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GENERALIZED PSEUDO B-GABOR FRAMES ON FINITE ABELIAN GROUPS

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Abstract. We seek for an invertible map B from $L^2(\Gamma)$ to $L^2(G)$, where G is a finite abelian group and Γ is the direct product of finite cyclic groups which is isomorphic to G, so that any Gabor frame in $L^2(G)$, is a generalized pseudo B-Gabor frame.

1. Introduction

The theory of frames in Hilbert spaces is one of the rapidly growing research area in mathematics due to its wide range applications [1, 10]. Time- frequency analysis of signals in $L^2(\mathbb{R})$, as suggested by Dennis Gabor in *Theory of Communication* [9], requires a special system of the form $\{E_{mb}T_{na}g : m, n \in \mathbb{Z}\}$, where $g \in L^2(\mathbb{R})$ and E_{mb}, T_{na} $(m, n \in \mathbb{Z}, a, b > 0)$ are the Modulation and Translation operators respectively. Frames in Hilbert spaces were introduced in 1952 by Duffin and Schaeffer [4] in their study of non harmonic Fourier series. In 1980's, Janssen designed it an independent topic of mathematical investigation by his outstanding work [11].

Frames were brought to life by Daubechies, Grossmann and Meyer in 1986 with the fundamental works [3] and put forth the idea of combining Gabor analysis with frame theory. Gabor analysis aims at representing functions (signals) $f \in L^2(\mathbb{R})$ as a superposition of translated and modulated versions of a fixed function $g \in L^2(\mathbb{R})$. A major component in frame theory is the frame operator associated with a given frame. In particular, Gabor frame operators, which are very special in their construction, are acquiring notable research attention and are of interest in this paper too, in the general perspective of finite dimensional Hilbert spaces.

The concept of generalized Weyl-Heisenberg frames, frame operators in $L^2(\mathbb{R})$ and its characterization are discussed in [5, 6]. The notion of *B-translation*, *B-modulation* and *pseudo B-Gabor like frame* on general Hilbert space $\mathcal{H}$ is introduced in [13, 15]. We begin with some basic definitions and results which are needed for the present work in Section 2. Gabor analysis in $L^2(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_p})$ and $L^2(G)$ are discussed in Section 3. Section 4 focused on identifying Gabor frames in $L^2(G)$, as a generalized pseudo B-Gabor frames in $L^2(G)$. Throughout in this article, $L^2(\Gamma)$,

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where $\Gamma = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \ldots \mathbb{Z}_{m_p}$ and $m_1, m_2, \ldots, m_p$ are positive integers, denotes the space of all complex valued functions on $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \ldots \mathbb{Z}_{m_p}$ and $L^2(G)$ denotes the space of all complex valued functions on finite abelian group G with respect to standard inner product. Our basic references for both abstract Hilbert frame theory and the theory of Weyl-Heisenberg frames are [2, 10].

2. Preliminaries

A sequence of elements $\{u_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ is said to be a *frame* in $\mathcal{H}$, if there are positive constants α, β such that

$$\alpha\|x\|^2 \;\le\; \sum_{k=1}^{\infty} |\langle x, u_k\rangle|^2 \;\le\; \beta\|x\|^2 \;\; \forall\, x \in \mathcal{H}.$$

These constants α, β are called *frame bounds.* A frame $\{u_k\}_{k=1}^{\infty}$ is called a *tight frame* if the frame bounds are the same. It is called a *parseval frame or normalized tight frame* when $\alpha = \beta = 1$. If a sequence $\{u_k\}_{k=1}^{\infty}$ of elements in $\mathcal{H}$ satisfies the upper frame inequality, then it is known as a *Bessel sequence* or a *semi-frame sequence.*

Remark 2.1. If $\{u_k\}_{k=1}^{\infty}$ is a frame in a Hilbert space $\mathcal{H}$, then the map S defined by $Sx = \sum_{k=1}^{\infty}\langle x, u_k\rangle u_k$ for all $x \in \mathcal{H}$ is a bounded linear operator on $\mathcal{H}$, called the *frame operator* associated with the frame $\{u_k\}_{k=1}^{\infty}$. The frame operator is bounded, invertible, self-adjoint and positive [2].

Let $\{u_k\}_{k=1}^{\infty}$ be a frame with frame operator S in $\mathcal{H}$, then the frame $\{S^{-1}u_k\}_{k=1}^{\infty}$ is called the *canonical dual frame* of the frame $\{u_k\}_{k=1}^{\infty}$. As follows, every frame together with its canonical dual frame in $\mathcal{H}$ admits the *frame decomposition* in two ways.

Theorem 2.2. [2] *Let $\{u_k\}_{k=1}^{\infty}$ be a frame with frame operator S in a Hilbert space $\mathcal{H}$. Then for all $x \in \mathcal{H}$, $x = \sum_{k=1}^{\infty}\langle x, S^{-1}u_k\rangle u_k$ and $x = \sum_{k=1}^{\infty}\langle x, u_k\rangle S^{-1}u_k$. Both the series converge unconditionally for all $x \in \mathcal{H}$.*

3. Gabor frames on finite products of finite cyclic groups

This section is meant to develop the basic properties of Gabor frames on $L^2(\Gamma)$, the space of all complex valued functions on a finite product $\Gamma = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_p}$ where $m_1, m_2, \ldots, m_p$ are positive integers. In the finite dimensional Hilbert space $L^2(\Gamma)$, the standard inner product is given by,

$$\langle f, g\rangle = \sum_{i_1=1}^{m_1}\sum_{i_2=1}^{m_2}\cdots\sum_{i_p=1}^{m_p} f(i_1, i_2, \ldots, i_p)\overline{g(i_1, i_2, \ldots, i_p)}, \qquad f,\; g \in L^2(\Gamma)$$

For each $k = (k_1, k_2, \ldots, k_p) \in \Gamma$ the *translation operator* $T_k : L^2(\Gamma) \to L^2(\Gamma)$ is defined by,

$$T_k g(r_1, r_2, \ldots, r_p) = g(s_1, s_2, \ldots, s_p) \qquad (r_1, r_2, ..., r_p) \in \Gamma$$

where, $s = (s_1, s_2, \ldots, s_p) \in \Gamma$ is such that $r_j - k_j \equiv s_j (mod\ m_j)$ for $j = 1, 2, \ldots, p$.

Similarly for each $l = (l_1, l_2, \ldots, l_p) \in \Gamma$ the *modulation operator* $M_l : L^2(\Gamma) \to L^2(\Gamma)$ is defined by,

$$M_l g(r_1, r_2, ..., r_p) = e^{2\pi i[\frac{l_1 r_1}{m_1} + \frac{l_2 r_2}{m_2} + ... + \frac{l_p r_p}{m_p}]} g(r_1, r_2, ..., r_p), \qquad (r_1, r_2, ..., r_p) \in \Gamma$$

.

