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CERTAIN SUBCLASS OF STRONGLY MEROMORPHIC CLOSE-TO-CONVEX FUNCTIONS

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Abstract. The purpose of this paper is to introduce a new subclass of strongly meromorphic close-to-convex functions by subordinating to generalized Janowski function. We investigate several properties for this class such as coefficient estimates, inclusion relationship, distortion property, argument property and radius of meromorphic convexity. Various earlier known results follow as particular cases.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disc $E = \{z :| z |< 1\}$. The class of functions $f \in \mathcal{A}$ and which are univalent in E, is denoted by $\mathcal{S}$.

The class of starlike univalent functions is denoted by $\mathcal{S}^*$ and is given by

$$\mathcal{S}^* = \left\{ f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in E \right\}.$$

The class $\mathcal{K}$ of convex univalent functions is defined as follows:

$$\mathcal{K} = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > 0, z \in E \right\}.$$

The concept of close-to-convex functions was given by Kaplan [7]. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}$ of close-to-convex functions if there exists a function $g \in \mathcal{S}^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0 (z \in E).$$

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A function w which has expansion of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n,$$

and satisfy the conditions $w(0) = 0$ and $|w(z)| \leq 1$, is called a Schwarz function. The class of Schwarz functions is denoted by $\mathcal{U}$.

Let f and g are two analytic functions in E, then f is said to be subordinate to g, if there exists a Schwarz function $w \in \mathcal{U}$ such that

$$f(z) = g(w(z)).$$

If f is subordinate to g, then it is denoted by $f \prec g$. Further, if g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

By $\mathcal{M}$, we denote the class of functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_k z^k,$$

which are meromorphic analytic in the open unit punctured disc

$$E^* = \{z : z \in \mathbb{C}, 0 <| z |< 1\} = E - \{0\}.$$

A function $f \in \mathcal{M}$ is said to be in the class $\mathcal{MS}^*$ of meromorphic starlike functions if it satisfies the condition

$$Re\left(\frac{zf'(z)}{f(z)}\right) < 0, z \in E^*.$$

The class $\mathcal{MK}$ of meromorphic convex functions is given by

$$\mathcal{MK} = \left\{f : f \in \mathcal{M}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) < 0, z \in E^*\right\}.$$

It is obvious that $f \in \mathcal{MK}$ if and only if $-zf'(z) \in \mathcal{MS}^*$.

A function $f \in \mathcal{M}$ is called meromorphic starlike function of order α $(0 \leq \alpha < 1)$ if it satisfies the condition

$$Re\left(-\frac{zf'(z)}{f(z)}\right) > \alpha, z \in E^*.$$

The class of meromorphic starlike functions of order α is denoted by $\mathcal{MS}^*(\alpha)$. In particular, $\mathcal{MS}^*(0) \equiv \mathcal{MS}^*$. Also for $\alpha = \frac{1}{2}$, the class $\mathcal{MS}^*(\alpha)$ reduces to the class $\mathcal{MS}^*\left(\frac{1}{2}\right)$.

By $\mathcal{MC}$, we denote the class of meromorphic close-to-convex functions. A function $f \in \mathcal{M}$ is called meromorphic close-to-convex function if there exists a meromorphic starlike function g such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) < 0, z \in E.$$

Gao and Zhou [4] studied the class $\mathcal{K}_S$ given by:

$$\mathcal{K}_s = \left\{ f : f \in \mathcal{A}, Re\left(\frac{-z^2 f'(z)}{g(z)g(-z)}\right) > 0, g \in \mathcal{S}^*\left(\frac{1}{2}\right), z \in E \right\}.$$

Knwalczyk and Les-Bomba [8] extended the class $\mathcal{K}_S$ by introducing the class $\mathcal{K}_S(\gamma), (0 \leq \gamma < 1)$ mentioned below:

$$\mathcal{K}_s(\gamma) = \left\{ f : f \in \mathcal{A}, Re\left(\frac{-z^2 f'(z)}{g(z)g(-z)}\right) > \gamma, g \in \mathcal{S}^*\left(\frac{1}{2}\right), z \in E \right\}.$$

For $\gamma = 0$, the class $\mathcal{K}_S(\gamma)$ reduces to the class $\mathcal{K}_S$.

