \mathrm{Im}\,(g) + \mathrm{Ntor}(R) = R$. Hence g is a nonnil-epimorphism, which is not an epimorphism.

Proposition 2.6. *Let $f : M \to N$ be an R-module homomorphism and S be a multiplicative subset of R. Then the following statements are equivalent:*

(1) *f is a nonnil-monomorphism,*

(2) *f_S is a nonnil-monomorphism.*

Proof. (1) $\Rightarrow$ (2) This is straightforward by Theorem 2.3.

(2) $\Rightarrow$ (1) Assume that f_S is a nonnil-monomorphism. Set $M' := \mathrm{NKer}(f)$. Then we have the following nonnil-exact sequence: $0 \to M' \overset{i}{\to} M \overset{f}{\to} N$. Thus $0 \to M'_S \overset{i_S}{\to} M_S \overset{f_S}{\to} N_S$ is also a nonnil-exact sequence. Hence $\mathrm{Ntor}(M_S) + \mathrm{Im}\,(i_S) = \mathrm{NIm}(i_S) = \mathrm{NKer}(f_S) = \mathrm{Ntor}(M_S)$, and so $M'_S \subseteq \mathrm{Ntor}(M_S)$. Now let $x \in M'$. Then $\frac{x}{1} \in \mathrm{Ntor}(M_S)$, whence $\frac{t}{s_1}\frac{x}{1} = 0$ for some $\frac{t}{s_1} \in R_S \setminus \mathrm{Nil}(R_S)$. Thus $stx = 0$ for some $s \in S$. Since $\mathrm{Nil}(R)$ is a prime ideal of R, $st \in R \setminus \mathrm{Nil}(R)$, and so $x \in \mathrm{Ntor}(M)$. Therefore, $\mathrm{NKer}(f) = \mathrm{Ntor}(M)$. Consequently f is a nonnil-monomorphism. □

Proposition 2.7. *Let $f : M \to N$ be an R-module homomorphism. Then the following statements are equivalent:*

(1) *f is a nonnil-epimorphism,*

(2) *$f_{\mathfrak{p}}$ is a nonnil-epimorphism for any prime ideal $\mathfrak{p}$ of R,*

(3) *$f_{\mathfrak{m}}$ is a nonnil-epimorphism for any maximal ideal $\mathfrak{m}$ of R.*

Proof. (1) $\Rightarrow$ (2) Assume that f is a nonnil-epimorphism. Then $M \overset{f}{\to} N \to 0$ is nonnil-exact. Let $\mathfrak{p}$ be a prime ideal of R. Then for $S := R \setminus \mathfrak{p}$, we have $M_{\mathfrak{p}} \overset{f_{\mathfrak{p}}}{\to} N_{\mathfrak{p}} \to 0$ is nonnil-exact according to Theorem 2.3. Thus $f_{\mathfrak{p}}$ is a nonnil-epimorphism for any prime ideal $\mathfrak{p}$ of R.

(2) $\Rightarrow$ (3) This is straightforward.

(3) $\Rightarrow$ (1) Let $y \in N$. Then $\frac{y}{1} \in N_{\mathfrak{m}} = \mathrm{NIm}(f_{\mathfrak{m}})$ for any maximal ideal $\mathfrak{m}$ of R. Thus for every $\mathfrak{m} \in \mathrm{Max}(R)$, there exist $\frac{t_{\mathfrak{m}}}{\alpha_{\mathfrak{m}}} \in R_{\mathfrak{m}} \setminus \mathrm{Nil}(R_{\mathfrak{m}})$, $x \in M$, and $s_{\mathfrak{m}} \in R \setminus \mathfrak{m}$ such that $\frac{t_{\mathfrak{m}}}{\alpha_{\mathfrak{m}}}\frac{y}{1} = \frac{t_{\mathfrak{m}}}{\alpha_{\mathfrak{m}}} f_{\mathfrak{m}}(\frac{x}{s_{\mathfrak{m}}})$. So $s'_{\mathfrak{m}}\alpha_{\mathfrak{m}}t_{\mathfrak{m}}s_{\mathfrak{m}}y = s'_{\mathfrak{m}}\alpha_{\mathfrak{m}}t_{\mathfrak{m}}f(x)$ for some $s'_{\mathfrak{m}} \in R \setminus \mathfrak{m}$. Set $S := \{s_{\mathfrak{m}} \mid \mathfrak{m} \text{ is a maximal ideal of } R\}$. Since S generates R, there exist finite elements $s_{\mathfrak{m}_1}, \dots, s_{\mathfrak{m}_n}$ of S and $\alpha_1, \dots, \alpha_n \in R$ such that $1 =$

