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RESEARCH ARTICLE

Investigating the substance and acceptability of empirical arguments: The case of maximum-minimum theorem and intermediate value theorem in Korean textbooks

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Abstract

Mathematical argument has been given much attention in the research literature as a mediating construct between reasoning and proof. However, there have been relatively less efforts made in the research that examined the nature of empirical arguments represented in textbooks and how students perceive them as proofs. Cases of point include Intermediate Value Theorem [IVT] and Maximum-Minimum theorem [MMT] in grade 11 in Korea. In this study, using Toulmin's framework (1958), the author analyzed the substance of the empirical arguments provided for both MMT and IVT to draw comparisons between the nature of datum, claims, and warrants among empirical arguments offered in textbooks. Also, an online survey was administered to learn about how students view as proofs the empirical arguments provided for MMT and IVT. Results indicate that nearly half of students tended to accept the empirical arguments as proofs. Implications are discussed to suggest alternative approaches for teaching MMT and IVT.

Keywords: argument, high school, intermediate value theorem, maximum-minimum theorem, proof, textbook analysis

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I. INTRODUCTION

Mathematical argument has been given much attention in the research literature as a mediating construct between reasoning and proof. Though the conceptualizations of and distinction between proof, reasoning, and argument are often made in various ways (e.g., Bieda, Conner, Kosko, & Staples, 2022; Knuth, Zaslavsky, & Kim, 2022), for this study, reasoning is defined as one's cognitive process of making a claim based on data (e.g., a preceding assumption, a set of examples) or through inferences. A sequence of reasoning leads to the development of a mathematical argument which "intends to show or explain a mathematical result is true" (Sriraman & Umland, 2014, p. 46). Despite the fact that Weber (2014) defined proof as a *cluster*, in this study, proof is conceptualized as "a *deductive* argument that *does not admit possible rebuttals*" (p. 357, italics added). In sum, reasoning is part of a mathematical argument and a mathematical argument may be conferred on the status of a proof provided that the mathematical argument is with no potential to admit any rebuttals. As well documented in Lakatos (1976), the essence of doing mathematics as a producer of mathematical knowledge is making efforts to derive a tentatively-true claim to one's eye and he or she defends the mathematical claim through seeking warrants, backings, or the use of a qualifier (Toulmin, 1958). In doing mathematics and understanding acceptable forms or representations of mathematical arguments, mathematical arguments represented in textbooks are of crucial importance in offering opportunities for students to understand what a mathematical proof is like (Cai, Ni, & Lester, 2011) and textbooks may be the venue through which students are first introduced to formal ideas of proof (Fan, 2013; Grouws, Smith, & Sztajn, 2004). It is important to learn about what types of proof students are introduced to and what the nature of substance of a proof is (Cai & Cirillo, 2014; Harel & Sowder, 1998; Miyakawa, 2017; Stein, Remillard, & Smith, 2007; Stylianides, 2014; Valverde et al., 2002). This study is another research effort in this area of research that examines mathematical arguments in Korean Mathematics II textbooks (Ministry of Education [MoE], 2015) which have been given little attention in the research.

In the research literature on proof and mathematical argument, there have been research efforts which investigated the nature of mathematical arguments presented in textbooks in relation to proof (Bergwall & Hemmi, 2017; Bieda, Ji, Drwencke, & Picard, 2014; Davis, Smith, Roy, & Bilgic, 2014; Fujita & Jones, 2014; McCory & Stylianides, 2014; Miyakawa, 2017; Otten, Males, & Gilbertson, 2014; Stylianides, 2009; Thompson, Senk, & Johnson, 2012). However, there have been relatively less efforts made in the research that examined the nature of empirical arguments presented in Korean textbooks and how likely students perceive them as proofs. In Korea, there are empirical arguments presented in the secondary textbooks: cases of point are Intermediate Value Theorem [IVT] and Maximum-Minimum theorem [MMT] in grade 11 (MoE, 2015). The corresponding proofs for the aforementioned theorems are beyond student's cognitive reach in compliance with the national curriculum (MoE, 2015). Though grade 11 is the time after students learn formal ideas of mathematical proof per the national curriculum in Korea (MoE, 2015), the content prescribed at the grade does not allow for proofs of both MMT and IVT, leaving both of the theorems not proved formally but empirically justified. This caught the author's attention to this matter and made him question how likely students were to be able to discern the limitations of the empirical arguments as proofs when presented with the arguments. As the substance (including datum, claims, warrants, and backings) of the empirical arguments for MMT and IVT may vary from one to another, this study was intended first to draw comparisons of the nature of the substance between Korean grade 11 textbooks and, then, to discuss implications suggesting alternative approaches for the teaching of these theorems. This study was particularly designed to address the following research questions:

