J. Korean Soc. Math. Ed. Ser. D. (2024) 27(1), 1–24
https://doi.org/10.7468/jksmed.2024.27.1.1
ISSN 1226-6191
Online ISSN 2287-9943

RESEARCH ARTICLE

How do one expert mathematics teacher in China implement deep teaching in problem-solving and problem-posing classroom: A case study

Yanhui Xu[1]

[1]Associated Professor, Department of Mathematics, Wenzhou University

Received: October 5, 2023 / Revised: November 21, 2023 / Accepted: January 30, 2024

Abstract

In this paper, the author analyzed characteristics of deep mathematics learning in problem solving and problem-posing classroom teaching. Based on a simple wrong plane geometry problem, the author describes the classroom experience how one expert Chinese mathematics teacher guides students to modify geometry problems from solution to investigation, and guides the students to learn how to pose mathematics problems in inquiry-based deep learning classroom. This also demonstrates how expert mathematics teacher can effectively guide students to teach deep learning in regular classroom.

Keywords: deep learning, mathematics problem, expert mathematics teacher, problem-solving, problem-posing, inquiry-based

• email: wzxuyanhui@126.com

I. INTRODUCTION

Chinese compulsory education mathematics curriculum standards (Ministry of Education of the People's Republic of China, 2022) and high school mathematics curriculum standards (Ministry of Education of the People's Republic of China, 2022) define mathematics core literacy as the students’ mathematics key abilities and qualities of thinking for adapting to needs of lifelong and social development. That is, the ultimate goal of mathematics education is to enable students to learn to see the real world by mathematicians’ perspective, ponder the real world by using the way of mathematicians’ thinking, and express the real world by mathematicians’ language. The proposal of mathematics core literacy has clarified the direction for mathematics education in China. But how to cultivate students' core mathematics literacy in classroom teaching has become an urgent problem that needs to be solved. The cultivation of mathematics core literacy requires students to immerse themselves in the process of experiencing and discovering mathematics knowledge, then the theory of deep learning in mathematics has emerged. Deep learning has become the focus of researchers' research with the proposal of core literacy, and deep learning is an important way to achieve core literacy (Li, & Wen, 2023).

II. THEORETICAL CONSIDERATIONS

In fact, as early as 1956, Bloom's classification of cognitive dimensions in his "Classification of Educational Objectives" already contained the view that "learning has deep and surface levels". Surface learning corresponds to the cognitive level of knowing and comprehending, and belongs to low-level thinking activities. It emphasizes externally driven learning and repetitive memory, simple description, and reinforcement training of knowledge. Deep learning corresponds to the cognitive level of application, analysis, synthesis, and evaluation, and belongs to higher-order thinking activities. In the mid-1950s, Ference Marton and Roger Saljo from University of Gothenburg in Sweden began conducting experimental research on students' learning processes. In 1976, the concept of deep learning was proposed in the hierarchical theory based on the essence of learning. Marton and Saljo (1976) believes that deep learning is not only a cognitive process of individual perception, memory, thinking, but also a social construction process rooted in social culture, historical background, and real life.

In the handbook, “How People Learn” (Bransford, Brown, & Cocking, 2000/2004, p. 8), the authors state that one of the hallmarks of what they call “the new science of learning” is the emphasis on learning with understanding used parallel to deep learning. New knowledge must be constructed from pre-existing knowledge and learners must be encouraged to be active and take control of their own learning (Bransford et al., 2000/2004). From research on learning sciences, Sawyer (2014, p. 4) emphasizes that one of the central underlying themes is that “students learn deeper knowledge when they engage in activities that are similar to the everyday activities of professionals who work in a discipline.” When studying how “new pedagogies find deep learning”, Fullan and Langworthy (2014) highlight that forces converging to produce deep learning outcomes are as an example (new) learning based in the real world of action and problem solving. They have from their research found that in the best teaching examples, teachers and students are working together in order to engage students in the learning by relating the learning to real-life problem solving. When engaging in deep learning, teachers and students partner with each other in learning processes where "high expectations are mutually negotiated and achieved through challenging deep learning tasks" (Fullan & Langworthy, 2014, p. ii).

