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APPLICATION OF GEGENBAUER POLYNOMIALS TO CERTAIN CLASSES OF BI-UNIVALENT FUNCTIONS OF ORDER $\nu + i\varsigma$

Omar Alnajar*, Ala Amourah, and Maslina Darus

Abstract. In this paper, a new class of bi-univalent functions that are described by Gegenbauer polynomials is presented. We obtain the estimates of the Taylor-Maclaurin coefficients $|m_2|$ and $|m_3|$ for each function in this class of bi-univalent functions. In addition, the Fekete–Szegö problems function new are also studied.

1. preliminaries

Legendre presented polynomials in his study in 1784 [6]. They are used extensively in analysis to solve boundary value problems like the Laplace equation and the Schrodinger equation by use of polynomial interpolation and approximation. They also have many further applications.

Legendre polynomials have several applications in physics, and are of them is the investigation of spherical harmonics and the construction of gravitational potentials.

Later in the 19th century, a scientist named Leopold Gegenbauer [9], who was of German and Czech ancestry, suggested a fresh set of orthogonal polynomials. is a nice example of a polynomial that has orthogonal qualities in its features. These orthogonal polynomials have a weight function that is more general and are specified on the interval $[-1, 1]$. The Fekete–Szegö inequality is a well-known inequality in mathematics that has applications to holomorphic one-to-one functions, complex analysis, and univalent (or holomorphic) functions. The inequality bears the names of the two mathematicians who independently proved it in the early 20th century, Michael Fekete and George–Szegö [21]. In 1967, Lewin [22] discovered that $|m_2| \leq 1.51$ while researching the bi-univalent function class Σ.

Many areas of mathematics, like complex analysis and geometry, use a class of analytical functions called "bi-univalent functions," which are defined in the complex plane. The bi-univalent functions are a kind of extension of the univalent functions.

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∗ Corresponding author.

They are responsible for mapping the unit disk of the complex plane to a certain area.

Continued study of bi-univalent functions has yielded several significant insights and advances in complex analysis and geometry [5, 20, 27]. In general

Let $\mathcal{A}$ be the class of functions f of the form

$$f(\Im) = \Im + m_2\Im^2 \cdots = \Im + \sum_{k=2}^{\infty} m_k \Im^k, \quad (\Im \in \mathfrak{h}). \tag{1}$$

which are analytical on the disk $\mathfrak{h} = \{\Im \in \mathbb{C} : |\Im| < 1\}$. We also call $\mathcal{S}$ the subclass of $\mathcal{A}$ made up of functions of Eq. (1) that are also univalent in $\mathfrak{h}$.

Differential subordination of analytical functions gives the field of geometric function theory many powerful tools that can be very helpful. Porwal and Darus presented a paper on the bi-univalent [25]. Additionally, see [10], which carries out analytical operations on the disk denoted by $\mathfrak{h} = \{\Im \in \mathbb{C} : |\Im| < 1\}$. $\mathcal{S}$ is also the name given to the subclass of $\mathcal{A}$ that is composed of functions of Eq (1) that are also univalent in $\mathfrak{h}$.

Every mathematical function $f \in \mathcal{S}$ has an inverse, denoted by f^{-1}.

$$f^{-1}(f(\Im)) = \Im \qquad (\Im \in \mathfrak{h})$$

and

$$\aleph = f(f^{-1}(\aleph)) \qquad (|\aleph| < r_0(f);\ r_0(f) \geq 0.25)$$

where

$$g(\aleph) = f^{-1}(\aleph) = \aleph - m_2\aleph^2 + (-m_3 + 2m_2^2)\aleph^3 - (m_4 + 5m_2^3 - 5m_3m_2)\aleph^4 + \cdots . \tag{2}$$

A function is said to be bi-univalent in $\mathfrak{h}$ if both $f(\Im)$ and $f^{-1}(\Im)$ are univalent in $\mathfrak{h}$. This is because both of these functions evaluate to the same value.