$\alpha_1 s_{\mathfrak{m}_1} + \cdots + \alpha_n s_{\mathfrak{m}_n}$. For all $i = 1, \ldots, n$, we have $s'_{\mathfrak{m}_i}\alpha_{\mathfrak{m}_i} t_{\mathfrak{m}_i} s_{\mathfrak{m}_i} y = s'_{\mathfrak{m}_i}\alpha_{\mathfrak{m}_i} t_{\mathfrak{m}_i} f(x)$, and so $s\alpha_{\mathfrak{m}_i} t_{\mathfrak{m}_i} s_{\mathfrak{m}_i} y = s\alpha_{\mathfrak{m}_i} t_{\mathfrak{m}_i} f(x)$ with $s := s'_{\mathfrak{m}_1} s'_{\mathfrak{m}_2} \cdots s'_{\mathfrak{m}_n}$. Then

$$\begin{aligned} s\alpha_{\mathfrak{m}_i} t_{\mathfrak{m}_i} y &= s\alpha_{\mathfrak{m}_i} t_{\mathfrak{m}_i}(\alpha_1 s_{\mathfrak{m}_1} + \cdots + \alpha_n s_{\mathfrak{m}_n}) y \\ &= s\alpha_{\mathfrak{m}_i} t_{\mathfrak{m}_i}\alpha_1 s_{\mathfrak{m}_1} y + \cdots + s\alpha_{\mathfrak{m}_i} t_{\mathfrak{m}_i}\alpha_n s_{\mathfrak{m}_n} y \\ &= s\alpha_{\mathfrak{m}_i} t_{\mathfrak{m}_i}\alpha_1 f(x) + \cdots + st\alpha_n f(x) \\ &= s\alpha_{\mathfrak{m}_i} t_{\mathfrak{m}_i} f(\alpha_1 x + \cdots + st\alpha_n x). \end{aligned}$$

Since $\operatorname{Nil}(R)$ is a prime ideal of R, it follows that $s\alpha_{\mathfrak{m}_i} t_{\mathfrak{m}_i} \in R \setminus \operatorname{Nil}(R)$. Therefore, $y \in \operatorname{NIm}(f)$. □

Recall that a ring R is called a ϕ-von Neumann regular ring if $R/\operatorname{Nil}(R)$ is a field [18, Theorem 4.1]. Note that if R is a ϕ-von Neumann regular ring, then every non-nilpotent element of R is a unit. We end this section with the following theorem, which characterizes when each nonnil-commutative diagram (resp., nonnil-exact sequence, nonnil-monomorphism, nonnil-epimorphism, nonnil-isomorphism) is commutative (resp., exact, monomorphism, epimorphism, isomorphism).

Theorem 2.8. *Let R be a ring. Then the following conditions are equivalent:*

(1) *Every nonnil-commutative diagram is commutative,*

(2) *Every nonnil-exact sequence is exact,*

(3) *Every nonnil-monomorphism is a monomorphism,*

(4) *Every nonnil-epimorphism is an epimorphism,*

(5) *Every nonnil-isomorphism is an isomorphism,*

(6) *R is a ϕ-von Neumann regular ring.*

Proof. (1) ⇒ (5), (2) ⇒ (3)&(5), and (6) ⇒ (2)&(3) are straightforward.