a) What is the nature of datum, claims, warrants and backings for MMT and IVT presented in the Korean textbooks?

b) How likely are students to find empirical arguments provided for MMT and IVT acceptable as proofs?

To address the first research question, the author analyzed the empirical arguments provided for MMT and IVT using Toulmin's framework (1958) and the substance of the empirical arguments provided for MMT and IVT to draw comparisons between the nature of datum, claims, warrants and backing among the empirical arguments offered in the Korean textbooks. In addition to that, for the second research question, an online survey was administered to learn about how likely students were to be able to discern the limitations of the empirical arguments as proofs when presented with the arguments. Implications for teaching these theorems along with the empirical arguments will also be discussed.

II. LITERATURE REVIEW

Importance of Proof and Reasoning in School Mathematics

Recently, researchers and recommendations call for proof and reasoning to play a central role in students' learning of mathematics and in school mathematics. NCTM (2000) maintains that proof and reasoning should be part of everyday instruction of mathematics at all grade levels without reserving it for special occasions or specific grades (Knuth, 2002). In a similar vein, Schoenfeld (1994) argued that proof and reasoning is a fundamental aspect of student's learning mathematics and he further stated that proof is "the soul" of mathematics (Schoenfeld, 2009). For clarity in the subsequent discussions, here and thereafter, I refer to *an empirical argument* as a mathematical argument of which data is a set of examples that are not comprehensive of the domain of the argument and of which truth is solely based on the result of testing the claim of the argument against a few chosen examples.

Researchers documented that proof and reasoning is challenging for both teachers and students despite the benefits of engaging in proof and reasoning. Fawcett (1938) suggested that geometric proofs serve to develop students' understandings about what it means to be a proof in mathematics rather than about geometric knowledge. Mathematicians read others' proofs first to understand and critique proofs and, then, to gain insight into other proofs (Epstein & Levy, 1995). However, students do not see the premium and values that mathematicians place on proof (Alcock & Inglis, 2008; Chazan, 1993) and teachers tend to consider proof as a mere subject of study at a certain time for a select few (Basturk, 2010; Kim, 2022a; Knuth, 2002). Another challenge in the learning of mathematics is concerned with difficulties in recognizing the limitations of empirical arguments as proofs (Coe & Ruthven, 1994; Harel & Sowder, 1998; Kim, 2022a; Knuth, Choppin, & Bieda, 2009). A similar tendency seems to persist among secondary students (Knuth et al., 2009). Difficulties of this nature seem to originate from the smudged picture of proof, evidence, and derivation (Aberdein, 2019) and the lack of connection between proof knowledge and proof construction (Schoenfeld, 1988). Also, the choice of form which one decides for an argument to take on hinges upon one's prior experiences with forms of proof (Harel & Sowder, 1998; Kim, 2022a; Knuth, 2002; Selden & Selden, 1995) and the consistency and coherence (or the lack thereof) of sociomathematical norms concerned with proof across grade levels (Jones, 2010; Knuth, 2002; Yackel & Cobb, 1996). In this regard, there is a need to engage students in proving-related opportunities across various topics of study and to appreciate proofs of diverse forms. Some researchers argue that mathematical argument is of the potential to engage students in proving-related opportunities as a routine part of everyday instruction (Kim, 2021, 2022b; Stylianou & Blanton, 2011).