Let Σ stand for the class of bi-univalent in $\mathfrak{h}$ given by (1). The following are some examples from the class Σ.

$$\frac{\Im}{1-\Im}, \qquad \log\frac{1}{1-\Im}.$$

On the other hand, the well-known Koebe function is not included in Σ. The following are some additional examples of standard functions found in $\mathfrak{h}$:

$$\frac{2\Im - \Im^2}{2} \text{ and } \frac{\Im}{1-\Im^2},$$

do not belong to Σ either. See reference [17], for interesting subclasses of functions in class Σ. Following Srivastava et al. [16], foundational work, several new subclasses of the bi-univalent function class Σ were established, was a catalyst for the creation of several unique subclasses inside the bi-univalent function class and estimations for the initial two coefficients, $|m_2|$ and $|m_3|$ in the Taylor-Maclaurin series expansion (1), which were obtained in [8, 12–15, 17–19, 26, 28].

Amourah et al. [4] The following Gegenbauer polynomial generating function was the subject of a study that was carried out in the year 2020.

$$H_\alpha(\gamma, \Im) = \frac{1}{(1 - 2\gamma\Im + \Im^2)^\alpha}, \tag{3}$$

where $\gamma \in [-1, 1]$ and $\Im \in \mathfrak{h}$ play a role. When γ is held constant, the function H_α is analytic in $\mathfrak{h}$, making its expansion through the Taylor series straightforward, as demonstrated by the following example.

$$H_\alpha(\gamma, \Im) = \sum_{k=0}^{\infty} C_k^\alpha(\gamma)\Im^k, \tag{4}$$

where $C_k^\alpha(\gamma)$ stands for a polynomial of degree k that belongs to the Gegenbauer family.

When $\alpha = 0$, it is very clear that H_α does not achieve anything. Because of this, the generating function of the Gegenbauer polynomial is defined as follows:

$$H_0(\gamma, \Im) = 1 - \log\left(1 - 2\gamma\Im + \Im^2\right) = \sum_{k=0}^{\infty} C_k^0(\gamma)\Im^k, \tag{5}$$

for $\alpha = 0$. Furthermore, it should be emphasized that, as stated in [23], it is preferred for the normalization to be greater than -0.5. The following recurrence relations are also applicable to the definition of Gegenbauer polynomials.

$$C_k^\alpha(\gamma) = \frac{1}{k}\left[2\gamma\left(k + \alpha - 1\right)C_{k-1}^\alpha(\gamma) - \left(k + 2\alpha - 2\right)C_{k-2}^\alpha(\gamma)\right], \tag{6}$$

keeping initial estimates in mind

$$C_0^\alpha(\gamma) = 1,\ C_1^\alpha(\gamma) = 2\alpha\gamma \text{ and } C_2^\alpha(\gamma) = 2\alpha\left(1 + \alpha\right)\gamma^2 - \alpha. \tag{7}$$

Chebyshev polynomials and Legendre polynomials are special cases of Gegenbauer polynomials $C_k^\alpha(\gamma)$. When $\alpha = 1$, and when $\alpha = 1/2$, Chebyshev polynomials and Legendre polynomials are special cases of Gegenbauer polynomials $C_k^\alpha(\gamma)$.

In recent years, a significant number of researchers have focused their attention on researching connections that exist between univalent functions, orthogonal polynomials, and bi-univalent functions, see references [1–3, 7, 8, 11, 24].

Before determining bounds for the $|m_2|$ and $|m_3|$ Taylor-Maclaurin coefficients, we first define and then construct a new subclass that is related to Gegenbauer polynomials. The Fekete-Szego functional difficulties are also addressed for functions that belong to this new class. which, part, are formed by Amourah et al. [4]

2. Definition and examples

In this section, the family $\mathfrak{G}_{\Sigma}^{\alpha}(\sigma,\beta,\nu,\varsigma,\gamma)$ is defined as follows:

