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COMMON FIXED POINT THEOREMS FOR THREE MAPPINGS IN GENERALIZED MODULAR METRIC SPACES

Sheela Arockiam* and Maria Joseph John

Abstract. In this paper, we obtain common fixed point theorems for three mappings of contractive type in the setting of generalized modular metric spaces. Our results generalize many results available in the literature including common fixed point theorems.

1. Introduction and Mathematical Preliminaries

Let (X, d) be a metric space and let $f : X \to X$ be a self-map. A point $x \in X$ is called a fixed point of f if the equation $f(x) = x$ is satisfied. In 1922, Steven Banach proved the very famnous theorem, "Banach Contraction Principle" in his thesis. He proved every contraction map on a complete metric space has a unique fixed point. Further, he proved in this theorem that if x_0 is fixed in the complete metric space X and if $f : X \to X$ is a contraction, then the sequence of iteration $\{x_n\}$ defined by $x_{n+1} = f(x_n)$ converges to th e fixed point. This theorem has applications in all branches of science. Thereafter, many authors extended this theorem in all directions extending either the domain or the function.

In particular, Perov [6] proved generalized Banach Contraction Principle in the setting of complete vector/matrix valued metric spaces.

Muhammad Usman Ali et al [15] proved matrix version of contraction principle as follows :

Let X be a nonempty set and R^m is the set of all $m \times 1$ matrices with real entries. If $\alpha, \beta \in R^m$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)^T$, $\beta = (\beta_1, \beta_2, ..., \beta_m)^T$ and $c \in R$, then define $\leq$ by $\alpha \leq \beta$ ($\alpha < \beta$) if and only if $\alpha_i \leq \beta_i$ ($\alpha_i < \beta_i$) for each $i \in \{1, 2, ..., m\}$ and $\alpha \leq c$, if and only if $\alpha_i \leq c$ for each $i \in \{1, 2, ..., m\}$.

A mapping $d : X \times X \to R^m$ is called a vector-valued / generalized metric on X if the following properties are satisfied :

1. $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

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* Corresponding author.

A set X equipped with a vector-valued / generalized metric d is called a vector-valued / generalized metric space and it is denoted by (X, d).

Throughout this paper we denote

1. $R_+ = [0, \infty)$;
2. $M_{m\times m}(R_+)$ = The set of all $m \times m$ matrices with real entries;
3. $D_{m\times m}([0, 1))$ = The set of all $m \times m$ diagonal matrices with entries $\in [0, 1)$;
4. Zero $m \times m$ matrix $= \overline{0}$;
5. Identity $m \times m$ matrix = I;
6. $A^0 = I$, $A \neq \overline{0}$.

Note that if $A \in D_{m\times m}([0, 1))$, then $A \leq I$.

A matrix A is said to be convergent to zero if and only if $A^n \to \overline{0}$ as $n \to \infty$.

Following are some matrices which converges towards zero :

1. Any matrix $A = \begin{pmatrix} b & b \\ a & a \end{pmatrix}$, where $a, b \in R_+$ and $a + b < 1$.
2. Any matrix $A = \begin{pmatrix} b & a \\ b & a \end{pmatrix}$, where $a, b \in R_+$ and $a + b < 1$.
3. Any matrix $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where $a, b, c \in R_+$ and $\max\{a, c\} < 1$.
4. If $A \in D_{m\times m}([0, 1))$, then A converges to $\overline{0}$.

THEOREM 1.1. [5] *Let $A \in M_{m\times m}(R_+)$. The following statements are equivalent.*

1. *A is convergent towards zero.*
2. *The eigenvalues of A are in the open unit disc, that is, $|\lambda| < 1$, for every $\lambda \in C$ with $det(A - \lambda I) = 0$.*
3. *The matrix $I - A$ is nonsingular and $(I - A)^{-1} = I + A + ... + A^n + ...$*

DEFINITION 1.2. [15] A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is known as a modular metric on X if the following axioms hold :

1. $\omega(\lambda, x, y) = 0$ for every $\lambda > 0$ if and only if $x = y$;
2. for each $x, y \in X$, $\omega(\lambda, x, y) = \omega(\lambda, y, x)$ for all $\lambda > 0$;
3. for each $x, y, z \in X$, $\omega(\lambda + \mu, x, z) \leq \omega(\lambda, x, y) + \omega(\mu, y, z)$ for all $\lambda, \mu > 0$.

A modular metric on X is said to be regular if (1) is replaced with the following axiom : $x = y$ if and only if $\omega(\lambda, x, y) = 0$ for some $\lambda > 0$.