Mathematical Argument Leading to Proof

The research of mathematical argumentation has been given attention in the research of mathematics education. In *Proofs and Refutations*, Lakatos (1976) demonstrated well how a mathematical discourse takes place in a hypothetical classroom and how a mathematical argument may be developed through different approaches such as qualifying definitions and claims and revising local arguments of a global argument. The volume shows the dynamic nature of doing mathematics and developing mathematical knowledge through mathematical argumentation (i.e. developing mathematical arguments), leading to a proof for a mathematical proposition. Mathematical argumentation seems to possess benefits such as enhancing one's conceptual understanding (Krummheuer, 1995) and one's taking an active role in producing mathematical knowledge (Koestler, Felton, Bieda, & Otten, 2013). Recognizing the importance of argumentation for student's learning of mathematics, researchers continue to make efforts to modify the Toulmin's model of argumentation (e.g., Conner, 2008; Krummheuer, 1995; Knipping & Reid, 2019). Despite the differences in the models they developed, what their models bear in common is that mathematical argumentation is a pathway to proof and, more importantly, that they adopt the constructs of Toulmin's (1958) model: *Data*, *claim*, *warrant*, and *backing*.

The Skeleton of Mathematical Arguments

Toulmin (1958) introduced, in *Uses of Arguments*, the skeleton of arguments and it has been translated into the field of mathematics education (e.g., Bieda et al., 2022; Conner et al., 2014; Knipping & Reid, 2019; Krummheuer, 1995). Toulmin (1958) analyzed arguments in various fields, he abstracted components with consideration of what role each component serves in an argument. Toulmin's (1958) skeleton of arguments consists of components: *Datum* [D], *warrants* [W], *backings* [B], *claims* [C], *rebuttals* [R], and *qualifiers* [Q]. Knipping & Reid (2019) pointed out that, in secondary textbooks, students are rarely encountered with R and Q. Accordingly, Knipping & Reid (2019) provided a diagram (see Figure 1 below) with D, C, W, and B. Figure 1 succinctly provides an overview of the structure of an argument.

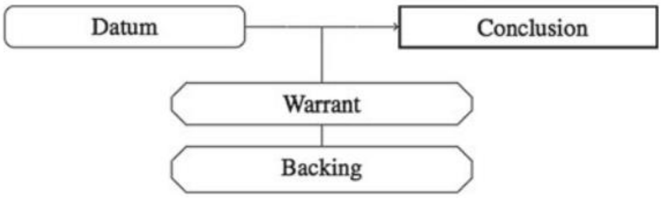


Figure 1. Toulmin's structure of argument (Knipping & Reid, 2019, p. 4)

III. METHODOLOGY

Data Collection

The corpus of data consists of excerpts taken from nine textbooks available in Korea. In Korea, the government has delegated its authority to a government-funded agency in approving curriculum materials for use in Korean schools and ensured the compliance of such curriculum materials with the national curricula. With the list of the approved titles for each subject, school officials and teachers make decisions about what a textbook for each subject is chosen for use once a new list of titles is released. The list of approved textbooks for Mathematics II in grade 11 is provided in Table 1.

Table 1. The list of approved mathematics II textbooks (Korea Textbook Research Foundation, n.d.)

Publisher	***Year of Publication***	***Authors***
Kyohak	2018	Kwon et al.
Keumsung	2018	Bae et al.
Donga	2018	Park et al.
MiraeN	2018	Hwang et al.
Visang	2018	Kim et al.
Sinsago	2018	Koh et al.
Jihaksa	2018	Hong et al.
Chunjae	2018	Lew et al.
Chunjae	2018	Lee et al.

According to the national mathematics curriculum (MoE, 2015, 2022), there are several theorems provided with no proofs: Cases of point include MMT and IVT. According to the national curriculum (MoE, 2015), it is generally stated that: students are able to "understand properties for continuous functions and apply them" (p. 74). It is also noted with caution that: "Definitions and properties of limit of function should be understood *intuitively* and technology may be used" (p. 75, italics added). With the foregoing statements concerned with the teaching of MMT and IVT, MoE places an emphasis on fostering the intuitive understanding of MMT and IVT in ways such as using graphs which are *generic examples* or *näive empiricism* (Balacheff, 1988).