Definition 2.1. For $\nu>0$, $\varsigma\in\mathbb{R}$, $\nu+i\varsigma\neq 0$, $0\leq\sigma\leq 1$ and $0\leq\beta\leq 1$, a function $f\in\Sigma$ is said to be in the family $\mathfrak{G}_{\Sigma}^{\alpha}(\sigma,\beta,\nu,\varsigma,\gamma)$ if it satisfies the subordinations:

$$\left(\frac{\Im f'(\Im)}{f(\Im)}\right)^{\nu+i\varsigma}\left[(1-\beta)\frac{\Im f'(\Im)}{f(\Im)}+\beta\left(1+\frac{\Im f''(\Im)}{f'(\Im)}\right)\right]^{\sigma}\prec H_{\alpha}(\gamma,\Im)$$

and

$$\left(\frac{\aleph g'(\aleph)}{g(\aleph)}\right)^{\nu+i\varsigma}\left[(1-\beta)\frac{\aleph g'(\aleph)}{g(\aleph)}+\beta\left(1+\frac{\aleph g''(\aleph)}{g'(\aleph)}\right)\right]^{\sigma}\prec H_{\alpha}(\gamma,\aleph).$$

The function $g(\aleph)=f^{-1}(\aleph)$ is defined by (2) where $\gamma\in(\frac{1}{2},1]$ and H_{α} is the generating function of the Gegenbauer polynomials given by (3).

Special cases:

i) Assume that $\nu>0$, $\varsigma\in\mathbb{R}$, $\nu+i\varsigma\neq 0$, $0\leq\sigma\leq 1$ and $\beta=0$, a function $f\in\Sigma$ is said to be in the family $\mathfrak{G}_{\Sigma}^{\alpha}(\sigma,0,\nu,\varsigma,\gamma)$ if it satisfies the subordinations:

$$\left(\frac{\Im f'(\Im)}{f(\Im)}\right)^{\nu+i\varsigma}\left[\frac{\Im f'(\Im)}{f(\Im)}\right]^{\sigma}\prec H_{\alpha}(\gamma,\Im)$$

and

$$\left(\frac{\aleph g'(\aleph)}{g(\aleph)}\right)^{\nu+i\varsigma}\left[\frac{\aleph g'(\aleph)}{g(\aleph)}\right]^{\sigma}\prec H_{\alpha}(\gamma,\aleph),$$

where $\gamma\in(\frac{1}{2},1]$.

ii) Assume that For $\nu>0$, $\varsigma\in\mathbb{R}$, $\nu+i\varsigma\neq 0$, $0\leq\sigma\leq 1$ and $\beta=1$, a function $f\in\Sigma$ is said to be in the family $\mathfrak{G}_{\Sigma}^{\alpha}(\sigma,1,\nu,\varsigma,\gamma)$ if it satisfies the subordinations:

$$\left(\frac{\Im f'(\Im)}{f(\Im)}\right)^{\nu+i\varsigma}\left[1+\frac{\Im f''(\Im)}{f'(\Im)}\right]^{\sigma}\prec H_{\alpha}(\gamma,\Im)$$

and

$$\left(\frac{\aleph g'(\aleph)}{g(\aleph)}\right)^{\nu+i\varsigma}\left[1+\frac{\aleph g''(\aleph)}{g'(\aleph)}\right]^{\sigma}\prec H_{\alpha}(\gamma,\aleph),$$

where $\gamma\in(\frac{1}{2},1]$.

3. Coefficient bounds of the class $\mathfrak{G}_{\Sigma}^{\alpha}(\sigma,\beta,\nu,\varsigma,\gamma)$

Here we gave the upper bound for $|m_2|$ and $|m_3|$ of the class $\mathfrak{G}_{\Sigma}^{\alpha}(\sigma,\beta,\nu,\varsigma,\gamma)$ provided in Definition 2.1 as follows.

Theorem 3.1. *For* $\nu>0$, $\varsigma\in\mathbb{R}$, $\nu+i\varsigma\neq 0$, $0\leq\sigma\leq 1$ *and* $0\leq\beta\leq 1$, *let* $f\in\mathcal{A}$ *be in the family* $\mathfrak{G}_{\Sigma}^{\alpha}(\sigma,\beta,\nu,\varsigma,\gamma)$. *Then*

$$|m_2|\leq\frac{2\,|\alpha\gamma|\,\sqrt{|\alpha\gamma|}}{\sqrt{\left|2\left[\begin{array}{c}\alpha\left((\nu+i\varsigma)\left(\nu+i\varsigma+1\right)+\Upsilon(\sigma,\beta,\nu,\varsigma)\right)\\-\left(1+\alpha\right)\left(\nu+i\varsigma+\sigma(\beta+1)\right)^{2}\end{array}\right]\gamma^{2}+\left(\nu+i\varsigma+\sigma(\beta+1)\right)^{2}\right|}},$$