Hattie and Donoghue (2016, p. 3) add that deep learning refers to "seeking meaning, relating and extending ideas, looking for patterns and underlying principles". Deep learning also involves coming to understand and utilize the relationship between concepts and procedural knowledge by being able to apply conceptual knowledge in new contexts (Hattie & Donoghue, 2016; Parker, Mosborg, Bransford, Vye, Wilkerson, & Abbott, 2011; Winch, 2017). Winch (2017) also argues that deep learning (mastery) is about understanding and utilizing the relationships between the propositions and concepts of an area of knowledge rather than accumulating facts. Epistemic knowledge with its embedded concepts, is pivotal because concepts are the key to learning to think abstractly through the process of objectification and generalizability; the mechanisms for acquiring deep understanding (Bruner, 1977). The crucial point is that one can generalize from concepts, but it is not possible to generalize from specific content or activities. Generalizing enables the student to move from the known to the unknown, from the familiar to the unfamiliar. Put very simply, without concepts students can't learn to think abstractly and generalize (McPhail, 2021). Fullan et al. (2018) suggest deep learning is exemplified by 'a strong sense of *identity* around a purpose or passion, *creativity and mastery* in relation to a valued pursuit, and *connectedness* with the world and others' (p. 5, italics in original).

Mathematics has both surface structure (surface meaning) and deep structure (deep meaning). Surface meaning refers to the content (concepts, propositions, theories) of the subject directly expressed by mathematics language and written symbols, while deep meaning refers to the spirit, value, and methodology contained in or behind mathematics knowledge content. The deep meaning is often invisible, penetrating, dispersed, and hidden. But it is the fundamental and decisive factor of forming students' mathematics core literacy. It is also what we strive for. Deep learning in mathematics is a cognitive learning approach that involves deep thinking and systematic integration of practical problems, which relies on the core content of mathematics, mathematical discipline ideas and methods, and existing cognitive foundations. Conceptual understanding, flexible thinking and an exploratory approach are all indicators of deep learning. Students who adopt a deep approach want to make sense of what they are doing and to build their own personalized knowledge structures. They tend to follow the general pattern of: endeavoring to understand material for themselves; interacting critically with content; relating ideas to previous knowledge and experience; examining the logic of arguments and relating evidence to conclusions (Mackie, 2002).

Deep learning in mathematics requires students to actively explore mathematics knowledge, think about mathematics problems deeply, creatively solve problems, actively communicate with peers, summarize and reflect on the acquired mathematics knowledge, and understand the essence of mathematics problems. When engaging in mathematics deep learning, it is beneficial for students to organize fragmented mathematical knowledge into a systematic and complete mathematical knowledge system, which can enhance their mathematical thought and scientific spirit. In deep mathematics learning, students often concentrate on their own thinking process and thinking about the essence of mathematics. Teachers should pay attention to giving students ample time to think, encouraging students to question and think from multiple perspectives. Deep learning in mathematics emphasizes the critical understanding and organic integration of knowledge, focuses on the construction and reflection of the learning process, and emphasizes the transfer and application of learning and problem-solving. Teachers should focus on designing problems that can reflect mathematicians' thinking style and drive classroom teaching by using the problems, and simultaneously inducing students to gradually think deeply about and comprehensively solve problems. Ultimately, students can form structured mathematical knowledge system and personalized knowledge network, and understand the essence of mathematics.

Problem is the heart of mathematics. The development of mathematics knowledge is largely driven by the problems. It is very important for the students to learn to solve problem and pose new problems when studying mathematics. Problem solving is often seen as the mean for reaching the goal deep learning (cf. the "challenging deep learning tasks" put forward by Fullan and Langworthy (2014, p. ii). Posing problems is more important than solving problems in the field of mathematics. Many mathematicians and mathematics educators believe that "posing mathematics problems" is an important mathematical activity, and students should get their own experience of posing mathematics problems as early as possible. As Kilpatrick (1987) pointed out, it is very important for every student to gain the experience of creating one's own mathematics problems, which should become an part of their mathematics education. Students should be given opportunities to formulate problems from given situations and create new problems by modifying the conditions of a given problem (NCTM, 1991, p. 95). It is critical for students with the ability of correctly posing mathematics problems to fully understand a concept.