DEFINITION 3.1. For $g \in L^2(\Gamma) \setminus \{0\}$ the collection of elements $\{M_l T_k g;\ k,l \in \Gamma\}$ is called a *Gabor System* generated by the *window function* g. A *Gabor frame* (also known as a *Weyl-Heisenberg frame*) in $L^2(\Gamma)$ is a Gabor system which spans $L^2(\Gamma)$. The frame operator of such a frame is given by $S(f) = \sum\limits_{k,l\in\Gamma} \langle f, M_l T_k g\rangle M_l T_k g$.

The following result gives a significant property of a Gabor frame operator and which is helpful for our subsequent discussions.

PROPOSITION 3.2. *The frame operator of a Gabor frame in $L^2(\Gamma)$ commutes with all translations and all modulations involved in that frame.*

Proof. Let $\{M_l T_k g : k,l \in \Gamma\}$ in $L^2(\Gamma)$ be a Gabor frame in $L^2(\Gamma)$ with frame operator S. Then for any $l' = (l'_1, l'_2, ..., l'_p) \in \Gamma$, we have

$$S(M_{l'} f) = \sum_{k,l\in\Gamma} \langle M_{l'} f, M_l T_k g\rangle M_l T_k g$$
$$= \sum_{k,l\in\Gamma} \langle f, M_q M_l T_k g\rangle M_l T_k g$$

where $q = (q_1, q_2, ..., q_p) \in \Gamma$ is such that $m_j - l'_j \equiv q_j (mod\ m_j)$ for $j = 1, 2, ..., p$.
Hence

$$S(M_{l'} f) = \sum_{k,r\in\Gamma} \langle f, M_r T_k g\rangle M_l T_k g$$

where $r = (r_1, r_2, ..., r_p) \in \Gamma$ is such that $q_j + l_j \equiv r_j (mod\ m_j)$ for $j = 1, 2, ..., p$, so that $M_l = M_{l'} M_r$.
Therefore,

$$S(M_{l'} f) = \sum_{k,r\in\Gamma} \langle f, M_r T_k g\rangle M_{l'} M_r T_k g$$
$$= M_{l'} \sum_{k,r\in\Gamma} \langle f, M_r T_k g\rangle M_r T_k g$$
$$= M_{l'} S(f)$$

Similarly we can show that for any $k' = (k'_1, k'_2, ..., k'_p) \in \Gamma$, $S(T_{k'} f) = T_{k'} S(f)$. $\square$

Now we discuss the effect of invertible bounded linear operators on $L^2(\Gamma)$ in the context of Gabor frames. Let $B : L^2(\Gamma) \longrightarrow \mathcal{H}$ be an invertible bounded linear operator. Then we can observe that for a Gabor frame $\{M_l T_k g : k,l \in \Gamma\}$ in $L^2(\Gamma)$

$$B(\{M_l T_k g\ : k,l \in \Lambda\}) = \{BM_l T_k g\ : k,l \in \Gamma\}$$
$$= \{BM_l B^{-1} BT_k B^{-1} Bg\ : k,l \in \Gamma\}$$
$$= \{(BM_l B^{-1})(BT_k B^{-1})(Bg)\ : k,l \in \Gamma\}$$

Thus the image of the family $\{M_l T_k g : k,l \in \Gamma\}$ in $L^2(\Gamma)$ under B gives a frame in $\mathcal{H}$ and it is interesting to see that its elements are generated by the action of a family of operators $\{M_l^B T_k^B\ : k,l \in \Gamma\}$ on a single generator Bg, where $M_l^B = BM_l B^{-1}$ and $T_k^B = BT_k B^{-1}$. Thus, such image frames are also structured frames in $\mathcal{H}$. We will formulate the following definitions which will be useful in our further discussions.

Definition 3.3. For an invertible bounded linear operator $B : L^2(\Gamma) \to \mathcal{H}$ and $k \in \Gamma$, *generalised B-translation* T_k^B on $\mathcal{H}$ is defined by $T_k^B = BT_kB^{-1}$ and for $l \in \Gamma$, *generalized B-modulation* M_l^B on $\mathcal{H}$ is defined by $M_l^B = BM_lB^{-1}$ where T_k and M_l are respectively the translation and modulation operators on $L^2(\Gamma)$. The family $\{M_l^BT_k^Bg \ : k, l \in \Gamma\}$ generated by $g \in \mathcal{H}$ is called a *generalized pseudo B-Gabor like system* in $\mathcal{H}$. Such a system is called a *generalized pseudo B-Gabor like frame (generalized pseudo B-Gabor like Bessel sequence)* if it forms a frame (Bessel sequence) in $\mathcal{H}$.

Remark 3.4. In view of Proposition 3.2 a generalized pseudo B-Gabor like frame $\{M_l^BT_k^Bg \ : k, l \in \Gamma\}$ in $\mathcal{H}$ acts similar to a Gabor frame in $L^2(\Gamma)$ if the frame operator of the generalized pseudo B-Gabor like frame $\{M_l^BT_k^Bg \ : k, l \in \Gamma\}$ commutes with the generalized B-translation T_k^B and generalized B-modulation M_l^B all $k, l \in \Gamma$.

Theorem 3.5. *Let $B : L^2(\Gamma) \longrightarrow \mathcal{H}$ be an invertible bounded linear operator. Then the frame operator of a generalized pseudo B-Gabor like frame $\{M_l^BT_k^Bg \ : k, l \in \Gamma\}$ commutes with the generalized B-translation T_k^B and generalized B-modulation M_l^B all $k, l \in \Gamma$ if and only if B^*B commutes with the translations T_k and modulations M_l for all $k, l \in \Gamma$.*

Proof. Let S be the frame operator of a generalized pseudo B-Gabor like frame $\{M_l^BT_k^Bg \ : k, l \in \Gamma\}$ in $\mathcal{H}$. Then $B^{-1} : \mathcal{H} \to L^2(\Gamma)$ maps this frame to the Gabor frame $\{M_lT_kB^{-1}g \ : k, l \in \Gamma\}$ whose frame operator is $B^{-1}S(B^{-1})^*$. Hence by Proposition 3.2 the operator $B^{-1}S(B^{-1})^*$ commutes with M_l and T_k for all $k, l \in \Gamma$.

Now, assume that S commutes with the generalized B-translation T_k^B and generalized B-modulation M_l^B all $k, l \in \Gamma$. Then,

$$
\begin{aligned}
SBM_lB^{-1} &= BM_lB^{-1}S(B^*)^{-1}B^* \\
&= BM_l(B^{-1}S(B^{-1})^*)B^* \\
&= B(B^{-1}S(B^{-1})^*)M_lB^* \\
&= S(B^{-1})^*M_lB^* \\
BM_lB^{-1} &= (B^{-1})^*M_lB^* \\
M_lB^{-1}(B^*)^{-1} &= B^{-1}(B^*)^{-1}M_l \\
M_l(B^*B)^{-1} &= (B^*B)^{-1}M_l
\end{aligned}
$$

Similarly, $T_k(B^*B)^{-1} = (B^*B)^{-1}T_k$. Thus $(B^*B)^{-1}$ commutes with translations T_k and modulations M_l for all $k, l \in \Gamma$ and hence B^*B.