and

$$|m_3| \leq \frac{2\,|\alpha\gamma|}{2\sqrt{(\nu+\sigma(2\beta+1))^2+\varsigma^2}} + \frac{4\alpha^2\gamma^2}{\left[(\nu+\sigma(\beta+1))^2+\varsigma^2\right]},$$

where

(8) $$\Upsilon(\sigma,\beta,\nu,\varsigma) = \sigma(\beta+1)\left[2(\nu+i\varsigma+1)+(\sigma-1)(\beta+1)\right].$$

Proof. Let $f \in \mathfrak{G}_{\Sigma}^{\alpha}(\sigma,\beta,\nu,\varsigma,\gamma)$. Then there are two holomorphic functions ϕ,φ given by

(9) $$\phi(\Im) = c_1\Im + c_2\Im^2 + c_3\Im^3 + \cdots \quad (\Im \in \mathfrak{h})$$

and

(10) $$\varphi(\aleph) = d_1\aleph + d_2\aleph^2 + d_3\aleph^3 + \cdots \quad (\aleph \in \mathfrak{h}),$$

with $\phi(0)=\varphi(0)=0$, $|\phi(\Im)|<1$, $|\varphi(\aleph)|<1$, $\Im,\aleph \in \mathfrak{h}$ such that

(11) $$\left(\frac{\Im f'(\Im)}{f(\Im)}\right)^{\nu+i\varsigma}\left[(1-\beta)\frac{\Im f'(\Im)}{f(\Im)}+\beta\left(1+\frac{\Im f''(\Im)}{f'(\Im)}\right)\right]^{\sigma} = H_\alpha(\gamma,\phi(\Im))$$

and

(12) $$\left(\frac{\aleph g'(\aleph)}{g(\aleph)}\right)^{\nu+i\varsigma}\left[(1-\beta)\frac{\aleph g'(\aleph)}{g(\aleph)}+\beta\left(1+\frac{\aleph g''(\aleph)}{g'(\aleph)}\right)\right]^{\sigma} = H_\alpha(\gamma,\varphi(\aleph)).$$

Combining (9), (10) and (11) yields

(13) $$\left(\frac{\Im f'(\Im)}{f(\Im)}\right)^{\nu+i\varsigma}\left[(1-\beta)\frac{\Im f'(\Im)}{f(\Im)}+\beta\left(1+\frac{\Im f''(\Im)}{f'(\Im)}\right)\right]^{\sigma} = 1 + C_1^{\alpha}(\gamma)c_1\Im + \left[C_1^{\alpha}(\gamma)c_2 + C_2^{\alpha}(\gamma)c_1^2\right]\Im^2 + \cdots$$

and

(14) $$\left(\frac{\aleph g'(\aleph)}{g(\aleph)}\right)^{\nu+i\varsigma}\left[(1-\beta)\frac{\aleph g'(\aleph)}{g(\aleph)}+\beta\left(1+\frac{\aleph g''(\aleph)}{g'(\aleph)}\right)\right]^{\sigma} = 1 + C_1^{\alpha}(\gamma)d_1\aleph + \left[C_1^{\alpha}(\gamma)d_2 + C_2^{\alpha}(\gamma)d_1^2\right])\aleph^2 + \cdots.$$

It is quite well-known that if $|\phi(\Im)|<1$ and $|\varphi(\aleph)|<1$, $\Im,\aleph \in \sigma$, then

(15) $$|c_i| \leq 1 \quad \text{and} \quad |d_i| \leq 1 \ \text{ for all } i \in \mathbb{N}.$$

