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A NEW QUARTERNIONIC DIRAC OPERATOR ON SYMPLECTIC SUBMANIFOLD OF A PRODUCT SYMPLECTIC MANIFOLD

Rashmirekha Patra* and Nihar Ranjan Satapathy

Abstract. The Quaternionic Dirac operator proves instrumental in tackling various challenges within spectral geometry processing and shape analysis. This work involves the introduction of the quaternionic Dirac operator on a symplectic submanifold of an exact symplectic product manifold. The self adjointness of the symplectic quaternionic Dirac operator is observed. This operator is verified for spin $\frac{1}{2}$ particles. It factorizes the Hodge Laplace operator on the symplectic submanifold of an exact symplectic product manifold. For achieving this a new complex structure and an almost quaternionic structure are formulated on this exact symplectic product manifold.

1. Introduction

Quaternionic Dirac operator is an useful tool for many problems such as spectral geometry processing and shape analysis. Quaternionic Dirac operator D_f, defined by Crane [3] for any conformally immersed surface $f : M \to Im(H)$ is a differential operator defined on the quaternion valued functions given by

$$D_f \nu = -\frac{df \wedge d\nu}{|df|^2}, \tag{1}$$

where M is a compact, oriented manifold and H is the space of quaternions. The solution of the respective Dirac equation

$$D_f \nu = \rho \nu, \tag{2}$$

provides a new immersed surface by means of spin transformation, where ρ is a real scalar quantity.

In shape analysis, conformal immersions on a compact, connected and orientable topological manifold preserve the length of the curve on a given surface. Immersions that preserve area of a curve are symplectic. Spectral methods are the main tool in the area of shape analysis and processing [3, 11]. In spectral methods, the Laplace-Beltrami operator acts on the scalar functions defined on the given shape. This is an elliptic and self adjoint operator, which provides the intrinsic information about

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* Corresponding author.

the shape. This operator is invariant with respect to the isometric deformation(rigid transformation) of the concerned geometry of the shape. Crane et al. [3] proposed the extrinsic Dirac operator to study the extrinsic information of the objects, which are intrinsically similar but shape wise different. This is called the extrinsic Dirac operator acting on the quaternion valued functions. Authors like Liu et al. [11], Hoffman [8], Wang, Solomon [16] and Ye et al. [18] have studied on the extrinsic Dirac operator, its discretization and applications in shape analysis. To deal with the immersion problem on surfaces and manifolds, the concept of the Dirac operator is used. If S is a surface with a metric g, then an isometric immersion in $\mathbb{R}^3$ with mean curvature H is the solution to the Dirac equation $D\psi = H\psi$. On the other hand, the extrinsic Dirac operator is introduced to develop the study of conformally immersed surfaces in $\mathbb{R}^3$. Wang and Solomon [16] found the relation between intrinsic and extrinsic Dirac operator, which is given by:

$$D = D_f + H. \tag{3}$$

The Dirac operator in quaternion setting has been discussed in many other ways [6, 7, 15]. The Dirac equation in the Euclidean space $\mathbb{R}^4$ in terms of quaternions is given in [7]. In [6, 7], Haydes discussed examples of some low dimensional Dirac operators and generalised Dirac operators in terms of quaternions. The Dirac operator on complexified quarternions is described by Tanisli et al. in [15].

Perez in [14] suggested a method for defining a quaternionic structure on a symplectic product manifold. The quantum mechanical view of the extrinsic Dirac operator may give the symplectic version of such kind of operators. Therefore, the objective of this work is to construct a symplectic version of extrinsic Dirac operator. As this is defined using quaternions, we shall construct a symplectic Dirac operator in quaternionic setting.

After a brief discussion on the quaternions, the details of the quaternionic structure on the submanifold of a symplectic product manifold is discussed in section two following the works of Crane [3] on conformally immersed surfaces. In section three, the concepts of quaternionic valued immersion and differential forms using quaternions are discussed along with the definition of symplectic quaternionic Dirac operator. The self adjointness property of this new operator is also verified. We have also proved that the symplectic quaternionic Dirac operator factorises the Hodge Laplace operator on symplectic submanifold of the product symplectic manifold. The new operator is equal to the spin Dirac operator for spin half particles.