Conversely suppose that B^*B commutes with the translations T_k and modulations M_l for all $k, l \in \Gamma$. Then

$$
\begin{aligned}
SM_l^B &= SBM_lB^{-1} = S(B^{-1})^*(B^*B)M_lB^{-1} \\
&= S(B^{-1})^*M_l(B^*B)B^{-1} = S(B^{-1})^*M_lB^* \\
&= BB^{-1}S(B^{-1})^*M_lB^* = BM_l(B^{-1}S(B^{-1})^*)B^* \\
&= BM_lB^{-1}S = M_l^BS \text{ for all } l \in \Gamma
\end{aligned}
$$

Also it is true that, $ST_k^B = T_k^BS$. Thus S commutes with the generalized B-translation T_k^B and generalized B-modulation M_l^B all $k, l \in \Gamma$. □

DEFINITION 3.6. A generalized pseudo B-Gabor like frame $\{M_l^B T_k^B g \ : k, l \in \Gamma\}$ in $\mathcal{H}$ is called a generalized pseudo B-Gabor frame if the frame operator of it commutes with the generalized B-translation T_k^B and generalized B-modulation M_l^B all $k, l \in \Gamma$.

3.1. Gabor frames on finite abelian groups. In this section we will discuss the basics of Gabor frames on finite abelian groups [13].

For a finite abelian group G, space $\mathbb{C}^G$ is identified with the space $L^2(G)$ of complex valued functions on G equipped with the standard inner product.The unitary translation operators $T_g : L^2(G) \to L^2(G)$, for $g \in G$, by $T_g f(h) = f(hg^{-1})$ for all $h \in G$ and the modulation operators on $L^2(G)$ are pointwise products with *characters* on G [7,8,14], where characters are precisely the group homomorphisms from G into the multiplicative group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. The set of all characters on G forms a group under pointwise multiplication and this group is called the *dual group of* G and is denoted by $\widehat{G}$ [13].Thus for $\xi \in \widehat{G}$, the modulation operator $M_\xi : L^2(G) \to L^2(G)$, is given by $M_\xi f(g) = \xi(g) f(g)$ for all $g \in G$.

We now take care of Gabor systems on $L^2(G)$, where G is a finite abelian group with dual group $\widehat{G}$. A Gabor systems on $L^2(G)$ is a family $\mathcal{G}(\varphi, \Lambda) = \{M_\xi T_g \varphi : (g, \xi) \in \Lambda\}$ where Λ is a subset (preferably subgroup) of the product group $G \times \widehat{G}$ and if φ is a non zero element in $L^2(G)$. A Gabor system which spans $L^2(G)$ is a frame and is called a *Gabor frame*. The frame operator corresponding to the Gabor frame $\mathcal{G}(\varphi, \Lambda)$ is the map $S : L^2(G) \to L^2(G)$, given by $S(f) = \sum\limits_{(g,\xi)\in\Lambda} \langle f, M_\xi T_g \varphi \rangle M_\xi T_g \varphi; \ \ \forall\ f \in L^2(G)$.

4. Generalized pseudo B-Gabor frames on finite abelian groups

Continuing from the previous sections, here we will explore how to view a Gabor frame in $L^2(G)$ using Gabor frames in $L^2(\Gamma)$. It is trivial that if G is a non cyclic abelian group of order N then by fundamental theorem of algebra, G is isomorphic to a direct product of the form $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_p}$, for some positive integers $m_1, m_2, \ldots, m_p$ with $m_1 m_2 \cdots_p = N$. Let us denote this product group corresponding to G by Γ_G. To proceed with our discussions in this direction, the following result is useful. Here onwards the abelian group G is taken as $G = \{g_0,\ g_1,\ ...,\ g_{N-1}\}$.

LEMMA 4.1. *For each isomorphism f from Γ_G onto G the map $B_f : L^2(\Gamma_G) \to L^2(G)$ defined by $B_f(x)(g_j) = x(f^{-1}(g_j))$ for all $g_j \in G$, is linear, bounded and unitary with adjoint $B_f^* : L^2(G) \to L^2(\Gamma_G)$ given by $B_f^*(y)(r_1, r_2, ..., r_p) = y(f(r_1, r_2, ..., r_p))$ for all $(r_1, r_2, ..., r_p) \in \Gamma_G$.*

Proof. For $x, y \in L^2(\Gamma_G)$ and $c \in \mathbb{C}$

$$\begin{aligned} B_f(cx + y)(g_j) &= (cx + y)(f^{-1}(g_j)) \\ &= (cx)(f^{-1}(g_j)) + y(f^{-1}(g_j)) \\ &= c(x(f^{-1}(g_j))) + y(f^{-1}(g_j)) \\ &= cB_f(x)(g_j) + B_f(y)(g_j) \\ &= (cB_f x + B_f y)(g_j) \end{aligned}$$

for $j = 0, 1, \ldots, n-1$.Thus B_f is linear.
Since both the spaces are finite dimensional it easily follows that B_f is bounded.
The map $B_f^* : L^2(G) \to L^2(\Gamma_G)$ defined by $B_f^*(y)(r_1, r_2, ..., r_p) = y(f(r_1, r_2, ..., r_p))$ for

all $(r_1, r_2, ..., r_p) \in \Gamma_G$, is the adjoint of B_f. This can be verified as follows.
For $x \in L^2(\Gamma_G)$ and $y \in L^2(G)$

$$\begin{aligned}\langle B_f x, y\rangle &= \sum_{j=0}^{N-1} B_f x(g_j)\overline{y(g_j)} \\ &= \sum_{j=0}^{N-1} x(f^{-1}(g_j))\overline{y(g_j)} \quad ....(1) \\ \langle x, B_f^* y\rangle &= \sum_{(r_1,r_2,...,r_p)\in\Lambda} x(r_1, r_2, ..., r_p)\overline{B_f^* y(r_1, r_2, ..., r_p)} \\ &= \sum_{(r_1,r_2,...,r_p)\in\Lambda} x(r_1, r_2, ..., r_p)\overline{y(f(r_1, r_2, ..., r_p))} \\ &= \sum_{j=0}^{N-1} x(f^{-1}(g_j))\overline{y(g_j)} \quad ....(2)\end{aligned}$$