Problem posing refers to both the generation of new problems and the re-formulation of given problem. One kind of problem posing, usually referred to as problem formulation or re-formulation, occurs within the process of problem solving. When solving a nontrivial problem, a solver engages in this form of problem posing by recreating a given problem in some ways to make it more accessible for solution. Problem formulation represents a kind of problem posing process, because the solver transforms a given statement of a problem into a new version that becomes the focus of solving (Silver, 1994). For relatively simple problems, problem formulation may occur primarily in the early stages of problem solving, but in extended mathematical investigations, "problem formulation and problem solution go hand in hand, each eliciting the other as the investigation progresses" (Davis, 1985, p. 23). Problem posing can also occur after having solved a particular problem, when one might examine the conditions of the problem to generate alternative related problems. This kind of problem posing is associated with the "looking back" phase of problem solving discussed by Polya (1957). Brown and Walter

(1983) have written extensively about a version of this type of problem posing, in which problem conditions and constraints are examined and freely changed through a process they refer to as "What-if?" and "What-if-not?". Kontorovicha, Koichua, Leikin, and Berman (2012) proposed eight kinds of problem-posing heuristics strategies, that is, (1) symmetry, (2) constraint manipulation, (3) numerical variation, (4) what-if-not, (5) goal manipulation, (6) targeting a particular solution, (7) generalization, (8) chaining.

When teachers pose a new interesting problem in classroom, it often gives students a strong attraction, and stimulate students' desire of learning mathematics, and inspire students' motivation of exploring and solving problems. By posing new mathematics problems, students can experience the process of creating mathematics, not just absorb mathematics knowledge taught by teachers. In this way, students can experience an important and exciting process of thinking about mathematics. Experience with mathematical problem posing can promote students' engagement in authentic mathematical activity; allow them to encounter many problems, methods, and solutions rather than only one of each; and promote students' creativity-a disposition to look for new problems, alternative methods, and novel solutions (Silver & Cai, 2005). Therefore, it is very important for the teachers to strengthen the students' consciousness of mathematics problems and pay attention to inspiring students to pose new problems.

When problem posing has been systematically incorporated into students' mathematics instruction, even something as simple as having students generate story problems, the reported results have generally been quite positive, including a positive effect on students' problem-solving achievement and their attitudes toward mathematics (see Hashimoto, 1987; Healy, 1993; Silverman, Wingrad & Strohauer, 1992; Whitin, 2000). Engaging in problem posing has the potential to change the learner's perspective regarding the essence of mathematics and what might be considered as "doing mathematics" (Lavy & Shriki, 2009; Shriki, 2006; Shriki & Lavy, 2004). In addition, such engagement develops a comprehensive view of mathematical phenomena and their generalizations (Lavy & Shriki, 2010). Involving students in problem posing activities and providing them with opportunities to pose their own problems, which in turn promotes more diverse and flexible thinking, improves their problem-solving skills, extends their perception of mathematics, and enriches and strengthens their knowledge of basic concepts (Brown & Walter, 1993; English, 1996, 2003). It is therefore important that teachers should integrate such activities in their deep mathematics teaching lessons.

However, it is often a great challenge for many teachers to pose a good mathematics problem. Teachers expressed a need for skills to enact tasks in class and specific ideas for handling posed problems from children. Without this knowhow, high-level tasks would not function in the way intended (Stein, Smith, Henningsen & Silver, 2009). There is a difference in the availability of instructional materials in "regular" versus investigation classes. In a "regular" geometry classroom, where proving is the main mathematical activity (e.g., Hanna & De Villiers, 2012), teachers choose proof problems from textbooks and other instructional materials. Inquiry-based learning, however, requires devolving investigation problems to the classroom (e.g., Da Ponte, 2007; Yerushalmy et al., 1990), yet often teachers cannot find investigation problems in regular instructional materials. Teachers cannot rely on textbooks for good investigation problems. In general, the majority of textbooks contain almost no investigation problems. One of the ways for designing investigation problems for geometry classes is transforming proof problems from regular textbooks into inquiry problems (Leikin & Grossman, 2013). Teachers must have the capacity of posing their own problems in classroom teaching. Based on fully studying mathematics textbooks, teachers should start with some basic problems, teach students to learn to pose some new related mathematics problems step by step. So, the students can experience the process of generating and creating mathematics problems, and can develop their own curiosity and desire of exploring mathematics.

The study addresses the following research questions: How can mathematics teachers implement deep teaching in problem-solving and problem-posing classroom?

What conditions do mathematics teachers need when implementing deep teaching in problem-solving and problem-posing classroom? This paper will illustrate how one expert Chinese mathematics teacher start from a basic problem to guide students to pose problems from solution to investigation in deep mathematics classroom teaching.