For fix $x_0 \in X$, the set $X_\omega = \{x \in X : \omega(\lambda, x, x_0) \to 0$ as $\lambda \to \infty\}$ is a modular space.

DEFINITION 1.3. [15] Let (X, ω) be a modular generalized metric space.

1. The sequence $\{x_n\}$ is $\omega - convergent$ to $x \in X_\omega$ if and only if $\omega(1, x_n, x) \to 0$ as $n \to \infty$.
2. The sequence $\{x_n\}$ is $\omega - Cauchy$ if $\omega(1, x_m, x_n) \to 0$ as $m, n \to \infty$.
3. A subset D of X_ω is $\omega - complete$ if any $\omega - Cauchy$ sequence in D is a $\omega - convergent$ in D.
4. A subset D of X_ω is $\omega - closed$ if $\omega - limit$ of each $\omega - convergent$ sequence of D always belongs to D.
5. A subset D of X_ω is $\omega - bounded$ if we have $\delta_\omega(D) = sup\{\omega(1, x, y); x, y \in D\} < \infty$.

6. A subset D of X_ω is $\omega-$*compact* if for any $\{x_n\}$ in D, there exists a subsequence $\{x_{n_k}\}$ and $x \in D$ such that $\omega(1, x_{n_k}, x) \to 0$ as $k \to \infty$.

Definition 1.4. [15] Let (X, ω) be a modular generalized metric space and $\{x_n\}$ be a sequence in X_ω. Then ω satisfies the Δ_M condition if $\lim_{m,n\to\infty} \omega(m-n, x_n, x_m) = 0$ for $m, n \in N$ with $m > n$ implies $\lim_{m,n\to\infty} \omega(\lambda, x_n, x_m) = 0$ for all $\lambda > 0$

Definition 1.5. [15] A modular generalized metric ω on X is strongly regular if the following conditions hold:

- condition (1) of modular generalized metric ω is replaced with $x = y$ if and only if $\omega(1, x, y) = 0$.
- $\lim_{n\to\infty} \omega(1, x_n, x) = 0$ and $\lim_{n\to\infty} \omega(1, x_n, y) = 0$ implies $\omega(1, x, y) = 0$.

Definition 1.6. [2] Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be Sequentially Convergent if for each sequence $\{y_n\}$ in X, the sequence $\{Ty_n\}$ converges implies $\{y_n\}$ is convergent.

Kannan [16] proved fixed point theorems for the mapping $T : X \to X$ defined on a complete metric space X satisfying the condition

$$d(Tx, Ty) \leq \alpha \, [d(x, Tx) + d(y, Ty)]$$

where $\alpha \in [0, 1/2)$.
Chatterjee [17] proved fixed point theorems for the self mapping T defined on a complete metric space X satisfying the condition

$$d(Tx, Ty) \leq \alpha \, [d(x, Ty) + d(y, Tx)]$$

where $\alpha \in [0, 1/2)$.
For further references, please refer all the papers in the reference including [1], [3], [8], [9], [10], [11], [12], [13] and [14].
Branciari [2] introduced the concept of sequentially convergent mapping.
Malceski et al [4] proved the following common fixed point theorem.

Theorem 1.7. [4] *Let (X, d) be a complete metric space, $T : X \to X$ be continuous, injective and sequentially convergent mapping and $S_1, S_2 : X \to X$. If there exist $\alpha > 0$, $\beta \geq 0$ such that $2\alpha + \beta < 1$ and*

$$d(TS_1x, TS_2y) \leq \alpha \, [d(Tx, TS_1x) + d(Ty, TS_2y)] + \beta \, [d(Tx, Ty)]$$

for all $x, y \in X$, then S_1 and S_2 have a unique common fixed point.

Definition 1.8. [18] Let $A \in (0, +\infty]$, $R_A^+ = [0, A)$. Let $F : R_A^+ \to R$ satisfy that

1. $F(0) = 0$ and $F(t) > 0$ for each $t \in (0, A)$;
2. F is nondecreasing on R_A^+;
3. F is continuous.

Define $\Im[0, A) = \{F \| F \text{satisfies (1) to (3)}\}$.

Definition 1.9. [18] Let $A \in (0, +\infty]$, Let $\psi : R_A^+ \to R^+$ satisfy that

1. $\psi(t) < t$ for each $t \in (0, A)$;
2. ψ is nondecreasing and right upper semi-continuous;
3. For each $t \in (0, A), \lim_{n\to\infty} \psi^n(t) = 0$.