$$where f(r_1, r_2, ..., r_p) = g_j.$$

from (1)and (2) we see that $\langle B_f x, y\rangle = \langle x, B_f^* y\rangle$; for all $x \in L^2(\Gamma_G)$ and $y \in L^2(G)$.
Now

$$\begin{aligned}(B_f^* B_f)(x)(r_1, r_2, ..., r_p) &= B_f x(f(r_1, r_2, ..., r_p)) \\ &= x(f^{-1}(f(r_1, r_2, ..., r_p))) \\ &= x(r_1, r_2, ..., r_p)\end{aligned}$$

for all $(r_1, r_2, ..., r_p) \in \Gamma_G$ and $x \in L^2(\Gamma_G)$, showing that $B_f^* B_f$ is identity on $L^2(\Gamma_G)$.
Similarly

$$\begin{aligned}(B_f B_f^*)(\eta)(g_j) &= B_f^* \eta(f^{-1}(g_j)) \\ &= \eta(f(f^{-1}(g_j))) \\ &= \eta(g_j)\end{aligned}$$

for all $g_j \in G$ and $\eta \in L^2(G)$, showing that $B_f B_f^*$ is identity on $L^2(G)$. □

Now for $l = (l_1, l_2, ..., l_p) \in \Gamma_G$ define $\xi_l : G \to S^1$ by

$$\xi_l(g) = e^{2\pi i[\frac{l_1 r_1}{m_1} + \frac{l_2 r_2}{m_2} + ... + \frac{l_p r_p}{m_p}]}$$

where $(r_1, ..., r_p) = f^{-1}(g)$, $g \in G$. It can be verified that ξ_l is a homomorphism.
For let $g_j, g_k \in G$, and $(r_1, r_2, ..., r_p) = f^{-1}(g_j)$ and $(r_1', r_2', ..., r_p') = f^{-1}(g_k)$.
Then

$$f(r_1 +_{m_1} r_1', r_2 +_{m_2} r_2', ..., r_p +_{m_p} r_p') = f(r_1, r_2, ..., r_p) + f(r_1', r_2', ..., r_p') = g_j * g_k$$

hence

$$\begin{aligned}\xi_l(g_j)\xi_l(g_k) &= e^{2\pi i[\frac{l_1 r_1}{m_1} + \frac{l_2 r_2}{m_2} + ... + \frac{l_p r_p}{m_p}]} e^{2\pi i[\frac{l_1 r_1'}{m_1} + \frac{l_2 r_2'}{m_2} + ... + \frac{l_p r_p'}{m_p}]} \\ &= e^{2\pi i[\frac{l_1(r_1 +_{m_1} r_1')}{m_1} + \frac{l_2(r_2 +_{m_2} r_2')}{m_2} + ... + \frac{l_p(r_p +_{m_p} r_p')}{m_p}]} \\ &= \xi_l(g_j * g_k)\end{aligned}$$

This proves that ξ_l is a homomorphism from G into S^1.

Also for $l, k \in \Gamma_G$, if $\xi_l = \xi_k$ then

$$e^{2\pi i[\frac{l_1 r_1}{m_1}+\frac{l_2 r_2}{m_2}+.....+\frac{l_p r_p}{m_p}]} = e^{2\pi i[\frac{k_1 r_1}{m_1}+\frac{k_2 r_2}{m_2}+.....+\frac{k_p r_p}{m_p}]}$$

for all $(r_1, r_2, ..., r_p) \in \Gamma_G$.
If we choose $r_1 = 1, r_2 = r_3 = ... = r_p = 0$ we will get $l_1 = k_1$. In a similar manner we can show that $l_j = k_j$ for all $j = 1, 2, ..., p$ and hence $l = k$.
Thus we conclude that $\xi_l \neq \xi_k$ for $l \neq k$. Hence $\{\xi_l : l \in \Gamma_G\}$, is precisely the collection of characters of G.
Upcoming Proposition reveals the action of the bounded linear transformation B_f and its adjoint on modulations and translations.

PROPOSITION 4.2. *Let G be a finite abelian group and $L^2(\Gamma_G)$, f and B_f are as in Lemma 4.1. Also let M_l and T_k denotes the modulations and translations in $L^2(\Gamma_G)$. Then for $k, l \in \Gamma_G$,*

(i) $B_f M_l = M_{\xi_l} B_f$
(ii) $B_f T_k = T_{g_k} B_f$
(iii) $B_f^* M_{\xi_l} = M_l B_f^*$
(iv) $B_f^* T_{g_k} = T_k B_f^*$

where M_{ξ_l} and T_{g_k} are respectively the modulations and translations in $L^2(G)$ and ξ_l are the characters of G as in previous discussion.

Proof. Let $l \in \Gamma$ and $g_j \in G$ with $f^{-1}(g_j) = (r_1, r_2, ..., r_p)$. Then for all $x \in L^2(\Gamma_G)$,

$$\begin{aligned} [(M_{\xi_l} B_f)(x)](g_j) &= M_{\xi_l}(B_f(x))(g_j) \\ &= e^{2\pi i[\frac{l_1 r_1}{m_1}+\frac{l_2 r_2}{m_2}+.....+\frac{l_p r_p}{m_p}]}(x(f^{-1}(g_j))) \end{aligned}$$

and

$$\begin{aligned} (B_f M_l)(x)(g_j) &= B_f(M_l(x))(g_j) \\ &= M_l(x)(f^{-1}(g_j)) \\ &= M_l(x)(r_1, r_2, ..., r_p) \\ &= e^{2\pi i[\frac{l_1 r_1}{m_1}+\frac{l_2 r_2}{m_2}+.....+\frac{l_p r_p}{m_p}]}(x(r_1, r_2, ..., r_p)) \\ &= e^{2\pi i[\frac{l_1 r_1}{m_1}+\frac{l_2 r_2}{m_2}+.....+\frac{l_p r_p}{m_p}]}(x(f^{-1}(g_j))) \end{aligned}$$

Thus $B_f M_l = M_{\xi_l} B_f$.
Similarly if $k \in \Gamma$, g_j, $g_k \in G$ with $f^{-1}(g_j) = (r_1, r_2, ..., r_p)$, $f^{-1}(g_k^{-1}) = (q_1, ..., q_p)$, $r_j - q_j \equiv s_j (mod\ m_j)$ for $j = 1, 2, ..., p$ and $x \in L^2(\Gamma_G)$ then,

$$\begin{aligned} [(T_{g_k} B_f)(x)](g_j) &= T_{g_k}(B_f(x))(g_j) \\ &= (B_f(x))(g_j g_k^{-1}) \\ &= x(f^{-1}(g_j g_k^{-1})) \\ &= x(f^{-1}(g_j) - f^{-1}(g_k^{-1})) \\ &= x(s_1, s_2, ..., s_p) \end{aligned}$$

and

$$\begin{aligned}(B_fT_k)(x)(g_j) &= B_f(T_k(x))(g_j)\\ &= T_k(x)f^{-1}(g_j)\\ &= T_k(x)(r_1, r_2, ..., r_p)\\ &= x(s_1, s_2, ..., s_p)\end{aligned}$$