Comparing the corresponding coefficients in (13), after simplifying, we have

(16) $$(\nu+i\varsigma+\sigma(\beta+1))\,m_2 = C_1^{\alpha}(\gamma)c_1,$$

(17) $$2\,(\nu+i\varsigma+\sigma(2\beta+1))\,m_3 + \frac{1}{2}\left[(\nu+i\varsigma)\,(\nu+i\varsigma-1)+\sigma(\beta+1)\begin{pmatrix} 2\,(\nu+i\varsigma) \\ +(\sigma-1)(\beta+1)\end{pmatrix} - 2\,(\nu+i\varsigma+\sigma(3\beta+1))\right]m_2^2 = C_1^{\alpha}(\gamma)c_2 + C_2^{\alpha}(\gamma)c_1^2,$$

Comparing the corresponding coefficients in (14), after simplifying, we have

(18) $$-\,(\nu+i\varsigma+\sigma(\beta+1))\,m_2 = C_1^{\alpha}(\gamma)d_1,$$

and

$$2\left(\nu+i\varsigma+\sigma(2\beta+1)\right)\left(2m_2^2-m_3\right) + \frac{1}{2}\left[\left(\nu+i\varsigma\right)\left(\nu+i\varsigma-1\right)+\sigma(\beta+1)\begin{pmatrix} 2\left(\nu+i\varsigma\right) \\ +(\sigma-1)(\beta+1)\end{pmatrix}-2\left(\nu+i\varsigma+\sigma(3\beta+1)\right)\right]m_2^2$$

(19)

$$= C_1^{\alpha}(\gamma)d_2 + C_2^{\alpha}(\gamma)d_1^2.$$

It follows from (16) and (18) that

$$c_1 = -d_1 \tag{20}$$

and

$$2\left(\nu+i\varsigma+\sigma(\beta+1)\right)^2 m_2^2 = \left[C_1^{\alpha}(\gamma)\right]^2\left(c_1^2+d_1^2\right). \tag{21}$$

If we add (17) to (19), we find that

$$\begin{aligned}&\left[\left(\nu+i\varsigma\right)\left(\nu+i\varsigma+1\right)+\sigma(\beta+1)\left(2(\nu+i\varsigma+1)+(\sigma-1)(\beta+1)\right)\right]m_2^2 \\ &= C_1^{\alpha}(\gamma)\left(c_2+d_2\right)+C_2^{\alpha}(\gamma)\left(c_1^2+d_1^2\right).\end{aligned} \tag{22}$$

Substituting the value of $c_1^2+d_1^2$ from (21) in the right hand side of (22), we deduce that

(23)

$$m_2^2 = \frac{\left[C_1^{\alpha}(\gamma)\right]^3\left(c_2+d_2\right)}{\left[C_1^{\alpha}(\gamma)\right]^2\left(\left(\nu+i\varsigma\right)\left(\nu+i\varsigma+1\right)+\Upsilon(\sigma,\beta,\nu,\varsigma)\right)-2C_2^{\alpha}(\gamma)\left(\nu+i\varsigma+\sigma(\beta+1)\right)^2}.$$

where $\Upsilon(\sigma,\beta,\nu,\varsigma)$ is given by (8).
Further computations using (7), (15) and (23), we obtain

$$|m_2| \le \frac{2\left|\alpha\gamma\right|\sqrt{\left|\alpha\gamma\right|}}{\sqrt{\left|2\left[\begin{array}{c}\alpha\left(\left(\nu+i\varsigma\right)\left(\nu+i\varsigma+1\right)+\Upsilon(\sigma,\beta,\nu,\varsigma)\right)\\ -\left(1+\alpha\right)\left(\nu+i\varsigma+\sigma(\beta+1)\right)^2\end{array}\right]\gamma^2+\left(\nu+i\varsigma+\sigma(\beta+1)\right)^2\right|}}.$$

Next, if we subtract (19) from (17), we can easily see that

$$4\left(\nu+i\varsigma+\sigma(2\beta+1)\right)\left(m_3-m_2^2\right) = C_1^{\alpha}(\gamma)(c_2-d_2)+C_2^{\alpha}(\gamma)(c_1^2-d_1^2). \tag{24}$$

In view of (20) and (21), we get from (24)

$$m_3 = \frac{C_1^{\alpha}(\gamma)(c_2-d_2)}{4\left(\nu+i\varsigma+\sigma(2\beta+1)\right)}+\frac{\left[C_1^{\alpha}(\gamma)\right]^2\left(c_1^2+d_1^2\right)}{2\left(\nu+i\varsigma+\sigma(\beta+1)\right)^2}.$$