2. An almost quaternionic structure on the symplectic product manifold

Definition 2.1. A sympletic manifold (Q, ω) is called an exact symplectic manifold if the symplectic form ω is not only closed but also exact, i.e. there exist a one form α such that $d\alpha = \omega$. That one form on symplectic manifold is known as Liouvillie one form.

Definition 2.2. The Liouvillie form α on an exact symplectic manifold (Q, ω) is a canonical tautological 1-form at any point (p, q). Here α is defined by $\alpha = \sum_i p_i dq^i$. The cannonical symplectic form(also known as poincare 2-form) is defined by

$$\omega = -d\alpha = \sum_i dq^i \wedge dp_i. \tag{4}$$

Let (Q, ω) be an exact symplectic manifold of dimension $2n$ with $d\alpha = \omega$, where α is an Liouville form on (Q, ω). A Liouville vector field Z is represented by the implicit equation $\alpha = i_Z\omega$. A diffeomorphism $\tau : (Q, \omega) \to (Q, \omega)$ is said to be a symplectomorphism if $\tau^*\omega = \omega$, where τ^* is the pullback of τ.

Let (Q_1, ω_1) and (Q_2, ω_2) be $2n$ dimensional exact symplectic manifolds, where $d\alpha_i|_{i=1,2} = \omega_i$ and $\alpha_i s$ are Liouvillie forms. Consider the product bundle $M = Q_1 \times Q_2$ of dimension $4n$ with cannonical projections $\eta_i : M \to Q_i, i = 1, 2$ respectively. Define 1-form α and 2-form ω on M as

$$\alpha = \eta_1^*\alpha_1 - \eta_2^*\alpha_2. \tag{5}$$

$$\omega = \eta_1^*\omega_1 - \eta_2^*\omega_2, \tag{6}$$

where $\eta_i^*, i = 1, 2$ is the pull back of η_i. Here, ω is the twisted symplectic form, that is used in the method of generating functions on the symplectic product manifold. We considered the local geometry of the symplectic product manifold M with the twisted symplectic form ω to define the following complex structure. We also verified the compatibility condition of the complex structure with the twisted symplectic form ω. The complex structure compatible to ω is given by

$$J = J_1 \oplus J_2^T, \tag{7}$$

where J_i are the complex structures compatible to $\omega_i, i = 1, 2$.

Let g^{Q_1} be the Reimannian metric on (Q_1, ω_1) and g^{Q_2} be the Reimannian metric on (Q_2, ω_2). The Riemannian metric on the product manifold is also defined in the following form

$$g^{Q_1\times Q_2} = \eta_1^* g^{Q_1} + \eta_2^* g^{Q_2}. \tag{8}$$

J_1 is the almost complex structure compatible to ω_1 on (Q_1, ω_1) and J_2 is the almost complex structure compatible to ω_2 on (Q_2, ω_2) with the following compatible conditions:

$$g^{Q_1}(u_1, u_2) = \omega_1(u_1, J_1 u_2) \tag{9}$$

$$g^{Q_2}(v_1, v_2) = \omega_2(v_1, J_2 v_2) \tag{10}$$

Lemma 2.3.

$$g^{Q_1\times Q_2}(u, v) = \omega(u, Jv) \tag{11}$$

Proof. Consider LHS of the equation (11)

$$\begin{aligned}
&g^{Q_1\times Q_2}(u, v)\\
&= \left(\eta_1^* g^{Q_1} + \eta_2^* g^{Q_2}\right)(u, v)\\
&= \eta_1^* g^{Q_1}(u, v) + \eta_2^* g^{Q_2}(u, v)\\
&= \eta_1^*\omega_1(u, J_1 v) + \eta_2^*\omega_2(u, J_2 v)\\
&= (\eta_1^*\omega_1 + \eta_2^*\omega_2)\left(u, \left(J_1 \oplus J_2^T\right) v\right)\\
&= \omega(u, Jv)
\end{aligned}$$

□

Remark 2.4. The complex structure $J = J_1 \oplus J_2^T$ may not exist for every symplectic manifold of dimension 4. But when $n = 1$, Q_1, Q_2 will be two 2 dimensional exact symplectic manifolds and $M = Q_1 \times Q_2$ will be the 4 dimensional exact symplectic manifold.