Define $\Psi[0, A) = \{\psi \| \psi \text{satisfies (1) to (3)}\}$.

Xian Zhang [18] proved a common fixed point theorem for two maps in 2007.

Theorem 1.10. [18] *Let X be a comlplete metric space and let $D = sup\{d(x,y)/x, y \in X\}$. Set $A = D$ if $D = \infty$ and $A > D$ if $D < \infty$. Suppose that $T, S : X \to X$, $F \in \Im[0, A)$ and $\psi \in \Psi[0, F(A-0))$ satisfy $F(d(Tx, Sy)) \leq \psi(F(M(x,y)))$ for each $x, y \in X$, where*

$$M(x,y) = \max\left(d(x,y), d(Tx,x), d(Sy,y), \frac{1}{2}[(d(Tx,y) + d(Sy,x)]\right)$$

Then T and S have a unique common fixed point in X. Moreover for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ converges to the common fixed point of T and S.

In this paper, we prove common fixed point theorems for three mappings in the setting of Generalized Modular Metric Spaces which are generalization of many results including Malceski et al [4] and Xian Zhang [18].

Definition 1.11. [7] $\omega(\lambda, a, B) = \inf\{\omega(\lambda, a, b) : b \in B\}$
$\Omega(\lambda, A, B) = \max[sup_{a\in A}(\omega(\lambda, a, B)), sup_{b\in B}(\omega(\lambda, A, b))]$
If (X, ω) is a modular metric space, then $(CB(X), \Omega)$ is a metric space.

Theorem 1.12. [7] *Let (X, ω) be a modular complete generalized metric space. Let $CB(X)$ denote the set of all nonempty closed and bounded subsets of X. Let $T : (X, \omega) \to CB(X)$ such that $\Omega(1, Tx, Ty) \leq A\,\omega(1, x, y)$ for some $m \times m$ matrix A converges to zero. If ω satisfies Δ_M condition, then T has a fixed point in X.*

2. Main Results

In this section, we prove the existence of common fixed points for three contractive type mappings.

Theorem 2.1. *Let X_ω be a strongly regular complete generalized modular metric space and $T : X_\omega \to X_\omega$ be continuous, injective, sequentially convergent mapping such that $T(X_\omega)$ is $\omega-$complete. Let $S_1, S_2 : X_\omega \to X_\omega$ be self maps such that*

$$\begin{aligned}\omega(1, TS_1x, TS_2y) \leq A\,[\omega(1, Tx, TS_1x) + \omega(1, Ty, TS_2y)] + B\,[\omega(1, Tx, Ty)] \\ + C\,[\omega(1, Ty, TS_1x) + \omega(1, Tx, TS_2y)]\end{aligned}$$

where $A, B, C \in D_{m\times m}([0,1))$ with $2A + B + 2C < I$ and $\omega(1, Tx, S_1y) \leq \omega(1, x, y)$ (or) $\omega(1, Tx, S_2y) \leq \omega(1, x, y)$ for all $x, y \in X_\omega$. Then T, S_1 and S_2 have a unique common fixed point.

Proof. Let T be continuous, injective and sequentially convergent mapping. Let $x_0 \in X_\omega$. Define a sequence $\{x_n\}$ by $x_{2n+1} = S_1x_{2n}$, $x_{2n+2} = S_2x_{2n+1}$ for $n = 0, 1, 2.....$
Let n be even.

$$\begin{aligned}\omega(1, Tx_n, Tx_{n+1}) &= \omega(TS_2x_{n-1}, TS_1x_n) \\ &\leq A\,[\omega(1, Tx_n, TS_1x_n) + \omega(1, Tx_{n-1}, TS_2x_{n-1})] + B\,[\omega(1, Tx_n, Tx_{n-1})] \\ &\quad + C\,[\omega(1, Tx_{n-1}, TS_1x_n) + \omega(1, Tx_n, TS_2x_{n-1})] \\ &\leq (A + C)\,\omega(1, Tx_n, Tx_{n+1}) + (A + B + C)\,\omega(1, Tx_{n-1}, Tx_n)\end{aligned}$$

Hence

$$\omega(1, Tx_n, Tx_{n+1}) \leq (A + B + C)\,[I - (A + C)]^{-1}\,\omega(1, Tx_{n-1}, Tx_n)$$