III. METHODS

The study is a case study. Case study is an in-depth analysis of the typical case, which summarizes the characteristics from case. The case study subject is selected according to the following criteria. Firstly, it is oriented towards obtaining the maximum information from the individual case. Secondly, it is oriented towards demonstrating the most typical from the individual case. Thirdly, it is oriented towards obtaining the maximum degree of participation from the individual case. The case study subject in this study is a senior high school mathematics teacher who teaches at the best junior high school in a prefecture level city in L province in China. The teaching class is composed of the best students in mathematics in this school, and the students are very good in mathematics. Considering whether the participating case study subject can actively cooperate with the researcher to complete this study, the selected case study subject in this study is the researcher's college classmate, who has a good relationship with the researcher and can actively cooperate with the researcher to complete the research task.

Basic Information of Case Study Subject

Teacher A is selected as a subject of this study, who has taught junior high school mathematics for 21 years. Currently, he is responsible for teaching mathematics in two classes in the eighth grade of junior high school and also serves as a class teacher for one class. Teacher A has been influenced by a good family atmosphere since childhood, and has been working hard to learn mathematics. His grades are always at the forefront of his peers, and he has successfully obtained admission to the key high schools in his region. Teacher A has always had excellent academic performance in high school and was successfully admitted to a key normal university in China. During his university years, Teacher A worked hard and achieved excellent results. He served as a class monitor from his sophomore year until graduation and was awarded multiple scholarships and honors such as outstanding student. Teacher A continued his studies and obtained a master's degree during his work.

After graduating from university, Teacher A successfully entered a public key junior high school with the highest score in the interview. Teacher A is highly personal in classroom teaching, which has a good teaching atmosphere. Students actively express their ideas, and their learning emotions are enthusiastic. The students in Teacher A's teaching class often have math scores at the top of their grades. Before class, Teacher A usually delves into lesson preparation from three aspects: students, textbooks, and the classroom. In class, Teacher A is able to promptly identify students' confusion and provide targeted prompts or answers. He is good at capturing the shining points of students' thinking, helping them grasp the key points and overcome difficulties of learning mathematics, and cultivating students' core mathematics literacy. After class, Teacher A is good at summarizing teaching experience and reflecting on shortcomings in time, and promptly improve any problems that arise. Teacher A has obtained provincial and municipal teaching honors multiple times, deeply loved and welcomed by students, and widely praised and unanimously recognized by school leaders. He has been rated as a special level teacher in L province and is a representative of excellent teachers in China.

Reasons for Choosing Teacher A as Subject of Case Study

Firstly, Z Junior High School, as the best key junior high school in a prefecture level city in L province, has many excellent teachers with relatively high teaching standards. The school has a good tradition of running schools and a good academic atmosphere. Many students in this school work hard and have excellent grades. The school places great emphasis on problem-based learning and advocates for a problem driven, teacher guided, and student-centered learning classroom model. Selecting research subject from such key junior high school, and studying the characteristics of mathematics core literacy oriented deep learning classroom teaching can provide demonstration and reference for other schools to implement mathematics deep learning in classroom teaching. It has a certain representation.

Secondly, Teacher A is a famous junior high school mathematics special level teacher in L province. The important achievements and excellent class performance achieved by Teacher A are representative to a certain extent. Choosing and analyzing the characteristics of Teacher A's core literacy oriented junior high school mathematics deep learning classroom teaching can not only analyze the characteristics of core literacy oriented junior high school mathematics deep learning classroom teaching, It can also provide reference for other mathematics teachers to implement deep learning classroom teaching of junior high school that focuses on mathematics core literacy.

Finally, in qualitative research, the scholar's research identity and their relationship with the subject play an undeniable role in promoting or limiting the entire research process and results. The researcher and Teacher A are classmates during their university years, who have known each other for four years and share the same academic background. During the research process, researcher can establish good communication and exchange with Teacher A, and obtain complete information to ensure the smooth progress of this study.

Validity of Case Study

A weak interference classroom observation method is adopted for the selected research subject, in order to obtain the most authentic teaching situation with minimal or even no interference in the regular teaching activities of the research subject. At the same time, by recording videos of the real teaching situations of the research subject, further detailed observation of their teaching behavior activities and recording relevant data are conducted to obtain more comprehensive information for in-depth analysis. Finally, conduct in-depth interviews after class to further collect relevant information about the research subject, and use this as a basis for supplementary analysis.