Hence, $B_fT_k = T_{g_k}B_f$. Now,

$$\begin{aligned}[(B_f^*M_{\xi_l})(\eta)](r_1, r_2, ..., r_p) &= M_{\xi_l}(\eta(f(r_1, r_2, ..., r_p)))\\ &= e^{2\pi i[\frac{l_1r_1}{m_1}+\frac{l_2r_2}{m_2}+.....+\frac{l_pr_p}{m_p}]}(\eta(f(r_1, r_2, ..., r_p)))\end{aligned}$$

and

$$\begin{aligned}(M_lB_f^*)(\eta)(r_1, r_2, ..., r_p) &= M_l(B_f^*\eta)(r_1, r_2, ..., r_p)\\ &= M_l(\eta(f(r_1, r_2, ..., r_p)))\\ &= e^{2\pi i[\frac{l_1r_1}{m_1}+\frac{l_2r_2}{m_2}+.....+\frac{l_pr_p}{m_p}]}(\eta(f(r_1, r_2, ..., r_p)))\end{aligned}$$

Thus, $B_f^*M_{\xi_l} = M_lB_f^*$
Again,

$$\begin{aligned}[(B_f^*T_{g_k})(\eta)](r_1, r_2, ..., r_p) &= T_{g_k}(\eta(f(r_1, r_2, ..., r_p)))\\ &= T_{g_k}(\eta(g_r))\\ &= (\eta(g_rg_k^{-1}))\\ &= (\eta(g_q))\end{aligned}$$

and

$$\begin{aligned}(T_kB_f^*)(\eta)(r_1, r_2, ..., r_p) &= T_k(B_f^*\eta)(r_1, r_2, ..., r_p)\\ &= B_f^*(\eta(q))\\ &= (\eta(f(q_1, q_2, ..., q_p)))\\ &= \eta(g_q)\end{aligned}$$

hence, $B_f^*T_{g_k} = T_kB_f^*$ □

Both unitary and non-unitary approaches hold equal importance in quantum field theory, as demonstrated in [12]. In this regard, images of Gabor frames under unitary and non unitary maps are equally significant, as is observed below.

THEOREM 4.3. *If G is a finite abelian group, then any Gabor frame in $L^2(G)$ can be identified as a generalized pseudo B-Gabor frame in $L^2(G)$.*

Proof. Let $\{M_{\xi_l}T_{g_k}\varphi : (g_k, \xi_l) \in G \times \widehat{G}\}$ be a Gabor frame in $L^2(G)$. From lemma 4.1, consider the map $B : L^2(\Gamma_G) \to L^2(G)$ defined by $B(x)(g_j) = x(f^{-1}(g_j))$ for all $g_j \in G$ where f is an isomorphism from Γ_G to G. Now using Proposition 4.2 we have,

$$\begin{aligned}\{B^{-1}(M_{\xi_l}T_{g_k}\varphi) : (g_k, \xi_l) \in G \times \widehat{G}\} &= \{(B^{-1}M_{\xi_l}B)(B^{-1}T_{g_k}B)B^{-1}\varphi) : (g_k, \xi_l) \in G \times \widehat{G}\}\\ &= \{M_lT_kB^{-1}\varphi : k, l \in \Gamma_G\}\end{aligned}$$

Since B is unitary and $\{M_{\xi_l}T_{g_k}\varphi : (g_k, \xi_l) \in G \times \widehat{G}\}$ is frame in $L^2(G)$, it is clear that $\{M_l T_k B^{-1}\varphi : k, l \in \Gamma\}$ is a frame in $L^2(\Gamma_G)$ and hence a Gabor frame in $L^2(\Gamma_G)$. Thus

$$\begin{aligned}\{M_{\xi_l}T_{g_k}\varphi : (g_k, \xi_l) \in G \times \widehat{G}\} &= \{B(M_l T_k(B^{-1}\varphi)) : k, l \in \Gamma\} \\ &= \{M_l^B T_k^B \varphi : k, l \in \Gamma\}\end{aligned}$$

That is, $\{M_{\xi_l}T_{g_k}\varphi : (g_k, \xi_l) \in G \times \widehat{G}\}$ is a generalised pseudo B-Gabor like frame in $L^2(G)$. Since B is unitary it is trivial that $\{M_{\xi_l}T_{g_k}\varphi : (g_k, \xi_l) \in G\times\widehat{G}\}$ is a generalised pseudo B-Gabor frame in $L^2(G)$ $\square$

Let $\Gamma = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times ... \times \mathbb{Z}_{m_p}$ for some positive integers $m_1, m_2, ..., m_p$. We make the following nice observation about the Fourier transform $\mathcal{F} : L^2(\Gamma) \to L^2(\Gamma)$ given in [13] by,

$$\mathcal{F}(g(r_1, r_2, ..., r_p)) = \sum_{i_1=1}^{m_1} ... \sum_{i_p=1}^{m_p} g(i_1, i_2, ..., i_p) e^{-2\pi i[\frac{i_1 r_1}{m_1} + ... + \frac{i_p r_p}{m_p}]}$$

for each $(r_1, r_2, ..., r_p) \in \Gamma$.

Proposition 4.4. *The Fourier transform $\mathcal{F}$ on $L^2(\Gamma)$ satisfies the relations; for $k, l \in \Gamma$,*
(i). $\mathcal{F}M_l = T_l\mathcal{F}$ and (ii). $\mathcal{F}T_k = M_{-k}\mathcal{F}$; where M_l, T_k are the modulation and translation in $L^2(\Gamma)$ respectively.

Proof. For $k, l \in \Gamma$ and for each $(r_1, r_2, ..., r_p) \in \Gamma$

$$\begin{aligned}&\mathcal{F}M_l(g(r_1, r_2, ..., r_p)) \\ &= \sum_{i_1=1}^{m_1} ... \sum_{i_p=1}^{m_p} M_l(g(i_1, i_2, ..., i_p)) e^{-2\pi i[\frac{i_1 r_1}{m_1} + ... + \frac{i_p r_p}{m_p}]} \\ &= \sum_{i_1=1}^{m_1} ... \sum_{i_p=1}^{m_p} e^{2\pi i[\frac{l_1 i_1}{m_1} + ... + \frac{l_p i_p}{m_p}]} g(i_1, i_2, ..., i_p) e^{-2\pi i[\frac{i_1 r_1}{m_1} + ... + \frac{i_p r_p}{m_p}]} \\ &= \sum_{i_1=1}^{m_1} ... \sum_{i_p=1}^{m_p} e^{-2\pi i[\frac{(r_1 - l_1) i_1}{m_1} + ... + \frac{(r_p - l_p) i_p}{m_p}]} g(i_1, i_2, ..., i_p) \\ &= T_l\mathcal{F}(g(r_1, r_2, ..., r_p)), \text{proving (i) and,}\end{aligned}$$