Since $2A+B+2C<I$, $T(X_\omega)$ is $\omega-complete$, T is sequentially convergent and T is continuous, there exists $x\in X_\omega$ such that $x_n\to x$ as $n\to\infty$. and hence $Tx_n\to Tx$. Now

$$\begin{aligned}\omega(1,Tx,TS_1x)&\le \omega(1,Tx,Tx_{2n})+\omega(1,Tx_{2n},TS_1x)\\&=\omega(1,Tx,Tx_{2n})+\omega(1,TS_2x_{2n-1},TS_1x)\\&\le \omega(1,Tx,Tx_{2n})+A\,[\omega(1,Tx,TS_1x)+\omega(1,Tx_{2n-1},TS_2x_{2n-1})]\\&\qquad+B\,\omega(1,Tx,Tx_{2n-1})+C\,[\omega(1,Tx_{2n-1},TS_1x)+\omega(1,Tx,TS_2x_{2n-1})]\\&\to (A+C)\,\omega(1,Tx,TS_1x)\,\mathrm{as}\,n\to\infty.\end{aligned}$$

Hence $\omega(1,Tx,TS_1x)\le (A+C)\,\omega(1,Tx,TS_1x)$. Since $(A+C)<I$, $Tx=TS_1x$. Since T is injective, $x=S_1x$. Similarly $x=S_2x$. Since $\omega(1,Tx,S_2y)\le\omega(1,x,y)$,

$$\begin{aligned}\omega(1,x,Tx)&\le\omega(1,x,x_{2n})+\omega(1,x_{2n},Tx)\\&=\omega(1,x,x_{2n})+\omega(1,S_2x_{2n-1},Tx)\\&=\omega(1,x,x_{2n})+\omega(1,x_{2n-1},x)\\&\to 0\,\mathrm{as}\,n\to\infty.\end{aligned}$$

Therefore $Tx=x$. Hence T,S_1,S_2 have a common fixed point.
Let us prove the uniqueness : Suppose there exists $y\in X_\omega$ such that $S_1y=S_2y=y$. Now

$$\begin{aligned}\omega(1,Tx,Ty)&=\omega(1,TS_1x,TS_2y)\\&\le A\,[\omega(1,Tx,TS_1x)+\omega(1,Ty,TS_2y)]+B\,[\omega(1,Tx,Ty)]\\&\qquad+C\,[\omega(1,Ty,TS_1x)+\omega(1,Tx,TS_2y)]\\&\le (B+2C)\,\omega(1,Tx,Ty)\end{aligned}$$

Since $B+2C<I$, $Tx=Ty$ and hence $x=y$. □

Example 2.2. Let $X=\{p,q,r\}\subseteq R^3$ where $p=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $q=\begin{pmatrix}2\\0\\0\end{pmatrix}$, $r=\begin{pmatrix}5\\0\\0\end{pmatrix}$ and $d(p,q)=1$, $d(q,r)=3$, $d(p,r)=4$. Then X is a complete metric space in R^3.
Define $T:X\to X$, the identity map by $T(x)=I(x)=x$ for every $x\in X$.
Then T is continuous, injective and sequentially convergent.
Define $S_1,S_2:X\to X$ by $S_1(p)=S_2(p)=q$, $S_1(q)=S_2(q)=q.$, $S_1(r)=S_2(r)=p$.
Define $\omega:(0,\infty)\times X\times X\to R^3$ by $\omega(\lambda,p,q)=\dfrac{1}{\lambda}\begin{pmatrix}d(p,q)\\0\\0\end{pmatrix}$ Then (X,ω) is a strongly regular complete generalized modular metric space. Now,

$$\omega(1,TS_1p,TS_2q)=\begin{pmatrix}d(TS_1p,TS_2q)\\0\\0\end{pmatrix}=\begin{pmatrix}d(q,q)\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$Let\,A=\frac{1}{8}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix};B=\frac{2}{3}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix};C=\frac{1}{30}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

Then $2A+B+2C = \frac{59}{60}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} < I_3$. Now,

$$A[\omega(1,Tp,TS_1p)+\omega(1,Tq,TS_2q)]+B(\omega(1,Tp,Tq)+C[\omega(1,Tq,TS_1p)+\omega(1,Tp,TS_2q)]$$
$$= A\begin{pmatrix}d(p,q)\\ 0\\ 0\end{pmatrix}+B\begin{pmatrix}d(p,q)\\ 0\\ 0\end{pmatrix}+C\begin{pmatrix}d(p,q)\\ 0\\ 0\end{pmatrix}=\begin{pmatrix}\frac{99}{120}\\ 0\\ 0\end{pmatrix}$$

Hence

$$\omega(1,TS_1p,TS_2q) \le A[\omega(1,Tp,TS_1p)+\omega(1,Tq,TS_2q)] + B(\omega(1,Tp,Tq)+C[\omega(1,Tq,TS_1p)+\omega(1,Tp,TS_2q)]$$

