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ON LACUNARY Δ^m-STATISTICAL CONVERGENCE IN G-METRIC SPACES

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Abstract. The aim of this research is to describe lacunary Δ^m-statistically convergent sequences with respect to metrics on generalised metric spaces (g-metric spaces) and to look into the fundamental characteristics of this statistical form of convergence. Also, the relationship between strong summability and lacunary Δ^m-statistical convergence in g-metric space is established at the end.

1. Inroduction and preliminaries

The idea of distance function can be generalised in a number of ways. One of them is the G-metric space notion, a new generalisation of the ordinary metric that has been researched by Mustafa et al. [26]. Distances between three points are the metrics in this space. For more generalization, Choi et al. [6] introduced g-metric with degree n, that is a distance between $n+1$ points.

In 1935, Zygmund [33] first brought the concept of statistical convergence. The notion of statistical convergence was formally introduced by Steinhaus [32] and Fast [12] in 1951. Since then, a number of mathematicians have looked into the characteristics of convergence and statistical convergence and applied them to a variety of fields, including approximation theory [9], finitely additive set functions [8], sequence space [14, 16–19], paranormed spaces [3, 4], b-metric spaces [15, 21], p-metric spaces [7,23,30], probability theory [13], quaternion G-metric [10], probabilistic generalised metric space [1], summability theory [6, 31], and recently in g-metric spaces [2]. Recently, a novel research has been conducted on weighted rough statistical convergence [22], weighted means of double sequence [24], dunkl analog of Szasz operators [25, 29], other approximating and fractional operators [5, 27, 28] .

Now, we recall some definitions and preliminaries that are needed in the rest of this paper.

We will use the standard notation. By the symbol $\mathbb{R}$ we will denote the set of real numbers while $\mathbb{N}$ stands for the set of natural numbers.

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Definition 1.1. [26] Let M be a non-empty set and let $G : M \times M \times M \to R_+$be a function satisfying the following axioms:

(i) If $G(x, y, z) = G(y, z, x) = G(z, x, y) = 0$ if $x = y = z$
(ii) $G(x, x, y) > 0$ for all $x, y \in M$, where $x \neq y$,
(iii) $G(x, x, z) \leq G(x, y, z)$ for all $x, y, z \in M$, with $z \neq y$,
(iv) $G(x, y, z) = G(p\{x, y, z\})$, where p is permutation of x, y, z (symmetry),
(v) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in M$ (rectangle inequality).

Then the function G is called a generalized metric, or more specifically G-metric on M, and the pair (M, G) is called G-metric space.

The following is an extension of G-metric space with degree l.

Definition 1.2. [6] Let M be a non-empty set. A function $g : M^{l+1} \to \mathbb{R}+$ is called a g-metric space with order l on M if it satisfies the following conditions:

(i) $g(x_0, x_1, x_2, \ldots, x_l) = 0$ if and only if $x_0 = x_1 = \ldots = x_l$,
(ii) $g(x_0, x_1, x_2, \ldots, x_l) = g\left(x_{\sigma(0)}, x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(l)}\right)$ for permutation σ on $\{0, 1, 2, \ldots l\}$,
(iii) $g(x_0, x_1, x_2, \ldots, x_l) \leq g(y_0, y_1, y_2, \ldots, y_l)$ for all $(x_0, x_1, x_2, \ldots, x_l), (y_0, y_1, y_2, \ldots, y_l) \in M^{l+1}$ with $\{x_i : i = 0, 1, \ldots l\} \subseteq \{y_i : i = 0, 1, \ldots l\}$,
(iv) For all $x_0, x_1, \ldots, x_s, y_0, y_1, \ldots y_t, w \in M$ with $s + t + 1 = l$,
$g(x_0, x_1, x_2, \ldots, x_s, y_0, y_1, y_2, \ldots, y_t)$
$\leq g(x_0, x_1, x_2, \ldots, x_s, w, w, \ldots w) + g(y_0, y_1, y_2, \ldots, y_t, w, w \ldots w)$.

The pair (M, g) is called g-metric space. For $l = 1, 2$ respectively, it is respectively equivalent to metric and G-metric space.

The following theorem is required in the proof of main results.