An online survey was administered to learn about students' views about the need of proof in mathematics and about the empirical arguments provided for both MMT and IVT. The survey was part of another study (Kim, 2022a) that was designed to learn about students' views about example use in relation to mathematical proof (see Table 2 for details). The survey questions used for this study were developed to address the second research question. Though sixty-two students initially opened the online survey, there were fifty participants who answered the survey in full (i.e. completed the survey until the very end). The participant students were recruited as a convenient sample from the high school where the author worked at the time when the study was underway and the students were not compensated for their participation.

Table 2. The online-administered survey questions

1. (After explaining the maximum-minimum theorem and the intermediate theorem with the examples and the narrative provided in the text) Do you think a proof for the maximum-minimum theorem may be replaced with the examples the teacher provided?
2*. Please explain why you think so.
3*. Generally speaking, do you think a few examples can replace proofs in mathematics?
4. Do you think a proof for the intermediate value theorem may be replaced with the examples the teacher provided?
5*. Please explain why you think so.
6*. Do you think a proof for the intermediate value theorem may be replaced with the examples the teacher provided?
(Asterisks denote that the responses to the questions were recorded in writing. The responses for the rest of the question numbers without asterisk were recorded on a four- or five-point Likert scale.)

Data Analysis

The datum, claims, warrants, and backings of the empirical arguments provided for MMT and IVT were analyzed. With the view (Knipping & Reid, 2019) that R and Q are rarely presented at secondary level, the author decided not to include these components for analysis from the initial round of coding since there was no instance of R and Q across all the textbooks.

The analysis took place in sequence. In the first path of the analysis, the author located the relevant pages where MMT and IVT are covered in the narrative of each textbook using the teacher guide and the table of contents. After locating the relevant pages, the pages were examined to confirm whether they indeed covered MMT and IVT. Then, the claims of MMT and IVT were identified and it was learned that all the claims made in all the textbooks were the same. Following this stage, the author analyzed the narrative appearing in advance of or following the claims to identify the datum, warrants, and backings for the claims.

The survey responses were recorded in writing, on a four- or five-point Likert scale. The responses recorded on a four- or five-point Likert scale were analyzed and reported statistically. Tendencies observed from the statistical results were triangulated with the responses recorded in writing. Some representative written responses were identified and reported to learn about students' views about how likely they are to accept the given empirical arguments of both MMT and IVT as proofs.

IV. RESULTS

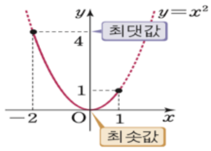
Examples as Datum Leading to Claims

All the claims were validated with a few chosen examples that confirm the claims for both MMT and IVT. It is hardly unexpected to see examples used to validate the claims for MMT and IVT given the compliance of textbooks with the national curriculum which recommends teachers "not to prove the theorems but to intuitively justify them" (MoE, 2022, p. 75). This choice of empirical arguments was not an option for textbook writers either. Instances of these empirical arguments are given below (see Figure 2).

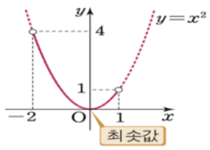
An example datum for MMT (Lee et al., 2018, p. 37)

주어진 구간에서 연속함수의 최댓값과 최솟값이 존재하는지 알아보자.

함수 $f(x)=x^2$은 모든 실수에서 연속이다.
이때 닫힌구간 $[-2, 1]$에서 함수 $f(x)$는
최댓값 $f(-2)=4$,
최솟값 $f(0)=0$
을 갖는다.



한편, 열린구간 $(-2, 1)$에서 함수 $f(x)$는
최솟값 $f(0)=0$
을 갖지만 최댓값은 갖지 않는다.



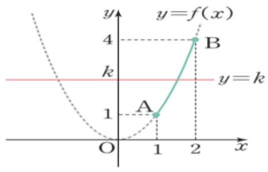
Translation: Let's figure out whether a continuous function has the maximum and minimum in a given interval. $f(x) = x^2$ is continuous at all real numbers. On the closed interval $[-2, 1]$, $f(x)$ has the maximum 4 at $x = -2$ and $f(x)$ has the minimum 0 at $x = 0$. In contrast, $f(x)$ has the minimum 0 at $x = 0$ but the maximum on the open interval $(-2, 1)$.