By the work of Massey [13] every $2n$ dimensional oriented manifold admits a complex structure when the structure group of the tangent bundle is reduced from $SO(2n)$ to $U(n)$, i.e. there exists a section of the associated $SO(2n)/U(n)$ bundle on the given manifold.

Every two dimensional manifold admits an almost complex structure if that is orientable. Every symplectic manifold is oriented. So every symplectic manifold of dimension 2 admits an almost complex structure [13]. Therefore, Q_1, Q_2 admits almost complex structures. The product symplectic manifold $M = Q_1 \times Q_2$ with the canonical projections $\eta_i : M \to Q_i, i = 1, 2$ admits a complex structure associated to the twisted symplectic form ω on M.

Avoiding the twisted diffeomorphisms and considering the local geometry of (M, ω), an almost quaternionic structure on M in Darboux coordinates is defined by $\{Id_{4n}, I_{4n}, J_{4n}, K_{4n}\} \subset End(TM)$. The almost quaternionic structure is defined as follows:
Consider

$$I_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2, \quad J_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_3, \quad K_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_1. \tag{12}$$

where $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices. We define I_{2n} as the matrix taking I_2 n times in the diagonal, J_{2n} as the matrix taking J_2 n times in the anti diagonal and K_{2n} as the matrix taking K_2 n times in the anti diagonal.
For example For $n = 3$, I_{2n}=

$$\begin{pmatrix} \begin{array}{ccc} I_2 & \multicolumn{2}{c}{\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & I_2 & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \multicolumn{2}{c}{\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}} & I_2 \end{array} \end{pmatrix}$$

For $n = 3$, J_{2n}=

$$\begin{pmatrix} \begin{array}{ccc} \multicolumn{2}{c}{\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}} & J_2 \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & J_2 & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ J_2 & \multicolumn{2}{c}{\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}} \end{array} \end{pmatrix}$$

For $n = 3$, K_{2n}=

$$\left(\begin{array}{cccc|cc}
0 & 0 & 0 & 0 & K_2 & \\
0 & 0 & 0 & 0 & & \\
\hline
0 & 0 & K_2 & & 0 & 0 \\
0 & 0 & & & 0 & 0 \\
\hline
K_2 & & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 0
\end{array}\right)$$

The almost quaternionic structure is the subset $\{Id_{4n}, I_{4n}, J_{4n}, K_{4n}\}$ of $End(TM)$ which is defined by $I_{4n} = \begin{pmatrix} I_{2n} & 0_{2n} \\ 0_{2n} & I_{2n} \end{pmatrix}$, $J_{4n} = \begin{pmatrix} 0_{2n} & J_{2n} \\ J_{2n} & 0_{2n} \end{pmatrix}$, $K_{4n} = \begin{pmatrix} 0_{2n} & K_{2n} \\ K_{2n} & 0_{2n} \end{pmatrix}$,

Remark 2.5.

1. The matrices I_{4n}, J_{4n}, K_{4n} satisfy the following conditions on M:

$$I_{4n}^2 = J_{4n}^2 = K_{4n}^2 = -Id_{4n} \tag{13}$$

$$I_{4n}J_{4n} = K_{4n}, J_{4n}K_{4n} = I_{4n}, K_{4n}I_{4n} = J_{4n}. \tag{14}$$

2. Exactly the set $\{I_{4n}, J_{4n}, K_{4n}\}$ is a local compatible frame for an almost quaternionic structure of a rank 3 subbundle, which is a subset of $End(T(Q_1 \times Q_2))$. The quaternionic action of this frame is dependent on the choice of the frames. Such a choice is local in nature.
3. A $4n$ dimensional manifold with $n \geq 2$ with a complex structure is called weakly quaternionic if the manifold admits a torsion free connection that preserves the rank 3 subbundle.
4. But for $n = 1$, the manifold $Q_1 \times Q_2$ is 4 dimensional and the holonomy group is $Sp(1)Sp(1) \simeq SO(4)$. In this case the holonomy does not create any additional structure but only an oriented Riemannian manifold. So the holonomy is weak for dimension 4. In dimension four the quaternionic structures can be extended to all of $Q_1 \times Q_2$ due to its weak holonomic nature. But it may not happen in higher dimensions.