The value of research depends on the reliability and validity of the study, among which the validity of the study is crucial. Generally speaking, the validity of qualitative research is ensured by enhancing the researcher's function as a research tool and the quality of research description and interpretation in the interactive analysis between researcher and data, and by avoiding various "validity threats" to ensure validity (Maxwell, 1992). In order to control and avoid the "validity threat", this study adopts triangular mutual verification to increase the validity and reliability of the study. The specific measures are as follows.

Firstly, search for research object with strong reliability and high compatibility for investigation. Teacher A is a research subject found by researcher in various aspects, with strong reliability, high cooperation, and general representation, which helps to carry out this study smoothly. After revealing the researcher's identity, research tasks, and research objectives, researcher collected as much relevant information and materials as possible from Teacher A.

Secondly, use the "triangle mutual verification method" to verify relevant conclusions with each other. By conducting classroom observations, post class video analysis, and in-depth interviews to the research subject, researcher conducted a "triangular mutual verification" to collect relevant information on Teacher A's classroom teaching from multiple perspectives, analyze and explain the data, and compare them from multiple perspectives, so that each party in the triangular mutual verification can repair each other to obtain more substantial and complete data to correct corresponding viewpoints.

Thirdly, in order to ensure the reliability and validity of the study, while avoiding inaccuracies caused by information omitted, researcher tries to collect relevant materials such as Teacher A's lesson preparation notebook as much as possible in the early stages of the study, and actively understand Teacher A's views on the situation that occurred in classroom teaching after class, to ensure that there are sufficient supporting materials behind each conclusion.

Fourthly, at the end of the study, the researcher had relevant exchanges and discussions with Teacher A on regarding the preliminary research results. After fully listening to Teacher A's suggestions and opinions on the research results, appropriate adjustments are made to the research results. Under the premise of respecting Teacher A, more objective and reasonable research results are provided.

Fifth, the researcher strives to objectively demonstrate the relevant situation of Teacher A and his classroom teaching, and minimize the emotional participation to the greatest extent, use the collected data to adopt a triangular mutual verification method, to avoid subjective judgments as much as possible, and ensure the authenticity and objectivity of research results.

On the one hand, qualitative research uses the researcher as research tool, use various data collection methods to organize and explore social phenomena in natural contexts, use inductive methods to analyze data and form theories, and obtain interpretive understanding of the behavior and meaning construction through interaction with the research object. On the other hand, Teacher A' classroom teaching has strong personal characteristics. The research conclusion on Teacher A’ core competence-oriented mathematics deep learning classroom teaching cannot represent other teachers’ mathematics deep learning classroom teaching. Therefore, the conclusions drawn from this study regarding mathematics deep learning classroom teaching that focuses on core competencies have certain limitations, and can only provide reference for teachers with similar backgrounds as Teacher A, and cannot be widely applied as a general conclusion.

Ethical Issues of Case Study

Qualitative research places more emphasis on humanistic spirit, and ethical issues are even more unavoidable. In the interaction process between researcher and subject, there is often an active and passive relationship. Only by fully respecting the subject can we better obtain the support and cooperation of the subject. To ensure the smooth progress of the research, researcher will take the following measures to properly address ethical issues.

Firstly, in the preparation stage of the study, the researcher actively confessed to the main tasks and objectives of the study to the subject, and signed a confidentiality agreement with Teacher A, promising to anonymize all matters that may involve personal privacy in the study, ensuring that no personal privacy of the subject will be exposed, and deleting sensitive data from the study. At the same time, taking into account the relevant experiences of the subject, sensitivity issues should be avoided as much as possible when designing research tools, especially during the interview process, which should not cause strong psychological burden on the subject.

Secondly, during the data collection stage of the study, on the one hand, it is necessary to ensure the informed consent of the research subject. The recording and video recording of the research process are conducted with the consent of Teacher A and students. On the other hand, the principle of intervention should be considered. When conducting in-depth interviews and classroom observations, researchers will communicate and negotiate with Teacher A in advance, respect and obey Teacher A's time arrangement, minimize interference with Teacher A's teaching work, and avoid affecting Teacher A's teaching thinking and behavior.

Finally, in the data analysis stage of the study, on the one hand, it is necessary to handle the relationship between researcher and subject well. In qualitative research, both researcher and subject have invested a lot of time and energy. Managing the relationship well not only helps to reduce the negative impact and harm of the subject, but also helps researcher quickly obtain research materials and data. On the other hand, rewards should be given to the subject. Considering the principle of reciprocity, researcher have obtained valuable research data. However, it is also a crucial ethical issue to determine what kind of rewards the subject should receive. Researcher use rewards to make the subject feel respected and cared for by the researcher, rather than blindly seeking and utilizing, which is more in line with the humanistic philosophy of qualitative research.