(3) ⇒ (6) Let $a \in R \setminus \operatorname{Nil}(R)$ and consider the following homomorphism $f : R/Ra \to 0$. Since $\operatorname{Ntor}(R/Ra) = R/Ra$, it follows that $R/Ra = \operatorname{Ntor}(R/Ra) \subseteq \operatorname{NKer}(f) \subseteq R/Ra$, and so $\operatorname{NKer}(f) = \operatorname{Ntor}(R/Ra)$. Hence f is a nonnil-monomorphism, and so it is a monomorphism by (3). Then $R/Ra = \operatorname{Ker}(f) = 0$, and hence a is a unit. Consequently $(R, \operatorname{Nil}(R))$ is a local ring. Hence $\operatorname{Nil}(R)$ is a divided prime ideal of R. Thus R is a ϕ-ring with $R/\operatorname{Nil}(R)$ being a field. Therefore, R is a ϕ-von Neumann regular ring by [18, Theorem 4.1].

(4) ⇒ (6) Let $a \in R \setminus \operatorname{Nil}(R)$ and consider the following homomorphism $f : 0 \to R/Ra$. Since $\operatorname{Ntor}(R/Ra) = R/Ra$, it follows that $\operatorname{NIm}(f) = Im(f) + \operatorname{Ntor}(R/I) = 0 + R/Ra = R/Ra$. Hence f is a nonnil-epimorphism, and so it is an epimorphism. Consequently $0 = \operatorname{Im}(f) = R/Ra$, and so a is a unit. Hence as in the above, R is a ϕ-von Neumann regular ring.

(5) ⇒ (6) Let $a \in R \setminus \operatorname{Nil}(R)$. Since $a(R/Ra) = 0$, it is easy to verify that $R/Ra \overset{N}{\simeq} 0$ (see Lemma 3.3), and so $R/Ra = 0$ by (5). Therefore, a is a unit, and so as in the above, R is a ϕ-von Neumann regular ring. □

3. Characterizing Nonnil-Projective Modules Using Projective Modules

The nonnil-projective module was studied in [20] using an N-free module, a right nonnil-split sequence, and a nonnil-projective basis. In particular, if an R-module P is nonnil-isomorphic to a projective module P_0, then P is nonnil-projective, and they conclude their paper by proposing the following problem.

Problem: Is every nonnil-projective module nonnil-isomorphic to some projective module?

The following theorem solves this difficulty by stating that an R-module is nonnil-projective if and only if it is nonnil-isomorphic to a projective module.

Theorem 3.1. *Let R be a ZN-ring. Then every nonnil-projective module is nonnil-isomorphic to some projective module.*

We need simple but necessary lemmas to prove Theorem 3.1.

Lemma 3.2. *If $A_1 \overset{N}{\simeq} B_1$ and $A_2 \overset{N}{\simeq} B_2$, then $A_1 \oplus A_2 \overset{N}{\simeq} B_1 \oplus B_2$.*

Proof. Let $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ be two nonnil-isomorphisms. Then there exist two homomorphisms $g_1 : B_1 \to A_1$ and $g_2 : B_2 \to A_2$ such that $\mathrm{NIm}(1_{A_1} - f_1 \circ g_1) = \mathrm{Ntor}(A_1)$, $\mathrm{NIm}(1_{B_1} - g_1 \circ f_1) = \mathrm{Ntor}(B_1)$, $\mathrm{NIm}(1_{A_2} - f_2 \circ g_2) = \mathrm{Ntor}(A_1)$, and $\mathrm{NIm}(1_{B_2} - g_2 \circ f_2) = \mathrm{Ntor}(B_2)$. Define

$$\begin{aligned} f : A_1 \oplus A_2 &\to B_1 \oplus B_2 \text{ by} \\ (x_1, x_2) &\mapsto f(x_1, x_2) = (f_1(x_1), f_2(x_2)) \end{aligned}$$

and

$$\begin{aligned} g : B_1 \oplus B_2 &\to A_1 \oplus A_2 \text{ by} \\ (x_1, x_2) &\mapsto g(x_1, x_2) = (g_1(x_1), g_2(x_2)). \end{aligned}$$