Let $g^{Q_1 \times Q_2}$ be the Riemannian metric on M corresponding to the Euclidean structure on $T_zM, z \in M$. The symplectic forms are defined as follows:

(15)

$$\omega_{I_{4n}}(\cdot,\cdot) = g^{Q_1\times Q_2}(\cdot, I_{4n}\cdot), \ \omega_{J_{4n}}(\cdot,\cdot) = g^{Q_1\times Q_2}(\cdot, J_{4n}\cdot), \ \omega_{K_{4n}}(\cdot,\cdot) = g^{Q_1\times Q_2}(\cdot, K_{4n}\cdot).$$

Let L be a $2n$ dimensional submanifold of M and $\pi : L \hookrightarrow M$ be an embedding. π is called a Lagrangian embedding, if $\pi^*\omega = 0$.

Theorem 2.6. [14] *If L is a Lagrangian submanifold with respect to $\omega_{I_{4n}}$ and $\omega_{J_{4n}}$, then it is a symplectic submanifold with respect to $\omega_{K_{4n}}$.*

In general, quaternions form a number system ahead of complex numbers. For the details about quaternionic calculus, we refer the work by Joyce [9, 10], Widdows [17] and Alekseevsky [1]. Imaginary part of quaternions usually describe the structure of $\mathbb{R}^3$ in an accurate way. Quaternions form a 4 dimensional vector space H with the basis $\{1, i, j, k\}$ with the associative and non commutative Hamiltonian product. The quatenionic structure defined above in the symplectic product bundle $\{Id_{4n}, I_{4n}, J_{4n}, K_{4n}\}$ forms a quaternionic vector space $\mathbf{H}$ of dimension 4, which is equivalent to $\mathbb{R}^4$. The general elements in 4 dimensional quaternions is expressed as:

$$m = p + qI_{4n} + rJ_{4n} + sK_{4n} \simeq (p, q, r, s)\,. \tag{16}$$

Here, p is called the real part of m and $qI_{4n} + rJ_{4n} + sK_{4n}$ is called the imaginary part of m. Therefore, $(q, r, s) \in Im\mathbf{H}$. In general, an arbitrary quaternion m on M is expressed as the combination of real scalar part $m_r \in \mathbb{R} \subset \mathbf{H}$ and an imaginary vector part $m_i \in \mathbb{R}^3 = Im\mathbf{H} \subset \mathbf{H}$:

$$m = m_r + m_i, \tag{17}$$

where $m_r = Re\,(m)$ and $m_i = Im\,(m)$. The conjugate of a quaternion $m = p + qI_{4n} + rJ_{4n} + sK_{4n}$ is given by

$$\overline{m} = p - qI_{4n} - rJ_{4n} - sK_{4n}. \tag{18}$$

In particular, if m is an imaginary quaternion, i.e. $m \in Im\mathbf{H}$, then $\overline{m} = -m$.

3. Quaternionic Dirac operator on symplectic submanifold

3.1. Quaternion valued immersion on symplectic submanifold. We consider the immersion in terms of quaternions. For the smooth symplectic manifolds (Q_1, ω_1) and (Q_2, ω_2) with the closed 2-forms ω_1, ω_2 respectively, a smooth immersion $f : (Q_1, \omega_1) \to (Q_2, \omega_2)$ is called symplectic, if f pulls back ω_2 into ω_1. The product symplectic manifold M is a $4n$ dimensional manifold. Therefore, we can not embed the manifold into the algebra of quaternions.

For calculation, we assume $n = 1$ and we take a symplectic submanifold of M (L, ω_{K_4}) of dimension 2. This symplectic submanifold can be embedded in the symplectic manifold M. Therefore, we consider an immersion of symplectic submanifold into the algebra of quaternions.

Let $\tilde{f}$ be a complex quaternion-valued function,

$$\tilde{f} : L \to span\{I_4, J_4, K_4\} = Im\mathbf{H} \tag{19}$$

The function $\tilde{f}$ is a symplectic immersion [4,5] like isometric immersion in Riemannian geometry [2]. A complex structure on L is a bundle automorphism $J_4 : TL \to TL$ such that $J_4^2 = -Id_4$. An immersion $\tilde{f}$ is said to be conformal if

$$d\tilde{f}\,(J_4 X) = \mathbf{N} \times d\tilde{f}\,(X)\,, \tag{20}$$

where $\mathbf{N}$ is the unit normal on the symplectic submanifold L.