Thus applying (7), we obtain

$$|m_3| \le \frac{2\left|\alpha\gamma\right|}{2\sqrt{\left(\nu+\sigma(2\beta+1)\right)^2+\varsigma^2}}+\frac{4\alpha^2\gamma^2}{\left[\left(\nu+\sigma(\beta+1)\right)^2+\varsigma^2\right]}.$$

Theorem's proof is now complete. □

4. Fekete–Szegö inequality for the class $\mathfrak{G}_{\Sigma}^{\alpha}(\sigma,\beta,\nu,\varsigma,\gamma)$

In this section, we prove the following Fekete–Szegö inequality for functions f in the class $\mathfrak{G}_{\Sigma}^{\alpha}(\sigma,\beta,\nu,\varsigma,\gamma)$, using the values of m_2^2 and m_3.

THEOREM 4.1. *For $\nu>0$, $\varsigma\in\mathbb{R}$, $\nu+i\varsigma\neq 0$, $0\le\sigma\le 1$, $0\le\beta\le 1$ and $\eta\in\mathbb{R}$, let $f\in\mathcal{A}$ be in the family $\mathfrak{G}_{\Sigma}^{\alpha}(\sigma,\beta,\nu,\varsigma,\gamma)$. Then*

$$\left|m_3-\eta m_2^2\right|\le\begin{cases}\frac{|\alpha\gamma|}{\sqrt{(\nu+\sigma(2\beta+1))^2+\varsigma^2}}, & |\eta-1|\le|\Omega(\alpha,\nu,\varsigma,\beta)|\\ \frac{8\alpha^2|\gamma|^3|1-\eta|}{\left|2\alpha\gamma^2((\nu+i\varsigma)(\nu+i\varsigma+1)+\Upsilon(\sigma,\beta,\nu,\varsigma))-(2(1+\alpha)\gamma^2-1)(\nu+i\varsigma+\sigma(\beta+1))^2\right|}, & |\eta-1|\ge|\Omega(\alpha,\nu,\varsigma,\beta)|\end{cases}$$

where

$$\Omega(\alpha,\nu,\varsigma,\beta)=\frac{2\alpha\gamma^2\left((\nu+i\varsigma)(\nu+i\varsigma+1)+\Upsilon(\sigma,\beta,\nu,\varsigma)\right)-\left(2(1+\alpha)\gamma^2-1\right)(\nu+i\varsigma+\sigma(\beta+1))^2}{8\alpha\gamma^2\left(\nu+i\varsigma+\sigma(2\beta+1)\right)}.$$

Proof. It follows from (23) and (24) that

$$\begin{aligned}&m_3-\eta m_2^2\\&=\frac{C_1^{\alpha}(\gamma)(c_2-d_2)}{4\left(\nu+i\varsigma+\sigma(2\beta+1)\right)}+(1-\eta)\,m_2^2\\&=\frac{C_1^{\alpha}(\gamma)(c_2-d_2)}{4\left(\nu+i\varsigma+\sigma(2\beta+1)\right)}+\frac{\left[C_1^{\alpha}(\gamma)\right]^3(c_2+d_2)(1-\eta)}{\left[C_1^{\alpha}(\gamma)\right]^2\begin{pmatrix}(\nu+i\varsigma)(\nu+i\varsigma+1)\\+\Upsilon(\sigma,\beta,\nu,\varsigma)\end{pmatrix}-2C_2^{\alpha}(\gamma)\left(\nu+i\varsigma+\sigma(\beta+1)\right)^2}\\&=C_1^{\alpha}(\gamma)\left[\left(\psi(\mu,r)+\frac{1}{4\left(\nu+i\varsigma+\sigma(2\beta+1)\right)}\right)c_2+\left(\psi(\mu,r)-\frac{1}{4\left(\nu+i\varsigma+\sigma(2\beta+1)\right)}\right)d_2\right],\end{aligned}$$

where

$$\psi(\eta,\gamma)=\frac{\left[C_1^{\alpha}(\gamma)\right]^2(1-\eta)}{\left[C_1^{\alpha}(\gamma)\right]^2\left((\nu+i\varsigma)(\nu+i\varsigma+1)+\Upsilon(\sigma,\beta,\nu,\varsigma)\right)-2C_2^{\alpha}(\gamma)\left(\nu+i\varsigma+\sigma(\beta+1)\right)^2}.$$