Theorem 1.3. [6] *Let (M, g) be a g-metric with order n on a nonempty set M. Then following are true:*

1. $g(\underbrace{x, \ldots, x}_{s \text{ times}}, y, \ldots, y) \leq g(\underbrace{x, \ldots, x}_{s \text{ times}}, v, \ldots, v) + g(\underbrace{x, \ldots, x}_{s \text{ times}}, y, \ldots, y)$,
2. $g(x, y, \ldots, y) \leq g(x, v, \ldots, v) + g(v, y, \ldots, y)$,
3. $g(\underbrace{x, \ldots, x}_{s \text{ times}}, v, \ldots, v) \leq sg(x, v, \ldots, v)$ *and* $g(\underbrace{x, \ldots, x}_{s \text{ times}}, v, \ldots, v)$
$\leq (n + 1 - s)g(v, x, \ldots, x)$,
4. $g(x_0, x_1, \ldots, x_n) \leq \sum_{i=0}^{n} g(x_i, v, \ldots, v)$
5. $|g(y, x_1, \ldots, x_n) - g(v, x_1, \ldots, x_n)| \leq \max\{g(y, v, \ldots, v), g(v, y, \ldots, y)\}$,
6. $g(\underbrace{x, \ldots, x}_{s \text{ times}}, v, \ldots, v) - g(\underbrace{x, \ldots, x}_{s' \text{ times}}, v, \ldots, v) \leq |s - s'|\, g(x, v, \ldots, v)$,
7. $g(x, v, \ldots, v) \leq (1 + (s - 1)(n + 1 - s))g(\underbrace{x, \ldots, x}_{s \text{ times}}, v, \ldots, v)$.

A subset E of the set $\mathbb{N}$ of natural numbers is said to have a "natural density" $\delta(E)$ if

$$\delta(E) = \lim_n \frac{1}{n} |\{k \leq n : k \in E\}|$$

where the vertical bars denote the cardinality of the enclosed set.

The number sequence $x = (x_k)$ is said to be statistically convergent to number l if for each $\epsilon > 0$,

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - l| \geq \epsilon\}| = 0$$

and x is said to be statistically cauchy sequence if for every $\epsilon > 0$ there exists a number $N = N(\epsilon)$ such that

$$\lim_n \frac{1}{n} |\{k \leq n : |x_k - x_N| \geq \epsilon\}| = 0.$$

The following definition was given by R.Abazari [1].

Definition 1.4. [1] Let $p \in \mathbb{N}$ and $k \in \mathbb{N}^P$ and $K(n) = \{(i_1, i_2, \ldots, i_p) \leq n (n \in \mathbb{N}) : (i_1, i_2, \ldots, i_p) \in K\}$, then

$$\delta_{(p)(K)} = \lim_{n\to\infty} \frac{p!}{n^p} |K(n)|$$

is called p-dimensional asymptotic (or natural density) of the set K.

Definition 1.5. [2] Let $\{x_p\}$ be a sequence in a g-metric space (M, g).
(i) $\{x_p\}$ is statistically convergent to x, if for all $\epsilon > 0$,

$$\begin{aligned} &\delta_t \left(\left| \left\{(j_1, j_2, \ldots, j_t) \in \mathbb{N}^t : \quad j_1, j_2, \ldots, j_t \leq p, \quad g\left(x, x_{j_1}, x_{j_2}, \ldots, x_{j_t} < \epsilon\right\} \right|\right) \\ &= \lim_{p\to\infty} \frac{t!}{p^t} \left| \left\{(j_1, j_2, \ldots, j_t) \in \mathbb{N}^t : \quad j_1, j_2, \ldots, j_t \leq p, \quad g\left(x, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right) < \epsilon\right\} \right| \\ &= 1 \end{aligned}$$

and is denoted by , $gs - \lim_{p\to\infty} x_p = x$ or $x_p \xrightarrow{gs} x$.
(ii) $\{x_p\}$ is statistically g-Cauchy, if for all $\epsilon > 0, \exists j_\epsilon \in \mathbb{N}$ such that

$$\lim_{p\to\infty} \frac{t!}{p^t} \left| \left\{(j_1, j_2, \ldots, j_t) \in \mathbb{N}^t : \quad j_1, j_2, \ldots, j_t \leq p, \quad g\left(x_{j_\epsilon}, \left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right) < \epsilon\right\} \right| = 1.$$

Kizmaz [20] introduced the difference sequence space $Z(\Delta)$ as given below

$$Z(\Delta) = \{y = (y_k) : (\Delta y_k) \in Z\}$$

for $Z = \ell_\infty, c, c_0$ i.e. spaces of all bounded, convergent and null sequences respectively, where $\Delta_y = (\Delta y_k) = (y_k - y_{k+1})$. In particular, $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ are also Banach spaces, relative to a norm induced by $\|y\|_\Delta = |y_1| + \sup_k |\Delta y_k|$.