And

$$\omega(1,TS_1q,TS_2r) = \begin{pmatrix}d(p,q)\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}$$

$$A[\omega(1,Tq,TS_1q)+\omega(1,Tr,TS_2r)+B(\omega(1,Tq,Tr)+C[\omega(1,Tr,TS_1q)+\omega(1,Tq,TS_2r)]$$
$$= A\begin{pmatrix}d(p,r)\\ 0\\ 0\end{pmatrix}+B\begin{pmatrix}d(q,r)\\ 0\\ 0\end{pmatrix}+C\begin{pmatrix}[d(q,r)+d(p,q)]\\ 0\\ 0\end{pmatrix}=\begin{pmatrix}\frac{79}{30}\\ 0\\ 0\end{pmatrix}$$

Hence

$$\omega(1,TS_1q,TS_2r) \le A[\omega(1,Tq,TS_1q)+\omega(1,Tr,TS_2r)+B(\omega(1,Tq,Tr) + C[\omega(1,Tr,TS_1q)+\omega(1,Tq,TS_2r)]$$

Similarly

$$\omega(1,TS_1r,TS_2p) \le A[\omega(1,Tr,TS_1r)+\omega(1,Tp,TS_2p)+B(\omega(1,Tp,Tr) + C[\omega(1,Tp,TS_1r)+\omega(1,Tr,TS_2p)]$$

Also $\omega(1,Tp,S_1q) \le \omega(1,p,q)$ (or) $\omega(1,Tp,S_2q) \le \omega(1,p,q)$ for all $p,q \in X_\omega$.
Thus T, S_1 and S_2 satisfy the hypothesis of Theorem 2.1.
Here q is the common fixed point of T, S_1 and S_2.

The following result of Malceski et al [4] is a special case of the Theorem 2.1.

Corollary 2.3. *Let X_ω be a complete strongly regular generalized modular metric space and $T : X_\omega \to X_\omega$ be continuous, injective, sequentially convergent mapping. Let $S_1, S_2 : X_\omega \to X_\omega$ be self maps such that*

$$\omega(1,TS_1x,TS_2y) \le A\left[\omega(1,Tx,TS_1x)+\omega(1,Ty,TS_2y)\right]+B\left[\omega(1,Tx,Ty)\right]$$

where $A, B \in D_{m\times m}([0,1))$ such that $2A+B<I$ and $\omega(1,Tx,S_1y) \le \omega(1,x,y)$ (or) $\omega(1,Tx,S_2y) \le \omega(1,x,y)$ for all $x,y \in X_\omega$, then T, S_1 and S_2 have a unique common fixed point.

Proof. The proof of the corollary follows from Theorem 2.1 by putting $C = 0$. □

The following result of Xian Zhang [18] is a special case of the Theorem 2.1.

COROLLARY 2.4. *Let X_ω be a comlplete strongly regular generalized modular metric space and let $D = sup\{\omega(1,x,y)/x,y \in X_\omega\}$. Set $A = D$ if $D = \infty$ and $A > D$ if $D < \infty$. Suppose that $T,S : X_\omega \to X_\omega$, $F \in \Im[0,A)$ and $\psi \in \Psi[0, f(A-0))$ satisfy $F(\omega(1,Tx,Sy)) \le \psi(F(M(x,y)))$ for each $x,y \in X_\omega$, where*

$$M(x,y) = \max\left(\omega(1,x,y), \omega(1,Tx,x), \omega(1,Sy,y), \frac{1}{2}[\omega(1,Tx,y) + \omega(1,Sy,x)]\right)$$

Then T and S have a unique common fixed point in X_ω. Moreover for each $x_0 \in X_\omega$, the iterated sequence $\{x_n\}$ with $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ converges to the common fixed point of T and S.