According to (7), we find that

$$\left|m_3-\eta m_2^2\right|\le\begin{cases}\frac{2\left|C_1^{\alpha}(\gamma)\right|}{4(\nu+i\varsigma+\sigma(2\beta+1))} & 0\le|\psi(\eta,\gamma)|\le\frac{1}{4(\nu+i\varsigma+\sigma(2\beta+1))},\\ 2\left|C_1^{\alpha}(\gamma)\right|\left|\psi(\eta,\gamma)\right| & |\psi(\eta,\gamma)|\ge\frac{1}{4(\nu+i\varsigma+\sigma(2\beta+1))}.\end{cases}$$

This completes the proof of Theorem 4.1. □

5. Corollaries and consequences

Here we find two corollaries where $\beta=0$ and $\beta=1$, respectively.

Corollary 5.1. *For* $\nu > 0$, $\varsigma \in \mathbb{R}$, $\nu + i\varsigma \neq 0$, $0 \leq \sigma \leq 1$ *and* $\beta = 0$, *let* $f \in \mathcal{A}$ *be in the family* $\mathfrak{G}^{\alpha}_{\Sigma}(\gamma, 0, \nu, \varsigma, \gamma)$. *Then*

$$|m_2| \leq \frac{2\left|\alpha\gamma\right|\sqrt{\left|\alpha\gamma\right|}}{\sqrt{\left|2\left[\begin{array}{c}\alpha\left(\left(\nu+i\varsigma\right)\left(\nu+i\varsigma+1\right)+\Upsilon(\sigma,0,\nu,\varsigma)\right)\\ -\left(1+\alpha\right)\left(\nu+i\varsigma+\sigma\right)^2\end{array}\right]\gamma^2+\left(\nu+i\varsigma+\sigma\right)^2\right|}},$$

$$|m_3| \leq \frac{2\left|\alpha\gamma\right|}{2\sqrt{\left(\nu+\sigma\right)^2+\varsigma^2}} + \frac{4\alpha^2\gamma^2}{\left[\left(\nu+\sigma\right)^2+\varsigma^2\right]},$$

and

$$\left|m_3 - \eta m_2^2\right| \leq \begin{cases} \frac{|\alpha\gamma|}{\sqrt{(\nu+\sigma)^2+\varsigma^2}}, & |\eta-1| \leq |\Omega(\alpha,\nu,\varsigma,0)| \\ \frac{8\alpha^2|\gamma|^3|1-\eta|}{\left|2\alpha\gamma^2((\nu+i\varsigma)(\nu+i\varsigma+1)+\sigma[2(\nu+i\varsigma+1)+(\sigma-1)])-(-1+(2+2\alpha)\gamma^2)(\nu+i\varsigma+2\sigma)^2\right|}, & |\eta-1| \geq |\Omega(\alpha,\nu,\varsigma,0)| \end{cases}$$

where

$$\Omega(\alpha,\nu,\varsigma,0) = \frac{2\alpha\gamma^2\left(\left(\nu+i\varsigma\right)\left(\nu+i\varsigma+1\right)+\Upsilon(\sigma,0,\nu,\varsigma)\right)-\left(-1+(2+2\alpha)\gamma^2\right)\left(\nu+i\varsigma+\sigma\right)^2}{8\alpha\gamma^2\left(\nu+i\varsigma+\sigma\right)}.$$