$$\begin{aligned}
&\mathcal{F}T_k(g(r_1, r_2, ..., r_p)) \\
&= \sum_{i_1=1}^{m_1} ... \sum_{i_p=1}^{m_p} T_k(g(i_1, i_2, ..., i_p))e^{-2\pi i[\frac{i_1 r_1}{m_1}+...+\frac{i_p r_p}{m_p}]} \\
&= \sum_{i_1=1}^{m_1} ... \sum_{i_p=1}^{m_p} e^{-2\pi i[\frac{i_1 r_1}{m_1}+...+\frac{i_p r_p}{m_p}]} g(i_1 - k_1, i_2 - k_2, ..., i_p - k_p) \\
&= \sum_{i_1=1}^{m_1} ... \sum_{i_p=1}^{m_p} e^{-2\pi i[\frac{(i_1-k_1)r_1}{m_1}+...+\frac{(i_p-k_p)r_p}{m_p}]} e^{-2\pi i[\frac{k_1 r_1}{m_1}+...+\frac{k_p r_p}{m_p}]} g(i_1 - k_1, ..., i_p - k_p) \\
&= e^{-2\pi i[\frac{k_1 r_1}{m_1}+...+\frac{k_p r_p}{m_p}]} \sum_{i_1=1}^{m_1} ... \sum_{i_p=1}^{m_p} e^{-2\pi i[\frac{(i_1-k_1)r_1}{m_1}+...+\frac{(i_p-k_p)r_p}{m_p}]} g(i_1 - k_1, ..., i_p - k_p) \\
&= M_{-k}\mathcal{F}(g(r_1, r_2, ..., r_p)), \text{ this proves (ii).}
\end{aligned}$$

□

From above Proposition we see that $\mathcal{F}$ on $L^2(\Gamma)$ does not commute with translations and modulations on $L^2(\Gamma)$ but since $\mathcal{F}$ is unitary it is trivially true that $\mathcal{F}^*\mathcal{F}$ commutes with translations and modulations on $L^2(\Gamma)$.

Following result shows the existence of a non unitary operator A on $L^2(\Gamma)$ which does not commute with translations and modulations on $L^2(\Gamma)$ but A^*A commutes with translations and modulations on $L^2(\Gamma)$.

THEOREM 4.5. *For any Gabor frame $\{M_l T_k g : l,\ k \in \Gamma\}$ in $L^2(\Gamma)$, there is a non unitary operator A on $L^2(\Gamma)$ which does not commute with modulations and translations involved in it but A^*A commutes with these involved modulations and translations.*

Proof. If $\{M_l T_k g : l,\ k \in \Gamma\}$ is a non-Parseval Gabor frame in $L^2(\Gamma)$, take S as the frame operator of the Gabor frame $\{M_l T_k g : l,\ k \in \Gamma\}$ otherwise take S as the frame operator of the Gabor frame $\{M_l T_k \alpha g : l,\ k \in \Gamma\}$ for some non zero complex number α with $|\alpha| \neq 1$. Then the frame operator S is non unitary. Define $A : L^2(\Gamma) \longrightarrow L^2(\Gamma)$ by $A = S\mathcal{F}$ where $\mathcal{F}$ is the Fourier transform on $L^2(\Gamma)$. Since S and $\mathcal{F}$ are invertible, it is clear that A is invertible with inverse $A^{-1} = \mathcal{F}^{-1}S^{-1}$. Since S is non unitary it is clear that A is a non unitary operator. Also it follows that for $k,\ l \in \Gamma$, $AT_k = S\mathcal{F}T_k = M_{-k}S\mathcal{F} = M_{-k}A$ and $AM_l == S\mathcal{F}M_l = T_l S\mathcal{F} = T_l A$. Thus A does not commute with the modulations and translations involved in the given frame.

Observe that, since S is self adjoint and $\mathcal{F}$ is unitary, $A^* = (S\mathcal{F})^* = \mathcal{F}^{-1}S^* = \mathcal{F}^{-1}S$ and hence $A^*A = \mathcal{F}^{-1}S^2\mathcal{F}$. Using commutator relation from Proposition 4.4 we have $\mathcal{F}^{-1}M_l = T_{-l}\mathcal{F}^{-1}$ and $\mathcal{F}^{-1}T_k = M_k\mathcal{F}^{-1}$ and hence, a direct simple calculation proves that, A^*A commutes with M_l and T_k, for all $k, l \in \Gamma$. □

Now we choose $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p) \in \Gamma$ with $gcd(\alpha_j, m_j) = 1$ and define two operators D_α and $D_{\frac{1}{\alpha}}$ on $L^2(\Gamma)$ by $(D_\alpha f)(r_1, r_2, \dots, r_p) = f(t_1, t_2, \dots, t_p)$ where $\alpha_j r_j \equiv t_j(mod\ m_j)$ and $(D_{\frac{1}{\alpha}} f)(r_1, r_2, \dots, r_p) = f(q_1, q_2, \dots, q_p)$ where $\alpha_j q_j \equiv r_j(mod\ m_j)$ for each $f \in L^2(\Gamma)$ and $(r_1, r_2, ..., r_p) \in \Gamma$. It is obvious that D_α and $D_{\frac{1}{\alpha}}$ are linear maps. Now for $f \in L^2(\Gamma)$ and $(r_1, r_2, ..., r_p) \in \Gamma$

$(D_\alpha D_{\frac{1}{\alpha}})(f)(r_1, r_2, ..., r_p) = D_\alpha(f(q_1, q_2, ..., q_p))$ where $\alpha_j q_j \equiv r_j (mod\ m_j)$
$= f(r_1, r_2, ..., r_p)$
Also $(D_{\frac{1}{\alpha}} D_\alpha)(f)(r_1, r_2, ..., r_p) = D_{\frac{1}{\alpha}}(f(t_1, t_2, ..., t_p))$; where $\alpha_j r_j \equiv t_j (mod\ m_j)$
$= f(r_1, r_2, ..., r_p)$
This shows that $D_\alpha D_{\frac{1}{\alpha}} = D_{\frac{1}{\alpha}} D_\alpha = I$. That is D_α and $D_{\frac{1}{\alpha}}$ are invertible maps with $D_\alpha^{-1} = D_{\frac{1}{\alpha}}$.
For all $f, g \in L^2(\Gamma)$,

$$\langle D_\alpha f, g\rangle = \sum_{i_1=1}^{m_1} \cdots \sum_{i_p=1}^{m_p} f(t_1, t_2, \ldots, t_p)\overline{g(i_1, i_2, \ldots, i_p)};\ \text{where } \alpha_j i_j \equiv t_j (mod\ m_j)$$
$$= \langle f, D_{\frac{1}{\alpha}} g\rangle$$

That is $D_\alpha^* = D_{\frac{1}{\alpha}} = D_\alpha^{-1}$. This proves D_α is unitary.