Further, Prajapat [12] established that, a function $f \in \mathcal{A}$ is said to be in the class $\chi_t(\gamma)(|t| \leq 1, t \neq 0, 0 \leq \gamma < 1)$, if there exists a function $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$, such that

$$Re\left[\frac{tz^2 f'(z)}{g(z)g(tz)}\right] > \gamma.$$

In particular $\chi_{-1}(\gamma) \equiv \mathcal{K}_S(\gamma)$ and $\chi_{-1}(0) \equiv \mathcal{K}_S$.

Analogously, Wang et al. [17] introduced the class $\mathcal{M}_\mathcal{K}$ which consists of the functions $f \in \mathcal{M}$ such that

$$Re\left[\frac{f'(z)}{g(z)g(-z)}\right] > 0,$$

where $g \in \mathcal{MS}^*\left(\frac{1}{2}\right)$.

As a generalization of the class $\mathcal{M}_\mathcal{K}$, Sim and Kwon [15] established the class $\mathcal{M}_\mathcal{K}(A, B)$ $(-1 \leq B < A \leq 1)$ defined as:

$$\mathcal{M}_\mathcal{K}(A, B) = \left\{ f : f \in \mathcal{M}, \frac{f'(z)}{g(z)g(-z)} \prec \frac{1 + Az}{1 + Bz}, g \in \mathcal{MS}^*\left(\frac{1}{2}\right), z \in E^* \right\}.$$

For $A = 1, B = -1$, the class $\mathcal{M}_\mathcal{K}(A, B)$ reduces to the class $\mathcal{M}_\mathcal{K}$.

Raina et al. [13] introduced the class of strongly close-to-convex functions of order β, as below:

$$\mathcal{C}'_\beta = \left\{ f : f \in \mathcal{A}, \left| arg\left\{\frac{zf'(z)}{g(z)}\right\}\right| < \frac{\beta\pi}{2}, g \in \mathcal{K}, 0 < \beta \leq 1, z \in E \right\},$$

or equivalently

$$\mathcal{C}'_\beta = \left\{ f : f \in \mathcal{A}, \frac{zf'(z)}{g(z)} \prec \left(\frac{1 + z}{1 - z}\right)^\beta, g \in \mathcal{K}, 0 < \beta \leq 1, z \in E \right\}.$$

For $-1 \leq B < A \leq 1$, Janowski [6] introduced the class of functions in $\mathcal{A}$ which are of the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ and satisfying the condition $p(z) \prec \dfrac{1 + Az}{1 + Bz}$. This class plays an important role in the study of various subclasses of analytic-univalent functions. As a generalization of Janowski's class, Polatoglu et al. [10] introduced the class $\mathcal{P}(A, B; \alpha)$ $(0 \leq \alpha < 1)$, the subclass of $\mathcal{A}$ which consists of functions of the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ such that $p(z) \prec \dfrac{1 + [B + (A - B)(1 - \alpha)]z}{1 + Bz}$. Also for $\alpha = 0$, the class $\mathcal{P}(A, B; \alpha)$ agrees with the class defined by Janowski [6].

Getting inspired by the above mentioned work, now we are going to define the following class:

Definition 1. Let $\mathcal{M}_{\mathcal{K}}(t; A, B; \alpha; \beta)(0 \leq \alpha < 1, 0 < \beta \leq 1, 0 < |t| \leq 1)$ denote the class of functions $f \in \mathcal{M}$ which satisfy the conditions,

$$-\frac{f'(z)}{tg(z)g(tz)} \prec \left(\frac{1+[B+(A-B)(1-\alpha)]z}{1+Bz}\right)^{\beta}, -1 \leq B < A \leq 1, z \in E^*,$$

where $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_k z^k \in \mathcal{MS}^*\left(\frac{1}{2}\right)$.

Particularly
(i) $\mathcal{M}_{\mathcal{K}}(-1; A, B; 0; 1) \equiv \mathcal{M}_{\mathcal{K}}(A, B)$, the class studied by Sim and Kwon [15].
(ii) $\mathcal{M}_{\mathcal{K}}(-1; 1, -1; 0; 1) \equiv \mathcal{M}_{\mathcal{K}}$, the class introduced by Wang et al. [17].