The generalized difference sequence spaces $Z(\Delta^m)$ was introduced by M.Et et.al. [11] as follows :

$$Z(\Delta^m) = \{y = (y_k) : (\Delta^m y_k) \in Z\}$$

for $Z = \ell_\infty, c, c_0$ where $\Delta^m(y) = (\Delta^m y_k) = (\Delta_{m-1} y_k - \Delta_{m-1} y_{k+1})$. So that $\Delta^m y_k = \sum_{r=0}^{p} (-1)^r \begin{pmatrix} m \\ r \end{pmatrix} x_{k+r}$.

2. Lacunary Δ^m statistical convergence

This section defines lacunary Δ^m-statistically convergent sequences in g-metric spaces and examines some of its fundamental properties.

By a lacunary sequence, we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $n \to \infty$.

Let $t \in \mathbb{N}$ and $k \in \mathbb{N}^t$ and $K(p) = \{(i_1, i_2, \ldots, i_t) \leq (p \in \mathbb{N}) : (i_1, i_2, \ldots, i_t) \in K\}$, then

$$\delta_t^\theta(K) = \lim_{p\to\infty} \frac{t!}{p^t}|K(p)|$$

is called t-dimensional asymptotic (or natural) θ-density of the set K.

DEFINITION 2.1. Let $\{x_p\}$ be a sequence in a g-metric space (M, g) and θ be a lacunary sequence.

(i) $\{x_p\}$ is lacunary Δ^m statistically convergent to x, if for all $\epsilon > 0$,

$$\delta_t^\theta\left(\left|\left\{(j_1, j_2, \ldots, j_t) \in \mathbb{N}^t : \quad j_1, j_2, \ldots, j_t \leq p, \quad g\left(l, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right) < \epsilon\right\}\right|\right)$$
$$= \lim_{p\to\infty} \frac{t!}{p^t}\left| \left\{(j_1, j_2, \ldots, j_t) \in \mathbb{N}^t : \quad j_1, j_2, \ldots, j_t \leq p, \quad g\left(l, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right) < \epsilon\right\}\right|$$
$$= 1$$

and is denoted by , $gt\Delta^m s - \lim_{p\to\infty} x_p = x$ or $x_p \xrightarrow{gt\Delta^m s} x$.

(ii) $\{x_p\}$ is lacunary Δ^m-statistically g-Cauchy, if for all $\epsilon > 0, \exists j_\epsilon \in \mathbb{N}$ such that

$$\delta_t^\theta\left(\lim_{p\to\infty} \frac{t!}{p^t}\left|\left\{(j_1, j_2, \ldots, j_t) \in \mathbb{N}^t : j_1, j_2, \ldots, j_t \leq p, g\left(x_{j_\epsilon}, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right)\right) < \epsilon\right\}\right|\right) = 1.$$

THEOREM 2.2. *Every convergent sequence is lacunary Δ^m-statistically convergent in g-metric spaces.*

Proof. Consider the metric space (M, g) and let $\{x_p\}$ be a sequence in it such that $\{x_p\}$ converges to x. Now, for $\epsilon > 0$ there exists $p_0 \in \mathbb{N}$ such that for all $j_1, j_2, \ldots, j_t \geq p_0$,

$$g\left(x, x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right) < \epsilon.$$

Set

$$Z(p) := \left\{(j_1, j_2, \ldots, j_t) \in\in \mathbb{N}^t : \quad j_1, j_2, \ldots, j_t \leq p, \quad \left(x, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right)\right) < \epsilon\right\}$$

then

$$|Z(p)| \geq \binom{p - p_0}{t}$$

and

$$\lim_{p\to\infty} \frac{t!|z(p)|}{p^t} \geq \lim_{p\to\infty} \frac{t!}{p^t}\binom{p - p_0}{t} = 1.$$

So

$$gt\Delta^m s - \lim_{p\to\infty} x_p = x.$$

□

The converse of the above theorem is provably false in the example that follows.

EXAMPLE 2.3. Let $M = \mathbb{R}$ and g be the metric as follows;

$$g : \mathbb{R}^3 \to \mathbb{R}$$
$$g(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\}.$$

Consider the following sequence,

$$\Delta^m x_l = \begin{cases} l, & \text{if } l \text{ is a square} \\ \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $\Delta^m \{x_l\}$ is lacunary Δ^m-statistically convergent while it is not convergent normally.