Proof. Let $T = I$, the identity map. Then I is sequentially convergent, continuous and injective. Let $T = S_1$, $S = S_2$. Let

$$M(x,y) = \max\left(\omega(1,x,y), \omega(1,Tx,x), \omega(1,Sy,y), \frac{1}{2}[\omega(1,Tx,y) + \omega(1,Sy,x)]\right)$$

Then for every $x,y \in X_\omega$, $\frac{1}{2}[\omega(1,Tx,y) + \omega(1,Sy,x)] \le M(x,y)$.
Let $A,B,C \in D_{m\times m}([0,1))$ with $2A+B+2C < I$. Define

$$\begin{aligned}\psi(t) &= \frac{1}{D}\left(A[\omega(1,Tx,x) + \omega(1,y,Sy)] + B[\omega(1,x,y)] + C[\omega(1,x,Sy) + \omega(1,y,Tx)]\right)t \\ &\le (2A+B+2C)t < t\end{aligned}$$

Thus $\Psi(t) < t$ for every $t \in (0,A)$, Ψ is non-decreasing, $\Psi^n(t) = (2A+B+2C)^n t \to 0$ and $F(t) = t$, then $F \in \Im[0,A)$. Hence

$$\begin{aligned}\omega(1,TS_1x,TS_2y) \le{}& A\left[\omega(1,Tx,TS_1x) + \omega(1,Ty,TS_2y)\right] + B\left[\omega(1,Tx,Ty)\right] \\ &+ C\left[\omega(1,Ty,TS_1x) + \omega(1,Tx,TS_2y)\right]\end{aligned}$$

Hence by Theorem 2.1, T and S have common fixed point. □

THEOREM 2.5. *Let K be a non-empty compact subset of a generalized modular metric space X_ω. Let $T: K \to K$ be continuous, injective mapping and let S_1, S_2 be self mappings of K. If there exists $A,B,C \in D_{m\times m}([0,1))$ such that $2A+B+2C \le I$ and*

$$\begin{aligned}\omega(1,TS_1x,TS_2y) \le{}& A\left[\omega(1,Tx,TS_1x) + \omega(1,Ty,TS_2y)\right] + B\left[\omega(1,Tx,Ty)\right] \\ &+ C\left[\omega(1,Ty,TS_1x) + \omega(1,Tx,TS_2y)\right]\end{aligned}$$

and $\omega(1,Tx,S_1y) \le \omega(1,x,y)$ (or) $\omega(1,Tx,S_2y) \le \omega(1,x,y)$ for all $x,y \in K$, then T, S_1 and S_2 have a common fixed point.

Proof. For each $n \in N$, let $x_{2n+1} = S_1x_{2n}$, $x_{2n+2} = S_2x_{2n+1}$ Then the sequence $\{x_n\} \subseteq K$. Since K is compact, $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\} \to x$ as $k \to \infty$. Let $\{x_{n_k}\} = x_n$ Since T is continuous, $\{Tx_{n_k}\} \to Tx$.

$$\begin{aligned}\omega(1,Tx,TS_1x) &\le \omega(1,Tx,Tx_{2n_k}) + \omega(1,Tx_{2n_k},TS_1x) \\ &= \omega(1,Tx,Tx_{2n_k}) + \omega(1,TS_2x_{2n_k-1},TS_1x) \\ &\le \omega(1,Tx,Tx_{2n_k}) + A\left[\omega(1,Tx,TS_1x) + \omega(1,Tx_{2n_k-1},TS_2x_{2n_k-1})\right] \\ &\quad + B\left[\omega(1,Tx,Tx_{2n_k-1})\right] + C\left[\omega(1,Tx_{2n_k-1},TS_1x) + \omega(1,Tx,TS_2x_{2n_k-1})\right] \\ &\to (A+C)\,\omega(1,Tx,TS_1x)\,\text{as}\,k \to \infty.\end{aligned}$$

Since $(A+C) < I$, T is injective, $x = S_1 x$. Similarly $x = S_2 x$.
Since $\omega(1, Tx, S_2 y) \leq \omega(1, x, y)$,
$\omega(1, x, Tx) \leq \omega(1, x, x_{2n_k}) + \omega(1, x_{2n_k}, Tx) \leq \omega(1, x, x_{2n_k}) + \omega(1, x_{2n_k - 1}, x) \to 0$ as $k \to \infty$. Therefore $Tx = x$. Hence T, S_1, S_2 have a common fixed point. □

Theorem 2.6. *Let K be a closed bounded convex subset of a generalized modular metric space and let $T : K \to K$ be a injective, continuous, sequentially convergent and affine with respect to $q \in F(T)$. Let $S_1, S_2 : K \to K$. If there exists $A, B, C \in D_{m\times m}([0,1))$ such that $(3A + B + 3C) \leq I$ and*

$$\begin{aligned}\omega(1, TS_1x, TS_2y) \leq A\left[\omega(1, Ty, TS_2y) + \omega(1, Tx, TS_1x)\right] + B\left[\omega(1, Tx, Ty)\right]\\ + C\left[\omega(1, Tx, TS_2y) + \omega(1, Ty, TS_1x)\right]\end{aligned}$$

and $\omega(1, Tx, S_1y) \leq \omega(1, x, y)$ (or) $\omega(1, Tx, S_2y) \leq \omega(1, x, y)$ for all $x, y \in K$, then T, S_1 and S_2 have a common fixed point.