IV. RESULTS

This section begins by giving a problem in Teacher A's class.

Provide Problem to Stimulate Students' Interest of Learning

Problem. In ΔABC, point E is midpoint of $\overline{AC}$, point D is on $\overline{BC}$, $\overline{BD} = 2\overline{DC}$, $\overline{AD}$ intersects $\overline{BE}$ at point G, $\overline{CG}$ intersects $\overline{AB}$ with point F, $S_{\Delta GCE} = 4$, $S_{\Delta GCD} = 5$, find the area of ΔABC.

Teacher: "Let's look at this problem. Who can solve this problem quickly?"

Student 1: "$\because \overline{BD} = 2\overline{DC}$, $\therefore \overline{BC} = 3\overline{DC}$. $\therefore S_{\Delta BCG} = 3S_{\Delta GCD} = 3 \times 5 = 15$.

$\because S_{\Delta GCE} = 4$, $\therefore S_{\Delta BCE} = S_{\Delta GBC} + S_{\Delta GCE} = 15 + 4 = 19$.

$\because$ Point E is midpoint of $\overline{AC}$, $\therefore S_{\Delta ABC} = 2S_{\Delta BCE} = 2 \times 19 = 38$.

Student 2: $\because$ Point E is midpoint of $\overline{AC}$, $\therefore S_{\Delta AEG} = S_{\Delta GCE} = 4$.

$\because S_{\Delta GCD} = 5$, $\therefore S_{\Delta ACD} = 13$.

$\because \overline{BD} = 2\overline{DC}$, $\therefore \overline{BC} = 3\overline{DC}$. $\therefore S_{\Delta ABC} = 3S_{\Delta ACD} = 3 \times 13 = 39$."

Teacher: "Why are there two different answers? Which answer is right?"

The students begin to examine the process of solving problem, and find the two solutions are both right.

Teacher: "What's wrong with it?"

Students' interest is raised at once, and the classroom atmosphere suddenly becomes intense. Teacher and students begin to explore this problem together.

Induce Students to Explore Problems and Encourage Them to Think from Multiple Perspectives

Teacher: "What can be done to verify the correctness of the two answers?"

The students are trying to find ways to verify the correctness of the results. But because of the difficulty of this problem, students can't find the error of the two solutions immediately. Thus, the teacher has to work with the students to find the error of the two solutions. At this moment, in order to enable students to intuitively determine the correctness of the two results, the teacher think they may explore the problem by using

Geometer's Sketchpad.

Then, Teacher opens Geometer's Sketchpad. First draw an arbitrary △ABC, point E is midpoint of $\overline{AC}$, point D is on $\overline{BC}$, $\overline{BD} = 2\overline{DC}$, $\overline{AD}$ intersects $\overline{BE}$ with point G, $\overline{CG}$ intersects $\overline{AB}$ with point F, as shown in figure 2.

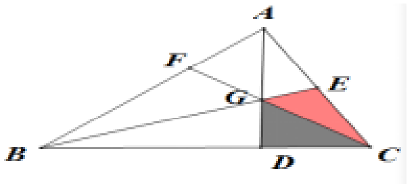

Figure 2

When dragging point C, the area of △GCE and the area of △GCD have changed. Take some information and the list is as follows.

$S_{\Delta GCE}$	1.00	2.00	3.00	3.75	4.00	5.00	6.00
$S_{\Delta GCD}$	1.34	2.66	4.01	5.00	5.34	6.66	8.00
$S_{\Delta ABC}$	10.01	20.01	29.97	37.47	40.05	49.96	59.57

Teacher asks students to look at this table carefully.

Teacher: “After watching this table carefully, what did you find?”

It is easy for students to find such a phenomenon. As shown in figure 3, although there may be some errors in the data displayed by Geometer's Sketchpad, such a great error is certainly not caused by random error. Therefore, when $S_{\triangle GCE}$=4, $S_{\triangle GCD}$=5.34 is not equal to 5.

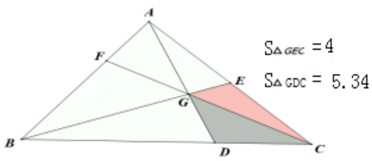

Figure 3

Teacher: “What else can you find from this table?”