The following theorem establishes the uniqueness of the lacunary Δ^m-statistical limit in g-metric space.

THEOREM 2.4. *Let $\{x_p\}$ be a sequence in g-metric space (M,g) and θ be the lacunary sequence such that $x_p \xrightarrow{gt\Delta^m s} x$ and $x_p \xrightarrow{gt\Delta^m s} y$, then $x = y$.*

Proof. Set,

$$P(\epsilon) := \left\{(j_1, j_2, \ldots, j_t) \in \mathbb{N}^t : \quad g\left(x, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right)\right) \geq \frac{\epsilon}{2t}\right\}$$
$$Q(\epsilon) := \left\{(j_1, j_2, \ldots, j_t) \in \mathbb{N}^t : \quad g\left(y, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right)\right) \geq \frac{\epsilon}{2t}\right\}$$

where $\epsilon > 0$ is arbitrary.

Since, $x_p \xrightarrow{gt\Delta^m s} x$ and $x_p \xrightarrow{gt\Delta^m s} y$, therefore $\delta_t^\theta(P(\epsilon)) = 0$ and $\delta_t^\theta(Q(\epsilon)) = 0$. Let $R(\epsilon) := P(\epsilon) \cup Q(\epsilon)$, then $\delta_t^\theta(R(\epsilon)) = 0$, hence $\delta_t^\theta\left(P^c(\epsilon)\right) = 1$.

Suppose $(j_1, j_2, \ldots, j_t) \in P^c(\epsilon)$, then by Theorem (1.3) we have

$$\begin{aligned} g(x, y, y, \ldots, y) &\leq g\left(x, \Delta^m\left(x_{j_1}, x_{j_1}, \ldots, x_{j_1}\right)\right) + g\left(\Delta^m\left(x_{j_1}\right), y, y, \ldots, y\right) \\ &\leq g\left(x, \Delta^m\left(x_{j_1}, x_{j_1}, \ldots, x_{j_1}\right)\right) + t\left(g\left(y, \Delta^m\left(x_{j_1}, x_{j_1}, \ldots, x_{j_1}\right)\right)\right) \\ &\leq g\left(x, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right)\right) + t\left(g\left(y, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right)\right)\right) \\ &\leq t\left(g\left(x, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right)\right)\right) + g\left(y, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right)\right) \\ &< t\left(\frac{\epsilon}{2t} + \frac{\epsilon}{2t}\right) = \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we get

$$g(x, y, y, \ldots, y) = 0$$

therefore $x = y$. □

DEFINITION 2.5. A set $P = \{p_l : l \in \mathbb{N}\}$ is said to be lacunary statistically dense in $\mathbb{N}$, if the set

$$P(p) = \left\{(j_1, j_2, \ldots, j_t) \in \mathbb{N}^t : j_i \in P \quad j_1, j_2, \ldots, j_t \leq p\right\}$$

has asymptotic θ-density 1 i.e.,

$$\delta_t^\theta = \lim_{p \to \infty} \frac{t!|P(p)|}{p^t} = 1.$$

DEFINITION 2.6. A subsequence $\{\Delta^m x_{p_l}\}$ of a sequence $\{\Delta^m x_p\}$ in a g-metric space (M, g) is lacunary Δ^m statistically dense, if the index set $\{p_l : l \in \mathbb{N}\}$ is lacunary Δ^m statistically dense subset of $\mathbb{N}$. i.e.,

$$\delta_t^\theta (p_l : k \in \mathbb{N}) = 1.$$

We now prove the following Theorem in g-metric spaces.

THEOREM 2.7. *Let $\{x_p\}$ be a sequence in a lacunary Δ^m-statistically convergent g-metric space (M, g) and θ be a lacunary sequence. Then the following are equivalent.*

1. *$\{x_p\}$ is lacunary Δ^m-statistically convergent in (M, g).*
2. *There is a convergent sequence $\{y_p\}$ in M such that $x_p = y_p$ for almost all $p \in \mathbb{N}$.*
3. *There is a lacunary Δ^m-statistically dense subsequence $\{x_{p_l}\}$ of $\{x_p\}$ such that $\{x_{p_l}\}$ is convergent.*
4. *There is a lacunary Δ^m-statistically dense subsequence $\{x_{p_l}\}$ of $\{x_p\}$ such that $\{x_{p_l}\}$ is lacunary Δ^m-statistically convergent.*

Proof. $(1 \Longrightarrow 2)$ i.e, if $\{x_p\}$ is lacunary Δ^m-statistically convergent in (M, g), then there is a convergent sequence $\{y_p\}$ in M such that $x_p = y_p$ for almost all $p \in \mathbb{N}$.