Proof. Fix $x_0 \in K$. Define a sequence $\{x_n\}$ by $x_{2n+1} = (1-\alpha_n)q + \alpha_n S_1 x_{2n}$ amd $x_{2n+2} = (1-\alpha_n)q + \alpha_n S_2 x_{2n+1}$, where $\alpha_n \in (0,1)$ such that $\alpha_n \to 1$ as $n \to \infty$. Suppose n is even. Using the same technique used in Theorem 2.5, one can easily prove that

$$\omega(1, Tx_n, Tx_{n+1}) \leq \alpha_n\, I\, \omega(1, Tx_{n-1}, Tx_n)$$

Since X_ω is $\omega -$ *complete*, T is sequentially convergent and T is continuous, there exists $x \in X_\omega$ such that $x_n \to x$ and hence $Tx_n \to Tx$. Similarly

$$\begin{aligned}\omega(1, Tx, TS_2x) &\leq \omega(1, Tx, Tx_{2n+1}) + (1-\alpha_n)\,\omega(1, Tq, TS_2x) + A\,\alpha_n\omega(1, Tx_{2n}, Tx_{2n+1})\\ &\quad + A\,\alpha_n\omega(1, Tx, TS_2x) + B\,\alpha_n\omega(1, Tx_{2n}, Tx) + C\,\alpha_n[\omega(1, Tx_{2n}, Tx)\\ &\quad + \omega(1, Tx, TS_2x) + \omega(1, Tx, Tx_{2n}) + \omega(1, Tx_{2n}, TS_1x_{2n})]\\ &\to (A+C)\,\omega(1, Tx, TS_2x)\end{aligned}$$

Since $(A+C) < I$, $Tx = TS_2x$. Since T is injective, $x = S_2x$. Similarly $x = S_1x$.

$$\begin{aligned}\omega(1, x, Tx) &\leq \omega(1, x, Tx_{2n+1}) + \omega(1, Tx_{2n+1}, Tx)\\ &= \omega(1, x, Tx_{2n+1}) + \omega(1, T((1-\alpha_n)q + \alpha_n S_1x_{2n}), Tx)\\ &\leq \omega(1, x, Tx_{2n+1}) + (1-\alpha_n)\omega(1, Tq, Tx) + \alpha_n\omega(1, TS_1x_{2n}, Tx)\\ &\leq \omega(1, x, Tx_{2n+1}) + (1-\alpha_n)\omega(1, Tq, Tx) + \alpha_n\omega(1, Tx_{2n}, Tx)\\ &\to 0\,\text{as}\,n \to \infty.\end{aligned}$$

Hence $x = Tx$. Therefore x is a common fixed point of T, S_1 and S_2. □

3. Applications

In this section, we obtain the existence theorem for the following system of integral equations :

$$x(t) = f(t) + \int_a^b g_1(t, s, x(s), y(s))ds; y(t) = f(t) + \int_a^b g_2(t, s, x(s), y(s))ds \tag{1}$$

for each $t, s \in I = [a, b]$, where $g_i : I \times I \times R \times R \to R$ is a continuous function for $i = 1, 2$. We denote $(C[a, b], R)$, the space of all continuous real valued functions defined on I = [a,b].

THEOREM 3.1. *Let $X = (C[a, b], R)$. Consider the operators $T_i, S_1, S_2 : X \times Y \to X$ given by the formula*

$$T_i(S_i(x(t), y(t)) = f(t) + \int_a^b g_i(t, s, x(s), y(s))ds,$$

where $g_i : I \times I \times R \times R \to R$ is a continuous function for $i = 1, 2$. Also, assume that for every $t, s \in [a, b]$ and $x, y, u, v \in X$, we have

$$\begin{aligned}|g_i(t, s, x(s), y(s)) - g_i(t, s, u(s), v(s))| \le\ & a_{i1}\{|T_1x(s) - T_1S_1x(s)| + |T_1y(s) - T_1S_2y(s)|\}\\ & + a_{i2}\{|T_2x(s) - T_2S_1x(s)| + |T_2y(s) - T_2S_2y(s)|\}\\ & + b_{i1}\{|x(s) - u(s)|\} + b_{i2}\{|y(s) - v(s)|\}\\ & + c_{i1}\{|T_1x(s) - T_1S_2y(s)| + |T_1y(s) - T_1S_1x(s)|\}\\ & + c_{i2}\{|T_2y(s) - T_2S_2y(s)| + |T_2y(s) - T_2S_1x(s)|\}\end{aligned}$$

where $A = (b-a)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$; $B = (b-a)\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}$; $C = (b-a)\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix}$ and $2A+B+2C$ converge to zero if T is a sequence and convergent and if $|TS_1x - TS_2y| \le |x - y|$, then the system of integral equations (1) has at least one solution.