Then it is easy to verify that:

$$\begin{aligned} \mathrm{NIm}(1_{A_1 \oplus A_2} - f \circ g) &= \mathrm{NIm}(1_{A_1} - f_1 \circ g_1) \oplus \mathrm{NIm}(1_{A_2} - f_2 \circ g_2) \\ &= \mathrm{Ntor}(A_1) \oplus \mathrm{Ntor}(A_2) \\ &= \mathrm{Ntor}(A_1 \oplus A_2) \end{aligned}$$

and

$$\begin{aligned} \mathrm{NIm}(1_{B_1 \oplus B_2} - g \circ f) &= \mathrm{NIm}(1_{B_1} - g_1 \circ f_1) \oplus \mathrm{NIm}(1_{B_2} - g_2 \circ f_2) \\ &= \mathrm{Ntor}(B_1) \oplus \mathrm{Ntor}(B_2) \\ &= \mathrm{Ntor}(B_1 \oplus B_2). \end{aligned}$$

Hence $A_1 \oplus A_2 \overset{N}{\simeq} B_1 \oplus B_2$. □

Lemma 3.3. *Let M be an R-module. Then $M \overset{N}{\simeq} 0$ if and only if M is a ϕ-torsion R-module.*

Proof. Let $f : M \to 0$ be a nonnil-isomorphism. Then $\mathrm{NIm}(1_M - f \circ 0) = \mathrm{Ntor}(M)$. Since $\mathrm{NIm}(1_M - f \circ 0) = \mathrm{NIm}(1_M) = M$, we get $M = \mathrm{Ntor}(M)$.

Conversely, assume that $M = \mathrm{Ntor}(M)$. Then $f : M \to 0$ is a nonnil-isomorphism since $\mathrm{NIm}(1_M) = M = \mathrm{Ntor}(M)$. □

For any submodule N of an R-module M and any multiplicative subset S of R, we define

$$S^M(N) := \{x \in M \mid sx \in N \text{ for some } s \in S\},$$

called the S-component of N in M. If no confusion can arise, we will also write $S(N)$ instead of $S^M(N)$. From this point on, set $S := R \setminus \mathrm{Nil}(R)$.

Lemma 3.4. *Let $f : A \to B$ be a nonnil-isomorphism and N be a submodule of A. Then $S(N) \overset{N}{\simeq} f(S(N))$.*

Proof. Let $g : B \to A$ such that $\mathrm{NIm}(1_A - g \circ f) = \mathrm{Ntor}(A)$ and $\mathrm{NIm}(1_B - f \circ g) = \mathrm{Ntor}(B)$. Define $f_{S(N)} : S(N) \to f(S(N))$ as the restriction of f on $S(N)$. Let $y = f(n') \in f(S(N))$ with $n' \in N$. Then there exists $t_1 \in R \setminus \mathrm{Nil}(R)$ such that $t_1 n' \in N$. On the other hand, since $\mathrm{NIm}(1_A - g \circ f) = \mathrm{Ntor}(A)$, we get $n' - (g \circ f)(n') \in \mathrm{Ntor}(A)$. Then $t_2 n' = t_2 (f \circ g)(n')$ for some $t_2 \in R \setminus \mathrm{Nil}(R)$, and hence $t_2 t_1 g(y) = t_2 t_1 n' \in N$. Therefore, $f(y) \in S(N)$ and it is easy to verify that $\mathrm{NIm}(1_{S(N)} - g_{f(S(N))} \circ f_{S(N)}) = \mathrm{Ntor}(S(N))$ and $\mathrm{NIm}(1_{f(S(N))} - f_{S(N)} \circ g_{f(S(N))}) = \mathrm{Ntor}(f(S(N)))$. Hence $S(N) \overset{N}{\simeq} f(S(N))$. □