Let $\epsilon > 0$ and $\{x_p\}$ be a sequence such that $\{x_p\}$ lacunary Δ^m-statistically converges to $x \in M$. i.e.,

$$\begin{aligned}
&\delta_t^\theta \left(\left| \left\{ (j_1, j_2, \ldots, j_t) \in \mathbb{N}^t : \quad j_1, j_2, \ldots, j_t \le p, \quad g\left(l, \Delta^m \left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right) < \epsilon \right\} \right| \right) \\
&= \lim_{p \to \infty} \frac{t!}{p^t} \left| \; \left\{ (j_1, j_2, \ldots, j_t) \in \mathbb{N}^t : \quad j_1, j_2, \ldots, j_t \le p, \quad g\left(l, \Delta^m \left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right) < \epsilon \right\} \right| \\
&= 1.
\end{aligned}$$

For every $l \in \mathbb{N}$, there exists $p_l \in \mathbb{N}$, such that for every $p > p_l$,

$$\frac{t!}{p^t} \left| \left\{ (j_1, j_2, \ldots, j_t) \in \mathbb{N}^t : \quad j_1, j_2, \ldots, j_t \le p, \quad g\left(l, \Delta^m \left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right) < \frac{1}{2^l} \right\} \right| > 1 - \frac{1}{2^l}.$$

We can choose $\{p_l\}$ as an increasing sequence in $\mathbb{N}$. Define $\{\Delta^m y_n\}$ as follows

$$\Delta^m y_n = \begin{cases} \Delta^m x_n, & 1 \le n \le p_1, \\ \Delta^m x_n, & p_k < n \le p_{l+1}, j_1, j_2, \ldots, j_{t-1} \\ & \le p, g\left(l, \Delta^m \left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right) < \frac{1}{2^l}\right. \\ 0, & \text{otherwise} . \end{cases}$$

Choose $l \in \mathbb{N}$ such that $\frac{1}{2^l} < \epsilon$. It is clear that $\{\Delta^m y_n\}$ to x. Fix $p \in \mathbb{N}$, for $p_l < p \le p_{l+1}$, we have,

$$\begin{aligned}
&\delta_t^\theta \left(\{ (j_1, j_2, \ldots, j_t) \; \in \mathbb{N}^t : j_1, j_2, \ldots, j_t \le p; \quad x_{j_t} \ne y_{j_t} \} \right) \\
&\quad \subseteq \left\{ (j_1, j_2, \ldots, j_t) \in \mathbb{N}^t : \quad j_1, j_2, \ldots, j_t \le p \right\} \\
&\qquad - \left\{ (j_1, j_2, \ldots, j_t) \in \mathbb{N}^t : \quad j_1, j_2, \ldots, j_t \le p_l, \quad g\left(l, \Delta^m \left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right) < \frac{1}{2^l} \right\}.
\end{aligned}$$

So

$$\lim_{n\to\infty}\frac{t!}{p^t}\left|\left\{(j_1,j_2,\ldots,j_t)\in\mathbb{N}^t : j_1,j_2,\ldots,j_t\le p;\quad x_{j_t}\neq y_{j_t}\right\}\right| \le \lim_{n\to\infty}\frac{t!}{p^t}\begin{pmatrix} p\\ t\end{pmatrix}$$
$$-\lim_{n\to\infty}\frac{t!}{p^t}\left|\left\{(j_1,j_2,\ldots,j_t)\in\mathbb{N}^t : j_1,j_2,\ldots,j_t\le p_l, \, g\left(l,\Delta^m\left(x_{j_1},x_{j_2},\ldots,x_{j_t}\right)<\frac{1}{2^l}\right\}\right|$$
$$\le 1-\left(1-\frac{1}{2^l}\right)=\frac{1}{2^l}\Bigg| <\epsilon.$$

Hence

$$\delta_t^\theta\left(\left\{(j_1,j_2,\ldots,j_t)\in\mathbb{N}^t : j_1,j_2,\ldots,j_t\le p;\quad x_{j_t}\neq y_{j_t}\right\}\right)=0 \quad \text{(almost all)}.$$

Suppose that $\Delta^m\{y_p\}$ be a convergent sequence in M such that $\Delta^m\{x_p\}=\Delta^m\{y_p\}$ for almost all $p\in\mathbb{N}$. Set $P=\{p\in\mathbb{N} : \Delta^m x_p=\Delta^m y_p$. Since $\Delta^m\{x_p\}=\Delta^m\{y_p\}$ for almost all p, hence $\delta_t^\theta(p)=1$ and therefore $\{\Delta^m y_p; p\in P\}$ is convergent and lacunary Δ^m-statistically dense subsequence of $\Delta^m\{x_p\}$.