An example datum for IVT (Hong et al., 2018, p. 39)

함수 $f(x)=x^2$은 닫힌구간 $[1, 2]$에서 연속이므로 이 함수의 그래프는 두 점 A(1, 1), B(2, 4) 사이에서 연결되어 있다.

따라서 $1<k<4$인 임의의 k에 대하여 x축에 평행한 직선 $y=k$와 함수 $f(x)=x^2$의 그래프는 반드시 만난다.

즉, $f(1)$과 $f(2)$사이에 있는 임의의 값 k에 대하여 $f(c)=k$인 c가 열린구간 $(1, 2)$에 존재한다.



Translation: Since $f(x) = x^2$ is continuous on $[1, 2]$, the graph connects two points, $A(1, 1)$ and $B(2, 4)$. Thus, for any k $(1 < k < 4)$, $y = k$ and $f(x) = x^2$ should intersect each other. That is, for any k between $f(1)$ and $f(2)$, there exists c such that $f(c) = k$.

Figure 2. Examples of datum used for MMT and IVT

The data used to make the claims for MMT and IVT were not the same for all textbooks. All the textbooks commonly used continuous functions within bounded and closed intervals to justify the claims for MMT and IVT. Examples used as data for the claims by each textbook are summarized below in Table 3.

Table 3. Examples used as datum for MMT and IVT

Textbook	***Examples for MMT***	***Examples for IVT***
Kwon et al. (2018)	The function $y = x^2 - 2x$ is considered on $[-1, 2]$ and $(-1, 2)$, respectively (p. 39).	The function $y = x^2$ is considered on $[0, 2]$ (p. 40).
Bae et al. (2018)	The function $y = x^2 + 1$ is considered on $[-1,2]$, $(-1, 2)$, and $(0, 2)$, respectively (p. 42).	The function $y = x^2 - 2x - 2$ is considered on $[1, 2]$ (p. 43). A continuous function $f(x)$ as *a generic example* is considered on $[a, b]$ (p. 44).
Park et al. (2018)	The function $y = x^2 + 1$ is considered on $[-1,2]$ and $(-1, 2)$, respectively (p. 38).	A continuous function $f(x)$ as *a generic example* is considered on $[a, b]$ (p. 40).
Hwang et al. (2018)	The function $y = -x^2 + 4$ is considered on $[-1, 2]$, $[-1, 2)$, and $(0, 2)$, respectively (p. 37).	The function $y = x^2$ is considered on $[1, 2]$ (p. 38).
Kim et al. (2018)	The function $y = x^2$ is considered on $[-2,1]$ (p. 37).	A continuous function $f(x)$ as *a generic example* is considered on $[a, b]$ (p. 38).
Koh et al. (2018)	The function $y = x^2 - 1$ is considered on $[-1, 2]$ and $(-1, 2)$, respectively (p. 37).	A continuous function $f(x)$ as *a generic example* is considered on $[a, b]$ (p. 38).
Hong et al. (2018)	The function $y = x^2$ is considered in $[1,2]$. Another continuous function $f(x)$ as *a generic example*[1] is considered on $[a, b]$ (p. 38).	The function $y = x^2$ is considered in $[1, 2]$. Another continuous function $f(x)$ as *a generic example* is considered on $[a, b]$ (p. 38).
Lew et al. (2018)	The function $y = x^2$ is considered on $[-1, 2]$, $(-1, 2)$, and $(0, 2]$, respectively (p. 37).	The function $y = (x - 1)^2$ is considered in $[1, 3]$. Also, another example of a discontinuous function is considered on $[1, 3]$ (p. 38).
Lee et al. (2018)	The function $y = x^2$ is considered on $[-2,1]$, $(-2, 1)$, and $(0, 2)$, respectively (p. 37).	The function $y = x^2$ is considered on $[1, 2]$ (p. 38).

Note: all the instances were adapted from the corresponding textbook appearing in the same row.

[1] Balacheff (1988) drew a distinction between example use with respect to one's recognition of the generality and specificity of examples used to justify a mathematical claim. By generic example, it refers to mathematical examples which are of generality in the sense that the operation or construction applied to them is readily applicable to other mathematical objects of the same class but still not general.