Proof. By the hypothesis of this theorem, we note that for each $t, s \in [a, b]$ and $x, y, u, v \in X$, we have

$$\begin{aligned}|T_i(x(t), y(t)) - T_i(u(t), v(t))| &\le \int_a^b |g_i(t, s, x(s), y(s)) - g_i(t, s, u(s), v(s))|ds\\ &\le \int_a^b [a_{i1}\{|T_1x(s) - T_1S_1x(s)| + |T_1y(s) - T_1S_2y(s)|\}\\ &\quad + a_{i2}\{|T_2x(s) - T_2S_1x(s)| + |T_2y(s) - T_2S_2y(s)|\}\\ &\quad + b_{i1}\{|x(s) - u(s)|\} + b_{i2}\{|y(s) - v(s)|\}\\ &\quad + c_{i1}\{|T_1x(s) - T_1S_2y(s)| + |T_1y(s) - T_1S_1x(s)|\}\\ &\quad + c_{i2}\{|T_2y(s) - T_2S_2y(s)| + |T_2y(s) - T_2S_1x(s)|\}]ds\end{aligned}$$

$$\begin{aligned}|T_i(x(t), y(t)) - T_i(u(t), v(t))| \le (b-a)\Bigg[& a_{i1}\Big(\max_{s\in I}\{|T_1x(s) - T_1S_1x(s)| + |T_1y(s) - T_1S_2y(s)|\}\Big)\\ & + a_{i2}\Big(\max_{s\in I}\{|T_2x(s) - T_2S_1x(s)| + |T_2y(s) - T_2S_2y(s)|\}\Big)\\ & + b_{i1}\Big(\max_{s\in I}\{|x(s) - u(s)|\}\Big) + b_{i2}\Big(\max_{s\in I}\{|y(s) - v(s)|\}\Big)\\ & + c_{i1}\Big(\max_{s\in I}\{|T_1x(s) - T_1S_2y(s)| + |T_1y(s) - T_1S_1x(s)|\}\Big)\\ & + c_{i2}\Big(\max_{s\in I}\{|T_2y(s) - T_2S_2y(s)| + |T_2y(s) - T_2S_1x(s)|\}\Big)\Bigg]ds\end{aligned}$$

Define the operator $T : W = X \times X \to W = X \times X$ *by* $T(\overline{x}) = T(x_1, x_2) = (T_1(x_1, x_2), T_2(x_1, x_2))$ for each $\overline{x} = (x_1, x_2) \in X \times X$, and consider the modular generalized metric space

$$\omega : (0, \infty) \times W \times W \to R^2 \ by$$

$$\omega(\lambda, \overline{x}(t), \overline{y}(t)) = \frac{1}{|\lambda|}\begin{pmatrix} \max_{t\in I}\{|x_1(t) - y_1(t)|\} \\ \\ \max_{t\in I}\{|x_2(t) - y_2(t)|\} \end{pmatrix}$$

It is trivial that W is $\omega - complete$ and

$$\omega(1, TS_1\overline{x}, TS_2\overline{y}) \le A[\omega(1, T\overline{x}, TS_1\overline{x}) + \omega(1, T\overline{y}, TS_2\overline{y})] + B[\omega(1, \overline{x}, \overline{y})] + C[\omega(1, T\overline{x}, TS_2\overline{y}) + \omega(1, T\overline{y}, TS_1\overline{x})]$$

Hence, by Theorem 1.12, there exist at least one $\overline{x} \in W$ such that $T\overline{x} = \overline{x}$. The system of integral equations (1) has at least one solution. □

4. Conclusion

In this paper, we proved Common fixed points for three Contractive type mappings. Further, we have provided an example for Theorem 2.1 and an application.

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Sheela Arockiam
Department of Mathematics, St. Joseph's College (Autonomous),
Tiruchirappalli, 620001, TN India
E-mail: sheela_arockiam@yahoo.com

Maria Joseph John
Department of Mathematics, St. Joseph's College (Autonomous),
Tiruchirappalli, 620001, TN India
E-mail: joseph80john@gmail.com