It is direct consequence of Theorem (2.5).
$(4\Longrightarrow 1)$.

Suppose $\{\Delta^m x_{p_l}\}$ be a lacunary Δ^m-statistically dense subsequence of the sequence $\Delta^m\{x_p\}$ such that lacunary Δ^m-statistically converges to $x\in M$, i.e., $gl\Delta^m s-\lim_{l\to\infty}x_{p_l}=x\in M$ and set $P=\{p_l; l\in\mathbb{N}\}$ then $\delta_t^\theta(p)=1$. For $\epsilon>0$

$$\left\{(j_1,j_2,\ldots,j_t)\in\mathbb{N}^t :\quad j_1,j_2,\ldots,j_t\le p,\quad g\left(l,\Delta^m\left(x_{j_1},x_{j_2},\ldots,x_{j_t}\right)<\epsilon\right\}\right)$$
$$\supseteq\left\{(j_1,j_2,\ldots,j_t)\in\mathbb{N}^t :\quad j_k\in P\quad j_1,j_2,\ldots,j_t\le p,\quad g\left(l,\Delta^m\left(x_{j_1},x_{j_2},\ldots,x_{j_t}\right)<\epsilon\right\},$$

and

$$\lim_{n\to\infty}\frac{t!}{p^t}\Bigg|\left\{(j_1,j_2,\ldots,j_t)\in\mathbb{N}^t :\quad j_1,j_2,\ldots,j_t\le p,\quad g\left(l,\Delta^m\left(x_{j_1},x_{j_2},\ldots,x_{j_t}\right)<\epsilon\right\}\right|$$
$$\ge\lim_{n\to\infty}\frac{t!}{p^t}\Bigg|\left\{(j_1,j_2,\ldots,j_t)\in\mathbb{N}^t : j_k\in P\quad j_1,j_2,\ldots,j_t\le p,\quad g\left(l,\Delta^m\left(x_{j_1},x_{j_2},\ldots,x_{j_t}\right)<\epsilon\right\}\right|$$
$$=1.$$

Hence $gt\Delta^m s-\lim_{p\to\infty}x_p=x$. □

The following corollary is a direct consequence of Theorem 2.7.

Corollory 2.8. *Every lacunary Δ^m-statistically convergent sequence in g-metric spaces has a convergent sunsequence.*

Theorem 2.9. *Every lacunary Δ^m-statistically convergent sequence is lacunary Δ^m statistically g-Cauchy.*

Proof. Let $\{\Delta^m x_p\}$ be a lacunary Δ^m-statistically convergent sequence in g-metric space (M,g) and $\epsilon>0$, the,

$$\lim_{n\to\infty}\frac{t!}{p^t}\left|\left\{(j_1,j_2,\ldots,j_t)\in\mathbb{N}^t : j_1,j_2,\ldots,j_t\le p, \, g\left(l,\Delta^m\left(x_{j_1},x_{j_2},\ldots,x_{j_t}\right)<\frac{\epsilon}{t(t+1)}\right\}\right|=1.$$

By the monotonicity condition for g-metric space and parts (4) and (7) of Theorem (1.3), it follows that

$$g\left(x_{i_\epsilon}, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right) \leq \sum_{l=0}^{t} g\left(\Delta^m x_{j_l}, x, x, \ldots, x\right) \leq \sum_{l=0}^{t} g\left(x, \Delta^m\left(x_{j_l}, x_{j_l}, \ldots, x_{j_l}\right)\right.\right..$$

So

$$\begin{aligned}&\left\{(j_1, j_2, \ldots, j_t) \in \mathbb{N}^t : j_1, j_2, \ldots, j_t \leq p, \quad g\left(x, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right) < \epsilon\right\}\right.\\ &\subseteq \left\{(j_1, j_2, \ldots, j_t) \in \mathbb{N}^t : j_k \in P j_1, j_2, \ldots, j_t \leq p, g\left(x_{j_\epsilon}, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right) < \epsilon\right\}\right..\end{aligned}$$

Therefore

$$\lim_{n\to\infty} \frac{t!}{p^t}\left| \left\{(j_1, j_2, \ldots, j_t) \in \mathbb{N}^t : j_1, j_2, \ldots, j_t \leq p, g\left(l, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right) < \epsilon\right\}\right| = 1.\right.$$

Thus, $\{\Delta^m x_p\}$ is lacunary Δ^m-statistically g-Cauchy in (M, g). □

Definition 2.10. Let (M, g) be a g-metric space, if every lacunary Δ^m-statistically Cauchy sequence be lacunary Δ^m-statistically convergent, then (M, g) is said to be lacunary Δ^m-statistically complete.