The differential forms in terms of quaternions are described as follows:

1. 0-form on the symplectic submanifold is the quaternion valued function $\tilde{f}$.
2. Let the vector field associated to the conformal immersion $\tilde{f}$ in symplectic submanifold be $X_{\tilde{f}}$ such that the $1-$forms $d\tilde{f}$ and the linear maps taking vector fields on L to an quaternionic valued function are defined by

$$d\tilde{f} = i_{X_{\tilde{f}}}\omega_{\mathbf{K}_4},$$

   where $i_{X_{\tilde{f}}}$ is the interior product with respect to the vector field $X_{\tilde{f}}$.
3. $\omega_{I_{4n}}, \omega_{J_{4n}}, \omega_{K_{4n}}$ are the quaternionic 2-forms in M.

4. In M the volume form σ' in the symplectic product manifold belongs to $\Gamma\left(\Omega^{4n}M\right) = \Lambda^{4n}\left(T_p^*M\right)$. So, this is defined as

$$n\sigma' = \frac{\Omega^2}{(n)!} = \frac{1}{(n)!}d \wedge \mathbf{N}d\tilde{f} \wedge \mathbf{N}d\tilde{f} \wedge ... \wedge \mathbf{N}d\tilde{f} = \frac{|d\tilde{f}|^2}{(n)!}, \tag{21}$$

where $\mathbf{N}d\tilde{f}$ is taken $2n-1$ times and

$$\Omega = \omega_{I_{4n}} \wedge \omega_{I_{4n}} + \omega_{J_{4n}} \wedge \omega_{J_{4n}} + \omega_{K_{4n}} \wedge \omega_{K_{4n}}. \tag{22}$$

In particular the volume form restricted to the symplectic submanifold (L, ω_{K_4}) of dimension 2 is given by

$$\frac{\omega_{K_4}}{1!} = \frac{1}{1!}d\tilde{f} \wedge \mathbf{N}d\tilde{f} = \frac{1}{1!}|d\tilde{f}|^2. \tag{23}$$

As $\tilde{f}$ is an immersion on symplectic submanifold, the Riemannian metric induced by $\tilde{f}$ is given by

$$g\left(X, Y\right) = \langle d\tilde{f}\left(X\right), d\tilde{f}\left(Y\right)\rangle. \tag{24}$$

The dimension of symplectic submanifold is 2. So, the volume form will be reduced to a quaternionic 2-form. If μ is the volume form on the symplectic submanifold, then $\mu = \frac{1}{2!}|df|^2$. Any 2-form ω with the scalar multiple with respect to the volume form of a symplectic submanifold is defined as follows:

$$\frac{\omega_{K_4}}{|d\tilde{f}|^2} := \omega_{K_4}\left(X, J_4X\right), \tag{25}$$

where J_4 is the complex structure on the symplectic submanifold and $\mu\left(X, J_4X\right) = \frac{1}{2!}|d\tilde{f}|^2$. Here, X has unit area with respect to $\tilde{f}$. For the 1-form $d\tilde{f}$ the Hodge star $\star_s : \Lambda^pT^*M \to \Lambda^{4-p}T^*M$, where $p = 0, 1, 2$ induced by the conformal immersion $\tilde{f}$ is expressed as

$$\star_s d\tilde{f} = \mathbf{N}d\tilde{f}, \tag{26}$$

where $\mathbf{N}$ is the unit normal for $\tilde{f}$. The following statements describe the role of the Hodge star induced by conformal immersion on (L, ω_{K_4}) on the quaternion valued differential forms:

1. $|d\tilde{f}|^2$ represents Hodge star on 0-forms, i.e. $\star_s\psi = \psi|d\tilde{f}|^2$.
2. The complex structure J_4 represents the Hodge star on 1-forms, i.e. $\star_s d\beta\left(X\right) = d\beta\left(J_4X\right)$. The wedge product of 1-forms with respect to the Hodge star can be expressed as follows.

$$\alpha_1 \wedge \alpha_2 = \alpha_1\alpha_2 - \star_s\alpha_1\alpha_2. \tag{27}$$

3. $\frac{1}{|d\tilde{f}|^2}$ represents Hodge star on 2-forms, i.e. $\star_s\omega_{K_4} = \frac{\omega_{K_4}}{|d\tilde{f}|^2}$.
4. The Hodge Laplacian induced by the conformal symplectic immersion $\tilde{f}$ or the 0-form on the symplectic submanifold is defined by $\triangle = \star_s d \star_s d$.