For $f \in L^2(\Gamma)$ and $k = (k_1, k_2, \ldots, k_p),\ l = (l_1, l_2, \ldots, l_p),\ \ (r_1, r_2, \ldots, r_p) \in \Gamma$

$$\begin{aligned} T_k D_\alpha f(r_1, r_2, \ldots, r_p) &= T_k(f(t_1, t_2, \ldots, t_p)); \text{where} \alpha_j r_j \equiv t_j (mod\ m_j) \\ &= f(s_1, s_2, \ldots, s_p),\ \text{where } t_j - k_j \equiv s_j (mod\ m_j) \\ &= D_\alpha f(q_1, q_2, \ldots, q_p);\ \text{where } \alpha_j q_j \equiv s_j (mod\ m_j) \end{aligned}$$

choose $k' = (k'_1, k'_2, \ldots, k'_p) \in \Gamma$ such that $\alpha_j k'_j \equiv k_j (mod\ m_j)$
so that $t_j - k_j \equiv s_j (mod\ m_j)$ becomes $\alpha_j r_j - \alpha_j k'_j \equiv \alpha_j q_j (mod\ m_j)$ and hence $r_j - k'_j \equiv q_j (mod\ m_j)$ and $T_k D_\alpha f(r_1, r_2, \ldots, r_p) = D_\alpha T_{k'} f(r_1, r_2, \ldots, r_p)$.
Thus $T_k D_\alpha = D_\alpha T_{k'}$; where $\alpha_j k'_j \equiv k_j (mod\ m_j)$
.

Further, $(D_\alpha M_l) f(r_1, r_2, \ldots, r_p) = M_l f(t_1, t_2, \ldots, t_p)$; where $\alpha_j r_j \equiv t_j (mod\ m_j)$
$$= e^{2\pi i[\frac{l_1 t_1}{m_1} + \cdots + \frac{l_p t_p}{m_p}]} f(t_1, t_2, \ldots, t_p)$$
$\alpha_j r_j \equiv t_j (mod\ m_j) \Rightarrow \alpha_j r_j - t_j = c_j m_j;$ for some integer c_j. $\Rightarrow t_j = \alpha_j r_j - c_j m_j;$ therefor from the above expression,

$$\begin{aligned} (D_\alpha M_l) f(r_1, r_2, \ldots, r_p) &= e^{2\pi i[\frac{l_1(\alpha_1 r_1 - c_1 m_1)}{m_1} + \cdots + \frac{l_p(\alpha_p r_p - c_p m_p)}{m_p}]} f(t_1, t_2, \ldots, t_p) \\ &= e^{2\pi i[\frac{l_1 \alpha_1 r_1}{m_1} + \cdots + \frac{l_p \alpha_p r_p}{m_p}]} e^{2\pi i[l_1 c_1 + \cdots + l_p c_p]} f(t_1, t_2, \ldots, t_p) \\ &= e^{2\pi i[\frac{l_1 \alpha_1 r_1}{m_1} + \cdots + \frac{l_p \alpha_p r_p}{m_p}]} D_\alpha f(r_1, r_2, \ldots, r_p) \\ &= M_{l'} D_\alpha f(r_1, r_2, \ldots, r_p); \end{aligned}$$
where $l' = (l'_1, l'_2, \ldots, l'_p) \in \Gamma$ with $\alpha_j l_j \equiv l'_j (mod\ m_j)$
Thus $D_\alpha M_l = M_{l'} D_\alpha$; where $\alpha_j l_j \equiv l'_j (mod\ m_j)$.

It follows from above discussions that if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) \in \Gamma$, with$\alpha_j \neq 1$ for some $j \in \{1,\ 2, \ldots, p\}$ and $gcd(\alpha_j, m_j) = 1$ for all $j \in \{1,\ 2, \ldots, p\}$ then D_α does not commute with modulations and translations in $L^2(\Gamma)$.

Interestingly, generalized pseudo C-Gabor like frames under non unitary maps C are equally significant in $L^2(G)$, as is shown in the following observation.

Theorem 4.6. *For a finite abelian group G, the canonical dual frame of a non-Parseval Gabor frame in $L^2(G)$ is a generalized pseudo C-Gabor like frame in $L^2(G)$ where the map $C : L^2(\Gamma_G) \longrightarrow L^2(G)$ is non unitary.*

Proof. Let G be a finite abelian group and $\mathcal{G} = \{M_{\xi_l} T_{g_k} \varphi : (g_k, \xi_l) \in G \times \widehat{G}\}$ be a non-Parseval Gabor frame in $L^2(G)$. By Theorem 4.3, $\mathcal{G}$ is the image of a Gabor frame $\mathcal{G}_{\Gamma_G}$ in Γ_G under a unitary transformation $U : L^2(\Gamma_G) \longrightarrow L^2(G)$. The frame $\mathcal{G}_{\Gamma_G}$ is also non-Parseval with frame operator $S \neq I$, for otherwise $\mathcal{G}$ also will be a Parseval frame since U is unitary.

Now, the frame operator of $\mathcal{G}$ is $USU^* = USU^{-1}$ and hence the canonical dual frame of $\mathcal{G}$ is $(USU^{-1})^{-1}(\mathcal{G}) = US^{-1}U^{-1}(\mathcal{G}) = U(S^{-1}(\mathcal{G}_{\Gamma_G}))$ since $U^{-1}(\mathcal{G}) = \mathcal{G}_{\Gamma_G}$. But $U(S^{-1}(\mathcal{G}_{\Gamma_G})) = C(\mathcal{G}_{\Gamma_G})$ where $C = US^{-1}$ and is nonunitary, as desired. □

Remark 4.7. By applying a multiplication function as in the proof of Theorem 4.5 to the Gabor frame $\mathcal{G}_{\Gamma_G}$ in above proof it is clear that the statement of theorem 4.6 is also valid if we replace non-Parseval Gabor frame as Gabor frame.