Corollary 5.2. *For* $\nu > 0$, $\varsigma \in \mathbb{R}$, $\nu + i\varsigma \neq 0$, $0 \leq \sigma \leq 1$ *and* $\beta = 1$, *let* $f \in \mathcal{A}$ *be in the family* $\mathfrak{G}^{\alpha}_{\Sigma}(\sigma, 1, \nu, \varsigma, \gamma)$. *Then*

$$|m_2| \leq \frac{2\left|\alpha\gamma\right|\sqrt{\left|\alpha\gamma\right|}}{\sqrt{\left|2\left[\begin{array}{c}\alpha\left(\left(\nu+i\varsigma\right)\left(\nu+i\varsigma+1\right)+\Upsilon(\sigma,1,\nu,\varsigma)\right)\\ -\left(1+\alpha\right)\left(\nu+i\varsigma+2\sigma\right)^2\end{array}\right]\gamma^2+\left(\nu+i\varsigma+2\sigma\right)^2\right|}},$$

$$|m_3| \leq \frac{2\left|\alpha\gamma\right|}{2\sqrt{\left(\nu+3\sigma\right)^2+\varsigma^2}} + \frac{4\alpha^2\gamma^2}{\left[\left(\nu+2\sigma\right)^2+\varsigma^2\right]},$$

and

$$\left|m_3 - \eta m_2^2\right| \leq \begin{cases} \frac{|\alpha\gamma|}{\sqrt{(\nu+3\sigma)^2+\varsigma^2}}, & |\eta-1| \leq |\Omega(\alpha,\nu,\varsigma,1)| \\ \frac{8\alpha^2|\gamma|^3|1-\eta|}{\left|2\alpha\gamma^2((\nu+i\varsigma)(\nu+i\varsigma+1)+2\sigma[2(\nu+i\varsigma+1)+2(\sigma-1)])-(-1+(2+2\alpha)\gamma^2)(\nu+i\varsigma+2\sigma)^2\right|}, & |\eta-1| \geq |\Omega(\alpha,\nu,\varsigma,1)| \end{cases}$$

where

$$\Omega(\alpha,\nu,\varsigma,1) = \frac{2\alpha\gamma^2\left(\left(\nu+i\varsigma\right)\left(\nu+i\varsigma+1\right)+\Upsilon(\sigma,1,\nu,\varsigma)\right)-\left(-1+(2+2\alpha)\gamma^2\right)\left(\nu+i\varsigma+2\sigma\right)^2}{8\alpha\gamma^2\left(\nu+i\varsigma+3\sigma\right)}.$$

Remark 5.3. The findings of this study would lead to a number of other novel findings for the classes $\mathfrak{G}^{1}_{\Sigma}(\sigma,\beta,\nu,\varsigma,\gamma)$for Chebyshev polynomials and $\mathfrak{G}^{0.5}_{\Sigma}(\sigma,\beta,\nu,\varsigma,\gamma)$for Legendre polynomials.

Concluding Remark: The problems with the coefficients that are associated with each new subclass $\mathfrak{G}^{\alpha}_{\Sigma}(\sigma,\beta,\nu,\varsigma,\gamma)$, of the class of bi-univalent functions on the open unit disk $\mathfrak{h}$, have been presented in this work and investigated. These problems can be found in the class of bi-univalent functions. The definitions for this bi-univalent function class can be found in reference 2.1. The estimates of the Taylor-Maclaurin coefficients $|m_2|$ and $|m_3|$, as well as the Fekete-Szego functional problems for functions belonging to this new subclass, have been obtained. The estimations presented here are applicable to functions that fall into each of the bi-univalent function classes. The number of factors that were evaluated for our original results was reduced, and as a consequence, a number of new outcomes have been demonstrated to follow. The problem of obtaining estimates on the boundaries of the value $|m_n|$ for $n \geq 4; n \in \mathbb{N}$, for the classes stated in this paragraph still exists.

6. Statements and Declarations

6.1. Ethical Approval. Not applicable.

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6.4. Availability of Data and Materials. A statement on how any datasets used can be accessed.

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Omar Alnajar
Department of Mathematical Sciences, Faculty of Science and Technology,
Universiti Kebangsaan Malaysia, Bangi 43600, Malaysia
E-mail: P117246@siswa.ukm.edu.my

Ala Amourah
Mathematics Education Program, Faculty of Education and Arts, Sohar University,
Sohar 3111, Oman.
E-mail: AAmourah@su.edu.om

Maslina Darus
Department of Mathematical Sciences, Faculty of Science and Technology,
Universiti Kebangsaan Malaysia, Bangi 43600, Malaysia
E-mail: maslina@ukm.edu.my