The inner product on the quarternion valued functions $\psi_1, \psi_2 : L \to \mathbf{H}$ is given by:

$$\langle\langle\psi_1, \psi_2\rangle\rangle = \int_L \overline{\psi_1}\psi_2|d\tilde{f}|^2. \tag{28}$$

3.2. Quarternionic Dirac operator on symplectic submanifold. The conformal immersion in the field of quaternions is defined as follows:

$$\tilde{f} : M \to Im\mathbf{H} \subset \mathbf{H}. \tag{29}$$

By choosing any quaternionic function $\xi : M \to \mathbf{H}$, this can be defined as the combination of the real and imaginary part, where the imaginary part is decomposed by tangent and normal parts respectively. Here, the quaternionic function ξ is expressed as:

$$\xi = b + d\tilde{f}\,(Y) + a\mathbf{N}, \tag{30}$$

for some vector field Y and the normal vector $\mathbf{N}$. The normal vector fields commute with each other but the normal and the tangent vector fields do not commute. Taking a pair of orthogonal tangent vector fields $d\tilde{f}\,(X)\,, d\tilde{f}\,(J_4X)$, the following relation is obtained.

$$d\tilde{f}\,(X)\, d\tilde{f}\,(J_4X) = d\tilde{f}\,(X) \times d\tilde{f}\,(J_4X) = \mathbf{N}|d\tilde{f}|^2 \tag{31}$$

We now define a Dirac operator as the following:

Definition 3.1. For a conformally immersed surface $\tilde{f}$ in the symplectic submanifold (L, ω_{K_4}), i.e. $\tilde{f} : M \to ImH$, we define the symplectic quaternionic Dirac operator as a first order differential operator $\mathcal{D}$ acting on smooth quaternion valued function ξ by

$$\mathcal{D}\xi := -2\frac{\left(i_{X_{\tilde{f}}}\omega_{K_4}\right) \wedge d\xi}{|d\tilde{f}|^2}, \tag{32}$$

where $i_{X_{\tilde{f}}}\omega_{K_4}$ is the Liouville one form on (L, ω_{K_4}).

This symplectic quaternionic Dirac operator is defined on the symplectic submanifold (L, ω_{K_4}). Here, $\mathcal{D}$ is a symplectic gradient operator. Similar to the complex function theory, the function and its derivative are both quaternionic valued functions. The symplectic gradient operator $\mathcal{D}$ also acts similarly on quaternionic valued functions.

3.3. Results.

Theorem 3.2. *For any two quaternion valued differentiable functions $\psi, \xi : L \to \boldsymbol{H}$ on (L, ω_{K_4}), $\mathcal{D}$ is self-adjoint, i.e.*

$$\langle\langle \mathcal{D}\psi, \xi \rangle\rangle = \langle\langle \psi, \mathcal{D}\xi \rangle\rangle. \tag{33}$$

Proof. For quaternionic valued function ψ, ξ, we have

$$\begin{aligned}
&\langle\langle \psi, \mathcal{D}\xi \rangle\rangle \\
&= -2\int_L \overline{\psi}\frac{\left(i_{X_{\tilde{f}}}\omega_{K_4}\right)\wedge d\xi}{|d\tilde{f}|^2}|d\tilde{f}|^2 \\
&= -2\int_L \{\overline{\psi}\left(i_{X_{\tilde{f}}}\omega_{K_4}\right)\wedge d\xi\} \\
&= 2\{\int_L d\overline{\psi}\wedge\left(i_{X_{\tilde{f}}}\omega_{K_4}\right)\xi - \int_L d\left(\overline{\psi}\left(i_{X_{\tilde{f}}}\omega_{K_{4n}}\right)\xi\right)\} \\
&= 2\int_L d\overline{\psi}\wedge\left(i_{X_{\tilde{f}}}\omega_{K_4}\right)\xi \;\; \{\text{the second term vanishes by Stoke's theorem}\} \\
&= -2\int_L \overline{\overline{\xi}\left(i_{X_{\tilde{f}}}\omega_{K_4}\right)\wedge d\psi} \\
&= \overline{\langle\langle \xi, \mathcal{D}\psi\rangle\rangle} = \langle\langle \mathcal{D}\psi, \xi\rangle\rangle, \;\; \text{which proves that } \mathcal{D} \text{ is self adjoint.}
\end{aligned}$$

□

DEFINITION 3.3. The spin Dirac operator on $\mathbb{R}^2$ is given by:

$$D = -i\left(\sigma_x\partial_x + \sigma_y\partial_y\right), \tag{34}$$

where σ_x and σ_y are Pauli matrices.