It is very difficult for students to answer this question. Many students don't know what to do. At this time, teacher gives students some hints appropriately.

Teacher: "Do you find what's the relationship between the areas of △GCE and △GCD?"

By calculating, students easily obtain a conclusion that the ratio between areas of △GCE and △GCD seems to be equal to 0.75. See the below table.

$S_{\Delta GCE}$	1.00	2.00	3.00	3.75	4.00	5.00	6.00
$S_{\Delta GCD}$	1.34	2.66	4.01	5.00	5.34	6.66	8.00
$\frac{S_{\Delta GCE}}{S_{\Delta GCD}}$	*0.746*	0.752	0.748	0.75	0.749	0.751	0.75

Teacher: "Now, do you know what's wrong with it?"

Students: "The condition ($S_{\Delta GCE} = 4$, $S_{\Delta GCD} = 5$) given in the original problem is wrong."

Teacher: "Is the ratio between areas of △GCE and △GCD 0.75? It is unreliable enough by relying solely on the data of Geometer's Sketchpad. We need to use strict methods to prove the relationship."

Then, students begin to try their best to find a rigorous proof of this problem. After a few minutes, student 3 gives a proof.

Student 3: "We may assume $S_{\Delta GCE} = x,\ S_{\Delta GCD} = y$.

According to the previous first solution, we can obtain that $S_{\Delta ABC} = 3S_{\Delta ACD} = 3(2S_{\Delta GCE} + S_{\Delta GDC}) = 3 \times (2x + y) = 6x + 3y$.

According to the previous second solution, we can obtain that

$S_{\Delta ABC} = 2S_{\Delta BCE} = 2(S_{\Delta GCE} + 3S_{\Delta GDC}) = 2 \times (x + 3y) = 2x + 6y$.

$\therefore 6x + 3y = 2x + 6y$, that is, $4x = 3y$, then $\frac{x}{y} = \frac{3}{4}$."

According to the above calculation, students find the ratio between areas of △GCE and △GCD is surely $\frac{3}{4}$.

At this point, the students know that the problem itself is wrong, which lead to two different result. At this time, teacher asks further question.

Teacher: "Watch figure 1 carefully, and see if you can prove the ratio between areas of △GCE and △GCD is $\frac{3}{4}$ in any other way."

After a while, the students gave several different proofs.

Student 4: "Let $S_{\triangle GCD}=1$. Because of BD=2DC, then $S_{\triangle GBD}=2$.

Because point E is midpoint of $\overline{AC}$, $S_{\triangle GEA}=S_{\triangle GEC}$, $S_{\triangle BEA}=S_{\triangle BEC}$, then $S_{\triangle BGA}=S_{\triangle BGC}=S_{\triangle GBD}+S_{\triangle GCD}=2+1=3$, $S_{\triangle BAD}=S_{\triangle BGA}+S_{\triangle GBD}=3+2=5$.

Because of BD=2DC, then $S_{\triangle ACD}=\frac{1}{2}S_{\triangle BAD}=\frac{5}{2}$, then $S_{\triangle GAC}=S_{\triangle ACD}-S_{\triangle GCD}=\frac{3}{2}$, then $S_{\triangle GCE}=\frac{1}{2}S_{\triangle GAC}=\frac{3}{4}$, that is, $\frac{S_{\triangle GCE}}{S_{\triangle GCD}}=\frac{3}{4}$."

Student 5: "Let $S_{\triangle GCD}=x$. Because of BD=2DC, then $S_{\triangle GBD}=2x$.

Because point E is midpoint of $\overline{AC}$, $S_{\triangle GEA}=S_{\triangle GEC}$, $S_{\triangle BEA}=S_{\triangle BEC}$, then $S_{\triangle BGA}=S_{\triangle BGC}=S_{\triangle GBD}+S_{\triangle GCD}=2x+x=3x$.

Let $S_{\triangle GCE}=y$, because point E is midpoint of $\overline{AC}$, then $S_{\triangle GEA}=S_{\triangle GEC}=y$, because of BD=2DC, then $S_{\triangle ABD}=2S_{\triangle ACD}$, that is, $5x=2(x+2y)$, we have $\frac{x}{y}=\frac{4}{3}$."