As $f \in \mathcal{M}_{\mathcal{K}}(t; A, B; \alpha; \beta)$, by definition of subordination, it follows that

$$(1) \qquad -\frac{f'(z)}{tg(z)g(tz)} = \left(\frac{1+[B+(A-B)(1-\alpha)]w(z)}{1+Bw(z)}\right)^{\beta}, w \in \mathcal{U}.$$

In this paper, we study the coefficient estimates, inclusion relationship, distortion theorem, argument theorem and radius of meromorphic convexity for the functions in the class $\mathcal{M}_{\mathcal{K}}(t; A, B; \alpha; \beta)$. The results proved by various authors follow as special cases.

Throughout this paper, we assume that $-1 \leq B < A \leq 1, 0 \leq \alpha < 1, 0 < \beta \leq 1, 0 < |t| \leq 1, z \in E^*$.

2. Preliminary Lemmas

For the derivation of our main results, we must require the following lemmas:

Lemma 1. [2,14] *Let,*

$$(2) \qquad \left(\frac{1+[B+(A-B)(1-\alpha)]w(z)}{1+Bw(z)}\right)^{\beta} = (P(z))^{\beta} = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

then

$$|p_n| \leq \beta(1-\alpha)(A-B), n \geq 1.$$

Lemma 2. [3] *For* $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_k z^k \in \mathcal{MS}^*$, *we have*

$$|b_n| \leq \frac{2}{n+1}.$$

Lemma 3. [13] *Let* $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, *then*

$$\left(\frac{1+A_1z}{1+B_1z}\right)^{\beta} \prec \left(\frac{1+A_2z}{1+B_2z}\right)^{\beta}.$$

Lemma 4. [11] *If* $g \in \mathcal{MS}^*$, *then for* $|z| = r, 0 < r < 1$, *we have*

$$\frac{(1-r)^2}{r} \leq |g(z)| \leq \frac{(1+r)^2}{r}.$$

Lemma 5. [5] *If* $f \in \mathcal{S}^*$, *then*

$$Re\left\{\frac{f(z)}{z}\right\}^{\frac{1}{2}} > \frac{1}{2}.$$

Lemma 6.5. [1,2] *If* $P(z) = \dfrac{1+[B+(A-B)(1-\alpha)]w(z)}{1+Bw(z)}, -1 \leq B < A \leq 1, w \in \mathcal{U}$, *then for* $|z| = r < 1$, *we have*

$$Re\frac{zP'(z)}{P(z)} \geq \begin{cases} -\frac{(A-B)(1-\alpha)r}{(1-[B+(A-B)(1-\alpha)]r)(1-Br)}, & \text{if } R_1 \leq R_2, \\ 2\frac{\sqrt{(1-B)(1-[B+(A-B)(1-\alpha)])(1+[B+(A-B)(1-\alpha)]r^2)(1+Br^2)}}{(A-B)(1-\alpha)(1-r^2)} & \\ -\frac{(1-[B+(A-B)(1-\alpha)]Br^2)}{(A-B)(1-\alpha)(1-r^2)} + \frac{(A+B)-\alpha(A-B)}{(A-B)(1-\alpha)}, & \text{if } R_1 \geq R_2, \end{cases}$$

where $R_1 = \sqrt{\frac{(1-[B+(A-B)(1-\alpha)])(1+[B+(A-B)(1-\alpha)]r^2)}{(1-B)(1+Br^2)}}$ *and* $R_2 = \frac{1-[B+(A-B)(1-\alpha)]r}{1-Br}$.