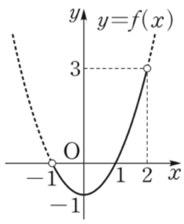
However, there was a difference in the use of examples from one textbook to another. The examples used are *partially* against the premises of the claims and thus may not make the same claims hold true. Cases of this point are provided below (see Figure 3). Given that the premise for MMT and IVT is "a continuous function on a closed and bounded interval $[a, b]$", the intervals such as $(-2, 1)$, $[-1, 2)$ are partially against the premise by considering intervals that are not closed but bounded. Also, other examples used were concerned with the continuity of a function: An example of this nature are *discontinuous* functions on a closed and bounded interval. However, there was no instance of examples which are against two or more conditions of the premise: continuity, boundedness and closedness of an interval.

An example that is partly against the premise of MMT (Koh et al., 2018, p. 37)

한편 함수 $f(x)$는 열린구간 $(-1, 2)$에서

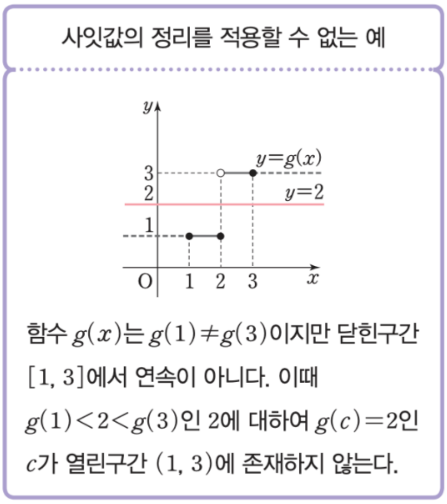
$x=0$일 때 최솟값 $f(0)=-1$

을 갖지만 최댓값은 갖지 않는다.



Translation: By contrast, $f(x)$ does not have the maximum while it has the minimum $f(0) = -1$ at $x = 0$ on $(-1, 2)$.

An example that is partly against the premise of IVT (Lew et al., 2018, p. 38)



Translation: The function $g(x)$ is not continuous on $[1, 3]$ while $g(1) \neq g(3)$. There does not exist c in $(1, 3)$ such that $g(c) = 2$ $(g(1) < 2 < g(3))$.

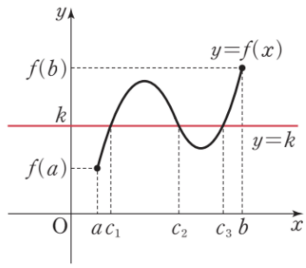
Figure 3. Instances of examples that are partly against the premises of IVT and MMT

The example shown at the top of Figure 3 is an instance which falls short of being the closed interval, indicating that the part of the premise (i.e. a closed interval) of MMT is not met. It is also true for the example shown at the bottom of Figure 3: the example does not satisfy the part of the premise of IVT (i.e. a continuous function on a given closed interval). This use of examples was not common across the textbooks. The vast majority (seven out of nine) presented a number of examples presumably to develop a sophisticated understanding of the premise in that the negation of any of the conditions contained in the premise would not guarantee the truth of the claim.

On the contrary, there was only one instance that is partly against the premise of IVT and is used as the data for the theorem. Lew et al. (2018) was the only aberration that a discontinuous function on a closed and bounded interval was considered for IVT. Except for the aberration, there were only two kinds of examples used as datum for the claim of IVT: one is particular examples and the other is generic examples. By *particular examples*, I refer to examples such as $y = x^2$ on $[1, 2]$ (Lee et al., 2018, p. 38). On the other hand, I refer to as a *generic example* a continuous function $f(x)$ on $[a, b]$ (Bae et al., p. 44) which is of generality in the sense that, on the surface, the example is a general case, however, the graph juxtaposed with the function is a particular case. The generic example of this nature only appeals to the intuition of the reader through the presentation of the graph for the continuous function $f(x)$ on $[a, b]$ (see Figure 4).

함수 $f(x)$가 닫힌구간 $[a,\ b]$에서 연속이면 이 구간에서 함수 $y=f(x)$의 그래프는 끊어지지 않고 연결되어 있다.