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The quaternion valued function $\phi : L \to \mathbf{H}$ can be represented as:

$$\phi = \phi_1 + j\phi_2, \tag{35}$$

where ϕ_1, ϕ_2 are the one dimensional complex valued functions.

THEOREM 3.4. *The symplectic quaternionic Dirac operator $\mathcal{D}$ is equal to the spin Dirac operator.*

Proof. Consider a quaternion valued function ϕ as in equation (35). ϕ is a function of two real variables. The derivative ∂ and the conjugate derivative $\overline{\partial}$ are defined as follows:

$$\partial\phi = \frac{1}{2}\left(\partial_x - i\partial_y\right)\phi, \;\; \overline{\partial}\phi = \frac{1}{2}\left(\partial_x + i\partial_y\right)\phi \tag{36}$$

In the expression of the quaternionic Dirac operator $\mathcal{D}$, the term $d\phi$ can be expressed as $d\phi = \partial\phi dz + \overline{\partial}\phi d\overline{z}$. Further, the expression is simplified as follows.

$$\begin{aligned}
d\phi &= \partial\phi dz + \overline{\partial}\phi d\overline{z} \\
&= \frac{1}{2}\left(\partial_x - i\partial_y\right)\phi dz + \frac{1}{2}\left(\partial_x + i\partial_y\right)\phi d\overline{z} \\
&= \frac{1}{2}\left(\partial_x - i\partial_y\right)\left(\phi_1 + j\phi_2\right)dz + \frac{1}{2}\left(\partial_x + i\partial_y\right)\left(\phi_1 + j\phi_2\right)d\overline{z} \\
&= \frac{1}{2}\{\partial_x\phi_1 + j\partial_x\phi_2 - j\partial_y\phi_1 - k\partial_y\phi_2\}dz + \frac{1}{2}\{\partial_x\phi_1 + j\partial_x\phi_2 + i\partial_y\phi_1 + k\partial_y\phi_2\}d\overline{z} \\
&= \frac{1}{2}\{\partial\phi_1 + j\partial\phi_2\}dz + \frac{1}{2}\{\overline{\partial}\phi_1 + j\overline{\partial}\phi_2\}d\overline{z} \\
&= \frac{1}{2}\{\left(\partial\phi_1 dz + \overline{\partial}\phi_1 d\overline{z}\right) + j\left(\partial\phi_2 dz + \overline{\partial}\phi_2 d\overline{z}\right)\}
\end{aligned}$$

Assume that for a given point, the immersion $\tilde{f}$ is a map to the j,k-plane. Here, we assume $\tilde{f} = kz$, where z is a 1-dimensional complex valued function on L. Now considering the numerator of symplectic quaternionic Dirac operator $i_{X_{\tilde{f}}}\omega_{K_4} \wedge d\phi$, which is equal to $d\tilde{f} \wedge d\phi$, we have

$$\begin{aligned}
d\tilde{f} \wedge d\phi &= kdz \wedge d\phi \\
&= \frac{1}{2}\{kdz \wedge \left(\left(\partial\phi_1 dz + \overline{\partial}\phi_1 d\overline{z}\right) + j\left(\partial\phi_2 dz + \overline{\partial}\phi_2 d\overline{z}\right)\right)\} \\
&= \frac{1}{2}\left(k\overline{\partial}\phi_1 + \partial\phi_2\right)\left(d\overline{z} \wedge dz\right) \\
&= \frac{1}{2}\left(k\overline{\partial}\phi_1 + \partial\phi_2\right)\left(i|d\tilde{f}|^2\right)
\end{aligned}$$

Dividing $\frac{-2}{|d\tilde{f}|^2}$ in the above equation we get

$$\mathcal{D}\phi = \left(k\overline{\partial}\phi_1 + \partial\phi_2\right)i \tag{37}$$

$$= \begin{pmatrix} 0 & \partial \\ k\overline{\partial} & 0 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} i, \tag{38}$$

which is similar to the spin Dirac operator mentioned in equation (34). □

Theorem 3.5. *The symplectic quaternionic Dirac operator satisfies*

$$\mathcal{D}^2\varphi = \triangle\varphi + \frac{d\mathbf{N} \wedge d\varphi}{|d\tilde{f}|^2}, \tag{39}$$

where the first term $\triangle\varphi$ is the Hodge Laplacian with respect to the conformal immersion function or the 0-form.