3. Main Results

Theorem 1. *If* $g \in \mathcal{MS}^*(\frac{1}{2})$ *and* $0 < |t| \leq 1$, *then*

$$tzg(z)g(tz) \in \mathcal{MS}^*.$$

Proof. As $g \in \mathcal{MS}^*(\frac{1}{2})$, we have

$$-Re\left\{\frac{zg'(z)}{g(z)}\right\} > \frac{1}{2}.$$

Let $h(z) = tzg(z)g(tz)$. Differentiating logarithmically, it yields

$$\frac{zh'(z)}{h(z)} = 1 + \frac{zg'(z)}{g(z)} + \frac{tzg'(tz)}{g(tz)}.$$

Therefore

$$-Re\left\{\frac{zh'(z)}{h(z)}\right\} = 1 - Re\left\{\frac{zg'(z)}{g(z)}\right\} - Re\left\{\frac{tzg'(tz)}{g(tz)}\right\},$$

which implies

$$-Re\left\{\frac{zh'(z)}{h(z)}\right\} > -1 + \frac{1}{2} + \frac{1}{2}.$$

Hence $Re\left\{\frac{zh'(z)}{h(z)}\right\} < 0$ and so $h \in \mathcal{MS}^*$. □

THEOREM 2. *If* $f(z) = \dfrac{1}{z} + \sum_{n=1}^{\infty} a_k z^k \in \mathcal{M}_{\mathcal{K}}(t; A, B; \alpha; \beta)$, *then*

$$|a_1| \leq 1$$

and

$$|a_n| \leq \frac{2}{n(n+1)} + \frac{\beta(1-\alpha)(A-B)}{n}\left[1 + \sum_{k=1}^{n-1} \frac{2}{k+1}\right]. \tag{3}$$

Proof. As $f \in \mathcal{M}_{\mathcal{K}}(t; A, B; \alpha; \beta)$, therefore (1) can be expressed as

$$-\frac{f'(z)}{tg(z)g(tz)} = (P(z))^{\beta},$$

which can be further represented as

$$\frac{-zf'(z)}{G(z)} = (P(z))^{\beta}, \tag{4}$$

where $G(z) = tg(z)g(tz)$.
For

$$q(z) = \frac{-zf'(z)}{G(z)}, \tag{5}$$

we have

$$q(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

Putting for f, G and q in (5), it yields

$$\frac{1}{z} - a_1 z - 2a_2 z^2 - ... - na_n z^n - ...$$

$$= \left(\frac{1}{z} + b_1 z + b_2 z^2 + ... + b_n z^n + ...\right)\left(1 + p_1 z + p_2 z^2 + ... + p_n z^n + p_{n+1} z^{n+1} + ...\right). \tag{6}$$

As f is univalent in E^*, it is well known that $|a_n| \leq 1$.
Comparing the coefficients of z^n in (6), we have

$$-na_n = b_n + b_{n-1}p_1 + b_{n-2}p_2 + ... + b_2 p_{n-2} + b_1 p_{n-1} + p_{n+1}. \tag{7}$$

Applying triangle inequality and using Lemma 1 and Lemma 2 in (7), it gives

$$n|a_n| \leq \frac{2}{n+1} + \beta(1-\alpha)(A-B)\left[\frac{2}{n} + \frac{2}{n-1} + ... + \frac{2}{3} + 1 + 1\right], \tag{8}$$

which proves Theorem 2. □

For $t = -1, \alpha = 0, \beta = 1$, Theorem 2 gives the following result due to Sim and Kwon [15].

COROLLARY 1. *If* $f \in \mathcal{M}_{\mathcal{K}}(A, B)$, *then*

$$|a_1| \leq 1$$

and

$$|a_n| \leq \frac{2}{n(n+1)} + \frac{A-B}{n}\left[1 + \sum_{k=1}^{n-1} \frac{2}{k+1}\right].$$

Putting $t = -1, A = 1,\ B = -1,\ \alpha = 0$ and $\beta = 1$ in Theorem 2, the following result due to Wang et al. [17] is obvious:

Corollary 2. *If* $f \in \mathcal{M}_{\mathcal{K}}$, *then*

$$|a_1| \leq 1$$

and

$$|a_n| \leq \frac{2}{n}\left[\frac{n+2}{n+1} + \sum_{k=1}^{n-1}\frac{2}{k+1}\right].$$

Theorem 3. *If* $-1 \leq B_2 = B_1 < A_1 \leq A_2 \leq 1$ *and* $0 \leq \alpha_2 \leq \alpha_1 < 1$, *then*

$$\mathcal{M}_{\mathcal{K}}(t; A_1, B_1; \alpha_1; \beta) \subset \mathcal{M}_{\mathcal{K}}(t; A_2, B_2; \alpha_2; \beta).$$