따라서 $f(a)\neq f(b)$일 때, 오른쪽 그림과 같이 $f(a)$와 $f(b)$ 사이의 임의의 값 k에 대하여 x축에 평행한 직선 $y=k$와 함수 $y=f(x)$의 그래프는 적어도 한 점에서 반드시 만나는 것을 알 수 있다.



Translation: If $f(x)$ is continuous on the closed interval $[a, b]$, then the graph of $f(x)$ is connected without being interfered. When $f(a) \neq f(b)$, as seen on the figure on the right-hand side, for any k between $f(a)$ and $f(b)$, the graphs of $y = k$ and $y = f(x)$ intersect at least once.

Figure 4. An instance of a generic example used as datum for IVT (Park et al., 2018, p. 40)

Warrant without Backing

Warrants used for the claims of MMT and IVT were the same for all the textbooks. The warrant was an authoritative statement that “generally, the MMT (or IVT) holds true for a continuous function on a closed interval” (Kwon et al., 2018, pp. 39-40). This statement simply put an end to discussions about the potential existence of rebuttals to the empirical arguments for both MMT and IVT in the textbooks, thus leaving redundant questions about the truth of the claims and the acceptability of the empirical arguments surfacing after the statement is read. Furthermore, there were no backings for the warrants for both MMT and IVT.

Empirical Arguments as Proofs for The Maximum-Minimum Theorem

Students were asked to express their agreement with the statement that a number of confirming cases suffice to prove MMT. Half (50%, accumulatively) of the students agreed with the sufficiency of the empirical argument as a proof to varying degrees while fourteen students (28%) were neutral and the other eleven students (22%, accumulatively) disagreed with the validity of the empirical statement as a proof to some degrees. Table 4 provides an overview of the students' responses.

Table 4. Students' views about empirical arguments as proofs for MMT

Degree of Agreement	***Count (percentage)***
Strongly agree	2 (4%)
Somewhat agree	23 (46%)
Neither agree nor disagree	14 (28%)
Somewhat disagree	8 (16%)
Strongly disagree	3 (6%)

Most of the students tended to somewhat agree with that the empirical arguments were sufficient to prove MMT. Among those students, there were students who pointed out the limitations of empirical arguments as proofs. However, there were also those who seemed to think that the empirical arguments might replace a proof without further examination. The following are some of the written responses from those students:

> *"Providing examples confirming the theorem is a way of proving the theorem."*
> *"A proof is something that is used to understand a theorem, so that confirming examples may replace the proof for a theorem."*
> *"Deriving from formulae is not always a way of proving a theorem. As long as it appeals to lay people, a convincing argumentation is a proof."*

These responses are concerned with issues related to perceptions about proof documented in the literature: misunderstanding evidence for proof (e.g., Chazan, 1993; Coe & Ruthven, 1994; Knuth et al., 2020) and failure to recognize the potential existence of counterexamples to the empirical arguments (Fischbein, 1982).

Empirical Arguments as Proofs for The Intermediate Value Theorem

Students were asked to express their agreement with the statement that the empirical arguments for IVT sufficed to prove the theorem. A slight majority (52%, accumulatively) of the students agreed with the statement to varying degrees while eleven students (22%) were neutral and the other thirteen students (26%, accumulatively) disagreed with the statement to certain degrees. In relation to the result in the case of MMT, there was a slight difference of four percent in disagreement and a slight difference of eight percent in strong agreement between MMT and IVT. Table 5 provides the detailed results.

Table 5. Students' views about empirical arguments as a proof for IVT

Degree of Agreement	***Count (percentage)***
Strongly agree	6 (12%)
Somewhat agree	20 (40%)
Neither agree nor disagree	11 (22%)
Somewhat disagree	10 (20%)
Strongly disagree	3 (6%)

Some of the written responses shed some light on the reasoning behind the results. The following excerpts were drawn from those students who showed somewhat or strong agreement with the statement that the empirical arguments provided for IVT were sufficient to prove IVT.

"It [the empirical argument] may be understood geometrically"
"It [the empirical argument] showed that proving it is possible."
"Since the only thing that matters is to be [understood]."

The excerpts of the students' written responses pointed to a persistent issue with students' views about proof: one's acceptance of a mathematical argument as a proof hinges upon the reader's understanding of the argument (Kim, 2022a).