Lemma 3.5. *If N is a direct summand of A, then $S(N) \overset{N}{\simeq} N$.*

Proof. Let $A = N \oplus L$ for some submodule L of A. Let $x = n + l \in S(N)$ with $n \in N$ and $l \in L$. Then $tx = tn + tl \in N$ for some $t \in R \setminus \mathrm{Nil}(R)$. Then $tl = tx - tn \in N \cap L = 0$, and so $tl = 0$, that is, $t \in \mathrm{Ntor}(L)$. Therefore, $S(N) \subseteq N \oplus \mathrm{Ntor}(L)$.

Conversely, let $x = n + l \in N \oplus \mathrm{Ntor}(L)$. Then $tl = 0$ for some $t \in R \setminus \mathrm{Nil}(R)$. Hence $tx = tn \in N$, and so $x \in S(N)$. Consequently $S(N) = N \oplus \mathrm{Ntor}(L)$. Since $\mathrm{Ntor}(L) \overset{N}{\simeq} 0$ by Lemma 3.3, $S(N) \overset{N}{\simeq} N$ according to Lemma 3.2. □

Proof of Theorem 3.1. Let P be a nonnil-projective module. Then by [20, Theorem 3.7], P is a direct summand of an N-free module. Hence there is a free R-module F such that $A = P \oplus L$ is nonnil-isomorphic to F. Let $f : A \to F$ be a nonnil-isomorphism. Our aim now is to show that $F = f(P) \oplus f(L)$. For this, let $g : F \to A$ such that $\mathrm{NIm}(1_A - g \circ f) = \mathrm{Ntor}(A)$ and $\mathrm{NIm}(1_F - f \circ g) = \mathrm{Ntor}(F)$. Since F is a free R-module and $Z(R) = \mathrm{Nil}(R)$, it follows from [13, Example 1.6.12 (1)] that

$\mathrm{Ntor}(F) = \mathrm{tor}(F) = 0$, and hence $\mathrm{Im}\,(1_F - f\circ g) \subseteq \mathrm{NIm}(1_F - f\circ g) = \mathrm{Ntor}(F) = 0$. Therefore, f is an epimorphism, that is, $F = f(A)$. Consequently $F = f(P) + f(L)$.

Let $y \in f(P) \cap f(L)$. Then there exist $x \in P$ and $l \in L$ such that $y = f(x) = f(l)$. Thus $f(x-l) = 0$, and so $x - l \in \mathrm{Ker}\,(f) \subseteq \mathrm{NKer}(f) = \mathrm{Ntor}(A)$. Then there exists a non-nilpotent element $t \in R$ such that $tx = tl$. Since $tx = tl \in P \cap L = 0$, it follows that $tx = 0$, whence $ty = f(tx) = 0$. Then $y = 0$ since F is a free R-module. Thus $F = f(P) \oplus f(L)$. Therefore, $f(P)$ is a projective R-module. By Lemma 3.5, $P \overset{N}{\simeq} S(P)$, and then $P \overset{N}{\simeq} f(S(P))$ according to Lemma 3.4. Note that $f(S(P)) = f(P \oplus \mathrm{Ntor}(L)) = f(P) + f(\mathrm{Ntor}(L))$. Since $f(\mathrm{Ntor}(L)) \subseteq \mathrm{Ntor}(F) = 0$, we get $f(S(P)) = f(P)$. Thus $P \overset{N}{\simeq} f(P)$ and $f(P)$ is a projective R-module. □

Note that Lemma 3.2 can be used to provide another demonstration of [20, Corollary 3.8] as shown below.