Proof. As $f \in \mathcal{M}_{\mathcal{K}}(t; A_1, B_1; \alpha_1; \beta)$, so

$$-\frac{f'(z)}{tg(z)g(tz)} \prec \left(\frac{1 + [B_1 + (A_1 - B_1)(1-\alpha_1)]z}{1 + B_1 z}\right)^{\beta}.$$

As $-1 \leq B_2 = B_1 < A_1 \leq A_2 \leq 1$ and $0 \leq \alpha_2 \leq \alpha_1 < 1$, we have

$$-1 \leq B_1 + (1-\alpha_1)(A_1 - B_1) \leq B_2 + (1-\alpha_2)(A_2 - B_2) \leq 1.$$

Thus by Lemma 3, it yields

$$-\frac{f'(z)}{tg(z)g(tz)} \prec \left(\frac{1 + [B_2 + (A_2 - B_2)(1-\alpha_2)]z}{1 + B_2 z}\right)^{\beta},$$

which implies $f \in \mathcal{M}_{\mathcal{K}}(t; A_2, B_2; \alpha_2; \beta)$. □

Theorem 4. *If* $f \in \mathcal{M}_{\mathcal{K}}(t; A, B; \alpha; \beta)$, *then for* $|z| = r, 0 < r < 1$, *we have*

$$\begin{aligned}\left(\frac{1 - [B + (A-B)(1-\alpha)]r}{1 - Br}\right)^{\beta} . \frac{(1-r)^2}{r^2} &\leq |f'(z)| \\ &\leq \left(\frac{1 + [B + (A-B)(1-\alpha)]r}{1 + Br}\right)^{\beta} . \frac{(1+r)^2}{r^2}\end{aligned} \tag{9}$$

and

$$\begin{aligned}\int_0^r \left(\frac{1 - [B + (A-B)(1-\alpha)]t}{1 - Bt}\right)^{\beta} . \frac{(1-t)^2}{t^2} dt &\leq |f(z)| \\ &\leq \int_0^r \left(\frac{1 + [B + (A-B)(1-\alpha)]t}{1 + Bt}\right)^{\beta} . \frac{(1+t)^2}{t^2} dt.\end{aligned} \tag{10}$$

Proof. From (4), we have

$$|f'(z)| = \frac{|G(z)|}{|z|}(P(z))^{\beta}. \tag{11}$$

Aouf [2] proved that

$$\frac{1 - [B + (A-B)(1-\alpha)]r}{1 - Br} \leq |P(z)| \leq \frac{1 + [B + (A-B)(1-\alpha)]r}{1 + Br},$$

which implies

$$\left(\frac{1-[B+(A-B)(1-\alpha)]r}{1-Br}\right)^{\beta} \leq |P(z)|^{\beta} \leq \left(\frac{1+[B+(A-B)(1-\alpha)]r}{1+Br}\right)^{\beta}. \tag{12}$$

Since $G \in \mathcal{MS}^*$, so by Lemma 4, we have

$$\frac{(1-r)^2}{r} \leq |G(z)| \leq \frac{(1+r)^2}{r}. \tag{13}$$

(11) together with (12) and (13) yields (9). On integrating (9) from 0 to r, (10) follows. □

For $t=-1, \alpha=0, \beta=1$, Theorem 4 gives the following result for the class $\mathcal{M}_{\mathcal{K}}(A,B)$.

Corollary 3. *If $f \in \mathcal{M}_{\mathcal{K}}(A,B)$, then for $|z|=r, 0<r<1$, we have*

$$\left(\frac{(1-r)^2(1-Ar)}{r^2(1-Br)}\right) \leq |f'(z)| \leq \left(\frac{(1+r)^2(1+Ar)}{r^2(1+Br)}\right)$$

and

$$\int_0^r \left(\frac{(1-t)^2(1-At)}{t^2(1-Bt)}\right) dt \leq |f(z)| \leq \int_0^r \left(\frac{(1+t)^2(1+At)}{t^2(1+Bt)}\right) dt.$$