Following theorem is a specific situation, which enables us to furnish the fact that the operator B in Theorem 4.3 need not be unitary rather an invertible operator is sufficient. For a finite abelian group G, let Γ_G be as in first paragraph of this section and B_f as in Lemma 4.1

Theorem 4.8. *Let G be a finite abelian group. Then corresponding to any non-Parseval Gabor frame $\{M_{\xi_l} T_{g_k} \varphi : (g_k, \xi_l) \in G \times \widehat{G}\}$ in $L^2(G)$ there is a non unitary invertible map $C : L^2(\Gamma_G) \longrightarrow L^2(G)$ with the following properties.*

(i) *$C = BA$ where A is a non-unitary map on $L^2(\Gamma_G)$ and B is a unitary map from $L^2(\Gamma_G)$ to $L^2(G)$ so that A^*A commutes with the elements of the family $\{M_l T_k : l,\ k \in \Gamma_G\}$.*

(ii) *If G is not isomorphic to $(\mathbb{Z}_2)^p$ for any positive integer p, then A does not commute with the elements of the family $\{M_l T_k : l,\ k \in \Gamma_G\}$.*

(iii) *C^*C commutes with the elements of the family $\{M_l T_k : l,\ k \in \Gamma_G\}$.*

(iv) *$\{M_{\xi_l} T_{g_k} \varphi : (g_k, \xi_l) \in G \times \widehat{G}\}$ is a generalized pseudo C-Gabor frame in $L^2(G)$*

Proof. Let G be a finite abelian group and $\{M_{\xi_l} T_{g_k} \varphi : (g_k, \xi_l) \in G \times \widehat{G}\}$ be a Gabor frame in $L^2(G)$. Let us consider the map B as in Proposition 4.4. Then B is a unitary map and $\{M_l T_k B^{-1} \varphi : k, l \in \Gamma_G\}$ is a Gabor frame in $L^2(\Gamma_G)$. Let S be the frame operator of $\{M_l T_k B^{-1} \varphi : k, l \in \Gamma_G\}$, then S commutes with the modulations and translations involved in this frame. Choose $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p) \in \Gamma$ with $gcd(\alpha_j, m_j) = 1$ and define the operators D_α as in discussion preceding to this theorem. If possible take α so that $\alpha_j \neq 1$ for some $j \in \{1,\ 2, \ldots, p\}$.

Define an operator $A : L^2(\Gamma_G) \to L^2(\Gamma_G)$ by $A = SD_\alpha$. Then A is an invertible linear operator with $A^{-1} = D_{\frac{1}{\alpha}} S^{-}1$. It is also trivial from the property of D_α that if G is not isomorphic to $(\mathbb{Z}_2)^p$ for any positive integer p, then A does not commutes with the elements of the family $\{M_l T_k : l,\ k \in \Gamma_G\}$.

The adjoint of A is given by $A^* = D_{\frac{1}{\alpha}} S$ and hence $A^*A = D_{\frac{1}{\alpha}} S^2 D_\alpha$. Thus for all $k, l \in \Gamma_G$, it is easy to see that, A^*A commutes with the elements of the family $\{M_l T_k : l,\ k \in \Gamma_G\}$.

Now define $C : L^2(\Gamma_G) \to L^2(G)$ by $C = BA$. Then the C is invertible and $C^* = (BA)^* = A^*B^*$ so that $C^*C = A^*B^*BA = A^*A$, because B is unitary as we

have seen in Lemma 4.1. Hence the commutativity property of C^*C with modulation M_l and translation T_k on $L^2(\Gamma_G)$ follows from that of A^*A.
Also for each $l, k \in \Gamma_G$

$$\begin{aligned} C^{-1}T_{g_k} &= (BA)^{-1}T_{g_k} = A^{-1}B^{-1}T_{g_k} \\ &= D_{\frac{1}{\alpha}}S^{-1}1T_kB^{-1} = D_{\frac{1}{\alpha}}T_kS^{-1}1B^{-1} \\ &= T_{k'}D_{\frac{1}{\alpha}}S^{-1}1B^{-1} \quad \text{where} \quad \alpha_j k'_j \equiv k_j (mod\ m_j) \\ &= T_{k'}(BA)^{-1} \\ &= T_{k'}C^{-1} \end{aligned}$$

and

$$\begin{aligned} C^{-1}M_{\xi_l} &= (BA)^{-1}M_{\xi_l} = A^{-1}B^{-1}M_{\xi_l} \\ &= D_{\frac{1}{\alpha}}S^{-1}1M_lB^{-1} = D_{\frac{1}{\alpha}}M_lS^{-1}1B^{-1} \\ &= M_{l'}D_{\frac{1}{\alpha}}S^{-1}1B^{-1} \quad \text{where} \quad \alpha_j l'_j \equiv l_j (mod\ m_j) \\ &= M_{l'}(BA)^{-1} \\ &= M_{l'}C^{-1} \end{aligned}$$

Since $\{M_{\xi_l}T_{g_k}\varphi : (g_k, \xi_l) \in G \times \widehat{G}\}$ is a Gabor frame in $L^2(G)$ and

$$\begin{aligned} &\{C^{-1}(M_{\xi_l}T_{g_k}\varphi)) : (g_k, \xi_l) \in G \times \widehat{G}\} \\ &\quad = \{M_{l'}T_{k'}C^{-1}\varphi : k, l \in \Gamma_G, \ \text{where} \ \ \alpha_j l'_j \equiv l_j (mod\ m_j) \ and \ \alpha_j k'_j \equiv k_j (mod\ m_j)\} \\ &\quad = \{M_lT_kC^{-1}\varphi : k, l \in \Gamma_G\}. \end{aligned}$$

It is clear that $\{M_lT_kC^{-1}\varphi : k, l \in \Gamma_G\}$ is a frame in $L^2(\Gamma_G)$ and hence a Gabor frame in $L^2(\Gamma_G)$.
Also

$$\begin{aligned} \{M_{\xi_l}T_{g_k}\varphi : (g_k, \xi_l) \in G \times \widehat{G}\} &= \{C(M_lT_k(C^{-1}\varphi)) : k, l \in \Gamma_G\} \\ &= \{(C(M_lC^{-1})(CT_kC^{-1})\varphi : k, l \in \Gamma_G\} \\ &= \{M_l^C T_k^C \varphi : k, l \in \Gamma_G\} \end{aligned}$$

That is, $\{M_{\xi_l}T_{g_k}\varphi : (g_k, \xi_l) \in G \times \widehat{G}\}$ is a generalized pseudo C-Gabor frame in $L^2(G)$. □

For a given finite abelian group G, not necessarily cyclic, Gabor frames in $L^2(G)$ are generated by the actions of modulation and translation operators on an appropriate element in $L^2(G)$. In this article we recognized such natural Gabor frames as generalized pseudo B- Gabor frames which are generated in $L^2(G)$ by mapping Gabor frames from $L^2(\Gamma)$ using invertible linear transformations $B : L^2(\Gamma) \to L^2(G)$, where G is isomorphic to a direct product Γ of finite cyclic groups. While mapping a Gabor frame from $L^2(\Gamma)$ to $L^2(G)$ through a linear transformation $B : L^2(\Gamma) \to L^2(G)$, the image frame also will exhibit a structure similar to that of the original frame whenever B is invertible. The new class of generalized pseudo B-Gabor frames seems to be larger than the class of Gabor frames in $L^2(G)$.

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