Remark 3.6. If P is nonnil-isomorphic to a projective module, then P is nonnil-projective.

Proof. Let K be a projective module such that $P \overset{N}{\simeq} K$. Since K is projective, it is a direct summand of a free module F, and so $F = K \oplus L$ for some L. Since $P \overset{N}{\simeq} K$, it follows from Lemma 3.2 that $P \oplus L \overset{N}{\simeq} K \oplus L = F$. Hence P is a direct summand of an N-free module. Then P is a nonnil-projective module by [20, Theorem 3.7]. □

Lemma 3.7. *Let R be a ring. If $A_1 \overset{N}{\simeq} B_1$ and $A_2 \overset{N}{\simeq} B_2$, then $A_1 \otimes A_2 \overset{N}{\simeq} B_1 \otimes B_2$.*

Proof. Let $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ be two nonnil-isomorphisms. Then there exist two homomorphisms $g_1 : B_1 \to A_1$ and $g_2 : B_2 \to A_2$ such that $\mathrm{NIm}(1_{B_1} - f_1 \circ g_1) = \mathrm{Ntor}(B_1)$, $\mathrm{NIm}(1_{A_1} - g_1 \circ f_1) = \mathrm{Ntor}(A_1)$, $\mathrm{NIm}(1_{B_2} - f_2 \circ g_2) = \mathrm{Ntor}(B_2)$, and $\mathrm{NIm}(1_{A_2} - g_2 \circ f_2) = \mathrm{Ntor}(A_2)$. Set $A := A_1 \otimes A_2$, $B := B_1 \otimes B_2$, $f := f_1 \otimes f_2$, and $g := g_1 \otimes g_2$. Then for every $(a_1 \otimes a_2) \in A_1 \otimes A_2$ (resp., $(b_1 \otimes b_2) \in B_1 \otimes B_2$) we have $f(a_1 \otimes a_2) = f_1(a_1) \otimes f_2(a_2)$ (resp., $g(b_1 \otimes b_2) = g_1(b_1) \otimes g_2(b_2)$). By [13, Example 2.2.10] we get that $f \circ g = (f_1 \circ g_1) \otimes (f_2 \circ g_2)$. Our aim now is to show that $\mathrm{NIm}(1_A - g \circ f) = \mathrm{Ntor}(A)$ and $\mathrm{NIm}(1_B - f \circ g) = \mathrm{Ntor}(B)$. Since $\mathrm{Im}(1_A - g\circ f) + \mathrm{Ntor}(A) = \mathrm{NIm}(1_A - g\circ f)$, to show that $\mathrm{NIm}(1_A - g\circ f) = \mathrm{Ntor}(A)$, it is enough to show that $\mathrm{Im}(1_A - g\circ f) \subseteq \mathrm{Ntor}(A)$. Let $a_1 \otimes a_2) \in A_1 \otimes A_2$. Then there exist $s_1, s_2 \in R\backslash \mathrm{Nil}(R)$ such that $s_1(g_1 \circ f_1(a_1) - a_1) = 0$ and $s_2(g_2 \circ f_2(a_2) - a_2) = 0$. Thus $g \circ f(a_1 \otimes a_2) - a_1 \otimes a_2 = g_1 \circ f_1(a_1) \otimes g_2 \circ f_2(a_2) - a_1 \otimes a_2$, which implies that $s(g \circ f(a_1 \otimes a_2) - a_1 \otimes a_2) = 0$ with $s = s_1 s_2 \in R \setminus \mathrm{Nil}(R)$. Similarly, we can deduce that $\mathrm{Im}(1_A - g\circ f) \subseteq \mathrm{Ntor}(A)$, since $\mathrm{Ntor}(A)$ is a submodule of A. Therefore $\mathrm{NIm}(1_A - g\circ f) = \mathrm{Ntor}(A)$. Likewise, we can deduce that $\mathrm{NIm}(1_B - f\circ g) = \mathrm{Ntor}(B)$. □