On putting $t=-1$, $A=1$, $B=-1$, $\alpha=0$ and $\beta=1$ in Theorem 4, the following result is obvious:

Corollary 1. *If $f \in \mathcal{M}_{\mathcal{K}}$, then for $|z|=r, 0<r<1$, we have*

$$\frac{(1-r)^3}{r^2(1+r)} \leq |f'(z)| \leq \frac{(1+r)^3}{r^2(1-r)}$$

and

$$\int_0^r \frac{(1-t)^3}{t^2(1+t)} dt \leq |f(z)| \leq \int_0^r \frac{(1+t)^3}{t^2(1-t)} dt.$$

Theorem 5. *If $f \in \mathcal{M}_{\mathcal{K}}(t;A,B;\alpha;\beta)$, then for $|z|=r, 0<r<1$, we have*

$$\left|arg(-z^2 f'(z)\right| \leq \beta sin^{-1}\left(\frac{(A-B)(1-\alpha)r}{1-[B+(A-B)(1-\alpha)]Br^2}\right) + 2sin^{-1}r.$$

Proof. From (4), we have

$$-f'(z) = tg(z)g(tz)(P(z))^{\beta},$$

which implies

$$\left|arg(-z^2 f'(z))\right| \leq \beta|argP(z)| + arg(zg(z)) + arg(tzg(tz)). \tag{14}$$

Aouf [2], established that,

$$|argP(z)| \leq sin^{-1}\left(\frac{(A-B)(1-\alpha)r}{1-[B+(A-B)(1-\alpha)]Br^2}\right). \tag{15}$$

As $g \in \mathcal{MS}^*(\frac{1}{2})$, so $g(z) \neq 0$ for $z \in E^*$ and $h \equiv \frac{1}{g} \in \mathcal{MS}^*(\frac{1}{2})$.
Let us define $k(z) = \frac{(g(z))^2}{z}$, then $k \in \mathcal{S}^*$ and applying Lemma 5, we have

$$Re\left\{\frac{k(z)}{z}\right\}^{\frac{1}{2}} > \frac{1}{2}.$$

The relation between g, h and k, yields

$$zg(z) \prec 1 + z,$$

which implies

$$|zg(z) - 1| \leq r,$$

and hence

$$|arg(zg(z))| \leq sin^{-1}r. \tag{16}$$

Now using the results (15) and (16) in (14), the proof of Theorem 5 is obvious. □

THEOREM 6. *Let $f \in \mathcal{M}_{\mathcal{K}}(t; A, B; \alpha; \beta)$, then*

$$-Re\frac{(zf'(z))'}{f'(z)} \geq \begin{cases} -\dfrac{1+r}{1-r} - \beta\dfrac{(A-B)(1-\alpha)r}{(1-[B+(A-B)(1-\alpha)]r)(1-Br)}, & \text{if } R_1 \leq R_2, \\ -\dfrac{1+r}{1-r} + \dfrac{(A+B)-\alpha(A-B)}{(A-B)(1-\alpha)} \\ +2\frac{\sqrt{(1-B)(1-[B+(A-B)(1-\alpha)])(1+[B+(A-B)(1-\alpha)]r^2)(1+Br^2)}}{(A-B)(1-\alpha)(1-r^2)} \\ -2\frac{(1-[B+(A-B)(1-\alpha)]Br^2)}{(A-B)(1-\alpha)(1-r^2)}, & \text{if } R_1 \geq R_2, \end{cases}$$

where R_1 and R_2 are defined in Lemma 6.

Proof. As $f \in \mathcal{M}_{\mathcal{K}}(t; A, B; \alpha; \beta)$, we have

$$-zf'(z) = G(z)(P(z))^{\beta}.$$

Differentiating logarithmically, we get

$$-\frac{(zf'(z))'}{f'(z)} = \frac{zG'(z)}{G(z)} + \beta\frac{zP'(z)}{P(z)}. \tag{17}$$

As $G \in \mathcal{MS}^*$, we have

$$Re\left(\frac{zG'(z)}{G(z)}\right) \geq -\frac{1+r}{1-r}. \tag{18}$$

Hence, using (18) and Lemma 6 in (17), the proof of Theorem 6 is obvious. □

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