Corollary 2.11. *Every lacunary Δ^m-statistically complete g-metric space is complete.*

Proof. Consider the lacunary Δ^m statistically complete g-metric space (M.g). Suppose $\{\Delta^m x_p\}$ be a Cauchy sequence in (M, g), then it is lacunary Δ^m statistically Cauchy sequencwe in (X, g). Since (M, g) is lacunary Δ^m statistically complete so $\{\Delta^m x_p\}$ is lacunary Δ^m statistically convergent. By corollary (2.11), there is a subsequence $\{\Delta^m x_{p_l}\}$ of $\{\Delta^m x_p\}$ that converges to a point $x \in M$. Since $\{\Delta^m x_p\}$ is Cauchy, hence, for $\epsilon > 0$, there exists $N \in \mathbb{N}$ and $x_{j_\epsilon} \in \{\Delta^m x_p\}$ such that for $j_1, j_2, \ldots, j_t \geq N$, we have

$$g\left(x_{j_\epsilon}, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right)\right) < \frac{\epsilon}{2t(t+1)}.$$

On the other hands, $\{\Delta^m x_{p_l}\}$ converges to x. Hence there exists $l_0 \geq N$ such that for $j_1, j_2, \ldots, j_t \geq l_0$,

$$g\left(x, \Delta^m\left(x_{p_{j_1}}, x_{p_{j_2}}, \ldots, x_{p_{j_t}}\right)\right) < \frac{\epsilon}{2}.$$

For $j_1, j_2, \ldots, j_t \geq N$ and applying parts (3) and (4) of Theorem (1.3), it follows that

$$\begin{aligned}&g\left(x, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right)\right) \leq g\left(x, x_{j_\epsilon}, x_{j_\epsilon}, \ldots, x_{j_\epsilon}\right) + \sum_{i=1}^{t} g\left(\Delta^m x_{j_i}, x_{j_\epsilon}, x_{j_\epsilon}, \ldots, x_{j_\epsilon}\right)\\ &\leq g\left(x, \Delta^m\left(x_{p_{j_1}}, x_{p_{j_2}}, \ldots, x_{p_{j_t}}\right)\right) + lt\left(g\left(x_{j_\epsilon}, \Delta^m\left(x_{p_{j_1}}, x_{p_{j_2}}, \ldots, x_{p_{j_t}}\right)\right)\right)\\ &\qquad + \sum_{i=1}^{t} tg\left(x_{j_\epsilon}, \Delta^m\left(x_{j_1}, x_{j_2}, \ldots, x_{j_t}\right)\right)\\ &< \frac{\epsilon}{2} + t\left(\frac{\epsilon}{2t(t+1)}\right) = t^2\left(\frac{\epsilon}{2t(t+1)}\right) = \epsilon.\end{aligned}$$

This completes the proof. □

3. Strong Summability

The relationship between strong summability and lacunary Δ^m-statistical convergence in g-metric space is established in this section.

DEFINITION 3.1. A sequence $\{\Delta^m(x_{i_1}, x_{i_2}, \ldots x_{i_t})\}$ is said to be strongly t-cesaro summable ($0 < t < \infty$) to limit x in (M, g) if $\lim_p \frac{1}{p}\sum_{j=1}^{n}(g)\left(\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t}), x)^t\right) = 0$ and we write it as $x_l \to t\,[c_1, g]_t$. In this case x is the $[c_1, g]_t$-limit of $\{\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t})\}$.

THEOREM 3.2. *Let (M, g) be a g-metric space and θ be a lacunary sequence.*
(a) If $0 < t < \infty$ and $\{\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t})\} \to x\,[c_1, g]_t$, then $\{\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t})\}$ is lacunary Δ^m-statistically g-convergent to x in (M, g).
(b) If $\{\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t})\}$ is bounded and lacunary Δ^m-statistically g-convergent to x in (M, g) then $\{\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t})\} \to x\,[c_1, g]_t$.