Corollary 3.8. *Let R is a ZN-ring and let P_1 and P_2 be nonnil-projective R-modules. Then $P_1 \otimes P_2$ is nonnil-projective.*

Proof. Let P_1' and P_2' be projective modules such that $P_1 \overset{N}{\simeq} P_1'$ and $P_2 \overset{N}{\simeq} P_2'$. Then by Lemma 3.7, $P_1 \otimes P_2 \overset{N}{\simeq} P_1' \otimes P_2'$. Since P_1' and P_2' are projective modules, $P_1' \otimes P_2'$ is projective by [13, Theorem 2.3.8]. Hence $P_1 \otimes P_2$ is nonnil-projective. □

Corollary 3.9. *Let R be a local ring. Then every nonnil-projective module is N-free.*

Proof. Let P be a nonnil-projective R-module. Then there exists a projective R-module P_0 such that $P \overset{N}{\simeq} P_0$. Since R is a local ring, P is free by [13, Theorem 2.3.17]. Hence P is nonnil-isomorphic to a free R-module. Thus P is N-free. □

Theorem 3.10. *Let R be a ZN-ring and I be a nonnil-projective nonnil-ideal of R. Then I is finitely generated.*

Proof. Let I be a nonnil-projective nonnil-ideal of R. Then by [20, Theorem 3.9], there exist elements $\{x_i \mid i \in \Gamma\} \subseteq I$ and R-homomorphisms $\{f_i \mid i \in \Gamma\} \subseteq \mathrm{Hom}_R(I, R)$ such that:

(1) If $x \in I$, then almost all $f_i(x) = 0$,

(2) If $x \in I$, then there exists an element $s \in R \setminus \mathrm{Nil}(R)$ such that $sx = s\sum_i f_i(x)x_i$.

Let $a \in I$ be a non-nilpotent element. Then there exists a finite subset K of Γ such that $f_i(a) = 0$ for all $i \in \Gamma \setminus K$. Now let $x \in I$. Then there exists an element $s \in R \setminus \mathrm{Nil}(R)$ such that $sx = s\sum_i f_i(x)x_i$. Hence $asx = as\sum_i f_i(x)x_i = s\sum_i xf_i(a)x_i = s\sum_{k\in K} xf_k(a)x_k = sa\sum_{k\in K} f_k(x)x_k$. Since sa is regular, we conclude that $x = \sum_{k\in K} f_k(x)x_k$. Therefore, $I = \sum_{k\in K} Rx_k$ is finitely generated. □

Let M be a nonnil-torsion-free R-module. Then M is nonnil-projective if and only if M is projective by [20, Lemma 4.1]. In particular, if R is a ZN-ring and I is an ideal of R, then I is nonnil-projective if and only if I is projective. Note that if I is a nil ideal (i.e, $I \subseteq \mathrm{Nil}(R)$), then I is not projective by [13, Proposition 6.7.12], and so it is not nonnil-projective. It is well known that in an integral domain every projective ideal is finitely generated according to [13, Corollary 5.2.7]. The following corollary gives a generalization of this fact.

Corollary 3.11. *Let R be a ZN-ring. Then every projective ideal of R is finitely generated.*

We know that every projective module is flat. So a natural question is whether a nonnil-projective module is ϕ-flat. The following example shows that a nonnil-projective module is not always ϕ-flat.

Example 3.12. Let R be a ring with $\mathrm{w.gl.dim}(R) \geq 2$ (for example $R = k[X, Y]$ with k a field). Then there exists a non-zero ideal I of R such that R/I is not flat.

Hence R/I is not a ϕ-flat module, but it is nonnil-projective since $R/I \overset{N}{\simeq} 0$.

Remark 3.13. Note that a nonnil-projective module is not necessarily ϕ-flat. However, if every R-module is nonnil-projective, then every R-module is ϕ-flat by [20, Theorem 4.5].

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