Proof. By the definition of symplectic quaternionic Dirac operator, we have

$$\begin{aligned}
&|d\tilde{f}|^2\mathcal{D}\varphi = -2d\tilde{f} \wedge d\varphi \\
&i.e.\ \ -d\tilde{f}d\tilde{f}\mathcal{D}\varphi = -d\tilde{f} \star_s d\varphi + \star_s d\tilde{f}d\varphi
\end{aligned}$$

Applying the equation (26), we have

$$-d\tilde{f}d\tilde{f}\mathcal{D}\varphi = -d\tilde{f} \star_s d\varphi - \mathbf{N}d\tilde{f}d\varphi. \tag{40}$$

Dividing equation (40) by $-d\tilde{f}$, the equation becomes

$$d\tilde{f}\mathcal{D}\varphi = \star_s d\varphi + \mathbf{N}d\varphi \tag{41}$$

Here $\tilde{f}$ is an immersion. Taking derivative of both the sides of equation (41), we have

$$d\left(d\tilde{f}\mathcal{D}\varphi\right) = d\star_s d\varphi + d\left(\mathbf{N}d\varphi\right)$$

$$i.e.\ d^2\tilde{f}\mathcal{D}\varphi - d\tilde{f}\wedge d\left(\mathcal{D}\varphi\right) = d\star_s d\varphi + d\mathbf{N}\wedge d\varphi - \mathbf{N}\wedge d^2\varphi$$

The first and the last term of the above equation vanishes as per the the property of exterior derivative. So, the equation becomes

$$-d\tilde{f}\wedge d\left(\mathcal{D}\varphi\right) = d\star_s d\varphi + d\mathbf{N}\wedge d\varphi \tag{42}$$

Dividing $|d\tilde{f}|^2$ on both sides of the equation (42), we get

$$\frac{1}{2}\mathcal{D}\left(\mathcal{D}\varphi\right) = \frac{d\star_s d\varphi}{|d\tilde{f}|^2} + \frac{d\mathbf{N}\wedge d\varphi}{|d\tilde{f}|^2}. \tag{43}$$

As per the action of Hodge star on 0-form on the equation (43), the equation becomes

$$\frac{1}{2}\mathcal{D}\left(\mathcal{D}\varphi\right) = \star_s d\star_s d\varphi + \frac{d\mathbf{N}\wedge d\varphi}{|d\tilde{f}|^2}, \tag{44}$$

which gives the following expression:

$$\mathcal{D}^2\varphi = \triangle\varphi + \frac{d\mathbf{N}\wedge d\varphi}{|d\tilde{f}|^2}, \tag{45}$$

where the first term $\triangle\varphi$ is the Hodge Laplacian with respect to the conformal immersion function or the 0-form. □

4. Conclusion

This proposed study introduces a quaternionic Dirac Operator on a symplectic submanifold of an exact symplectic product manifold. We verified the self-adjoint property of this symplectic quaternionic Dirac operator. Its applicability extends to representing spin half particles within a symplectic framework. Furthermore, it serves as a factorization tool for the Hodge Laplacian on the symplectic submanifold. Within the exact symplectic product manifold, we also introduced an almost quaternionic structure. The quaternionic Dirac operator has substantial significance in shape analysis and spectral geometry processing. These methodologies may be extended to discrete setups, enabling solutions for problems in the space retrieval and surface segmentation.

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Rashmirekha Patra
Department of Mathematics, Sambalpur University, Jyoti Vihar, Burla, Sambapur, Odisha, India
E-mail: rashmath12@gmail.com

Nihar Ranjan Satapathy
Department of Mathematics, Sambalpur University, Jyoti Vihar, Burla, Sambapur, Odisha, India
E-mail: niharmath@suniv.ac.in