Proof. (a). Let

$$L_\epsilon(t) = \{(j_1, j_2, \ldots, j_t) \in \mathbb{N}^t, j_1, j_2, \ldots, i_t \le p (p \in \mathbb{N}) : |g\left(\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t}), x\right)|^t \ge \epsilon\}.$$

Now, since $\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t}) \to x\,[c_1, g]_t$, then

$$\begin{aligned}
0 \leftarrow &\frac{1}{p}\sum_{j=1}^{\infty}|\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t}), x|^t \\
&= \frac{1}{p}\left\{\sum_{\substack{j=1\\ j\notin L_\epsilon(t)}}^{p}|\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t}), x|^t + \sum_{\substack{i=1\\ j\in L_\epsilon(p)}}^{n}|\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t}), l|^p\right\} \\
&\ge \frac{1}{p}|K_\epsilon(t)|\,\epsilon^t, \quad \text{as} \quad p \to \infty.
\end{aligned}$$

That is, $\lim_{p\to\infty}\frac{1}{p}|L_\epsilon(t)| = 0$ and $\delta_t^\theta(L_\epsilon(t)) = 0$. Hence $\{\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t})\}$ is lacunary Δ^m-statistically g-convergent to x in (M, g).

(b). Suppose that $\{\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t})\}$ is bounded and statistically g-convergent to x in (X, g). Then for $\epsilon > 0$, we have $\delta(L_\epsilon(t)) = 0$. Since $\{\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t})\} \in \ell_\infty$, there exists $T > 0$ such that $|g\left(\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t}), x\right)|^t \le T$. We have

$$\begin{aligned}
&\frac{1}{p}\sum_{j=1}^{p}|\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t}), x|^t \\
=&\frac{1}{p}\sum_{\substack{j=1\\ j\notin L_\epsilon(t)}}^{p}|g\left(\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t}), x\right)|^t + \frac{1}{p}\sum_{\substack{j=1\\ j\in L_\epsilon(t)}}^{p}|g\left(\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t}), x\right)|^t \\
=&S_1(p) + S_2(p),
\end{aligned}$$

where

$$S_1(p) = \frac{1}{p}\sum_{\substack{j=1\\ j\notin L_\epsilon(t)}}^{p}|g\left(\Delta^m(x_{j_1}, x_{j_2}, \ldots x_{j_t}), x\right)|^t$$

and

$$S_2(p) = \frac{1}{p} \sum_{\substack{j=1 \\ j \in L_\epsilon(t)}}^{p} \left| g\left(\Delta^m \left(x_{j_1}, x_{j_2}, \ldots x_{j_t}\right), x\right) \right|^t.$$

Now if $\{j_1, j_2, \ldots, j_t\} \notin L_\epsilon(t)$ then $S_1(p) < \epsilon^t$. For $\{j_1, j_2, \ldots, i_t\} \in L_\epsilon(t)$, we have $S_2(p) \leq \sup \left| g\left(\Delta^m \left(x_{j_1}, x_{j_2}, \ldots x_{j_t}\right), x\right) \right| \left(\frac{|L_\epsilon(t)|}{p}\right) \leq T\frac{|L_\epsilon(t)|}{p} \to 0$ as $p \to \infty$, since $\delta_t^\theta(L(\epsilon)) = 0$. Hence $\{\Delta^m \left(x_{j_1}, x_{j_2}, \ldots x_{j_t}\right)\} \to x\,[c_1, g]_t$.

This completes the proof. □

4. Conclusion

The research aimed to provide a comprehensive description of lacunary Δ^m-statistically convergent sequences within the context of generalised metric spaces (g-metric spaces). The investigation delved into the fundamental characteristics of this statistical form of convergence, shedding light on its properties and behavior within the broader framework of metric spaces.

Moreover, the research successfully established a noteworthy connection between strong summability and lacunary Δ^m-statistical convergence in g-metric spaces. This contribution enhances our understanding of the interplay between different modes of convergence and summability in the context of generalized metric spaces, providing valuable insights into the relationships and implications of these concepts.

The findings of this research contribute not only to the theoretical aspects of convergence in g-metric spaces but also hold potential applications in various mathematical and scientific domains. The established relationships and characteristics pave the way for further exploration and utilization of lacunary Δ^m-statistical convergence in the broader mathematical landscape.

Declarations

Conflicts of interests: There is no conflict of interest.

Availibility of data and materials: This paper has no associated data.

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