V. DISCUSSIONS

This study examined the nature of empirical arguments for two theorems in the Korean national curriculum. The cases of point are MMT and IVT which are neither formally justified nor expected to be justified through proofs during K-12 education in compliance with the national curriculum (MoE, 2015, 2022). The datum used for the theorems were either specific examples or generic examples and the same warrant was provided: it is generally known that the claim (of MMT or IVT) holds true. Also, there were no backings presented for the warrants to both MMT and IVT. Given the focus on the teaching of MMT and IVT in the national curriculum, teachers are expected to appeal for students' *intuitions* as to the truth of and understanding of the theorems. However, the online-administered survey revealed that a slight majority (50% for the case of MMT and 52% for IVT) of the students showed the empirical arguments given for the theorems sufficed to prove the theorems while 22% and 26% of the students showed disagreements with the empirical arguments provided as proofs for MMT and IVT, respectively. This brings to the fore issues concerned with the *acceptability* of a mathematical argument and care must be given to the teaching of both MMT and IVT with empirical arguments. The former case is related to the role of the readers who act as judges to evaluate the acceptability of a mathematical argument as a proof (Wittgenstein, 1974). The latter necessitates alternative approaches in teaching MMT and IVT with empirical arguments as represented in textbooks. It also suggests that there would be revisions or support for teachers as to enhancing students' understandings about the limitations of empirical arguments as proofs so that students understand MMT and IVT intuitively in grade 11 while acknowledging the limited nature of empirical arguments as proofs. In class, teachers may find helpful the results of Stylianides & Stylianides (2014): *Milestones* for leading whole-class discussions from an empirical argument toward a proof. Although the study was conducted with different tasks, with the results of the study in mind, teachers may have students question their convictions about the validity of the empirical arguments. In turn, students would be able to see the need of deductive proofs for IVT and MMT. In that moment, teachers and students could benefit from using a dynamic geometry environment to examine MMT and IVT. Engaging in a dynamic geometry setting to search for counterexamples to MMT and IVT, students would gradually learn that it is unlikely to find any counterexamples to these theorems. This alternative approach may help teachers plan their classes and, at the same time, it may also enhance students' understandings about the limitations of empirical arguments.

Another issue concerned with the provision of empirical arguments for MMT and IVT with an external authority relates to what a mathematical proof means to students. For MMT and IVT, all the textbooks offered purely empirical arguments that only justified the claims with a few chosen examples confirming the claims. Several textbook authors went further to use generic examples that involve a general case $f(x)$ with the specificity of the graph. This use of a generic example could further be enhanced through the consideration of various shapes of graphs. From viewing this result from Harel & Sowder (1998), for students who encounter proof-like empirical arguments without recognizing the limitations of empirical arguments, empirical arguments justified with an external authority (as is the case for IVT and MMT) may become part of their proof schemes and they may think of empirical arguments as proofs at a later time in their studies of mathematics, arguing that they have seen these arguments before and that they were proofs.

Attention and care must be given to the results of the students' written responses to the empirical arguments for MMT and IVT. Despite the slight majority of the students seemed to recognize the limitations of the empirical arguments as proofs for both MMT and IVT, nearly half of the students tended to accept the empirical arguments as proofs. The written responses of these students without a robust understanding of mathematical proof also confirmed the results of the extant literature: one is concerned with the fact that one's acceptance of a mathematical argument as a proof hinges upon her or his understanding of the argument (Knuth, 2002); another is the tendency that students misunderstand evidence for proof (Chazan, 1993). It is teachers who can address these issues in close contact with individual students. Teachers need support from researchers and textbook writers potentially with the provision of guidance that includes *cautious points on managing student approaches to proof* (Stylianides, 2008). They may place empirical arguments in a precarious position where empirical arguments are likely to be rejected due to the potential existence of counterexamples to them as rebuttals (Toulmin, 1958). With this support, teachers could better develop students' understandings about the limitations of empirical arguments as proofs. This study contributes to the literature by offering insights into students' views about the empirical arguments as proofs that have received little attention and by calling teachers' and teacher educators' attention to the need for alternative approaches in the teaching of IVT and MMT.

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