Student 6: "Let $S_{\triangle GCD}=x$. Because of BD=2DC, then $S_{\triangle GBD}=2x$,

Because point E is midpoint of $\overline{AC}$, $S_{\triangle GEA}=S_{\triangle GEC}$, $S_{\triangle BEA}=S_{\triangle BEC}$, then $S_{\triangle BGA}=S_{\triangle BGC}=S_{\triangle GBD}+S_{\triangle GCD}=2x+x=3x$.

Let $S_{\triangle GCE}=y$, because point E is midpoint of $\overline{AC}$, then $S_{\triangle GEA}=S_{\triangle GEC}=y$,

Let $S_{\triangle GAF}=a$, $S_{\triangle GBF}=b$, then $a+b=3x$, $a+b+2x=2(x+2y)$, that is, $\frac{x}{y}=\frac{4}{3}$."

Student 7: "Let $S_{\triangle GCD}=x$, because of BD=2DC, then $S_{\triangle GBD}=2x$.

Because point E is midpoint of $\overline{AC}$, $S_{\triangle GEA}=S_{\triangle GEC}$, $S_{\triangle BEA}=S_{\triangle BEC}$, then $S_{\triangle BGA}=S_{\triangle BGC}=S_{\triangle GBD}+S_{\triangle GCD}=2x+x=3x$, then $\frac{S_{\triangle BGA}}{S_{\triangle GBD}}=\frac{3}{4}$, that is, $\frac{S_{\triangle GAC}}{S_{\triangle GCD}}=\frac{3}{2}$.

Let $S\triangle GCE=y$. Because point E is midpoint of $\overline{AC}$, then $S_{\triangle GEA}=S_{\triangle GEC}=y$, then $\frac{2y}{x}=\frac{3}{2}$, that is, $\frac{x}{y}=\frac{4}{3}$."

Students gave so many different proofs that the teacher was very surprised. At this time, the teacher summarized it, and continued to inspire and guide students to pose new problem.

Focus on Problem Variants, Extensions, and Applications

Teacher: "As Klamkin (1986) pointed outed, George Polya makes the analogy of finding a precious uncut stone on the shore and tossing it away since it is not recognized as being valuable. One has to do a certain amount of cutting and polishing before the value of the stone is recognized, although an expert usually can get away with a careful examination. So, in regard to a problem which has just been solved or whose solution has been looked up, we should not immediately pass on to something else. Rather, we should "stand back" and re-examine the problem in light of its solution and ask ourselves whether or not the solution really gets to the "heart" of the problem. Mathematically, one of the points being made here is to check whether or not the hypotheses of the problem are necessary for the result. (That the hypotheses are sufficient follows from the validity of the result). Additionally, although our solution may be valid, there may be usually better ways of looking at the problem which make the result and the proof more transparent and can as well lead to extensions. Consequently, it should be easier to understand the result as well as to give a non-trivial generalization."

After that, Teacher guides students to further explore this problem, and ask whether students can find a simpler answer of the problem. Because it is a very typical figure, watching this picture, some students think of Ceva theorem and Menelaus' theorem. Then, the teacher asks students to apply the Ceva theorem and Menelaus' theorem to deduce the quantitative relation between areas of △GCE and △GCD by themselves. In this way, it is easy for students to get another way to deduce the relationship between areas of △GCE and △GCD. Finally, students finally understand the essence of the problem. The following answer is given by the students themselves under the guidance of the teacher.

Student 8: "By using Ceva theorem, we have $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1$, because of BD=2DC, and because point E is midpoint of $\overline{AC}$, we have $\frac{\overline{AF}}{\overline{FB}} = \frac{1}{2}$. By using Menelaus' theorem, we have $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BC}}{\overline{CD}} \cdot \frac{\overline{DG}}{\overline{GA}} = 1$, then $\frac{\overline{GA}}{\overline{DG}} = \frac{3}{2}$, then $\frac{S_{\Delta GCE}}{S_{\Delta GCD}} = \frac{\frac{1}{2}S_{\Delta ACG}}{S_{\Delta GCD}} = \frac{3}{4}$."

Teacher: "We correct the mistake of the original problem, that is, change the conditions of the original problem. What new math problems can we get?

Student 1: "In ΔABC, point E is midpoint of $\overline{AC}$, point D is on $\overline{BC}$, $\overline{BD} = 2\overline{DC}$, $\overline{AD}$ intersects $\overline{BE}$ with point G, $\overline{CG}$ intersects $\overline{AB}$ with point F, $S_{\Delta GCE} = 4$, seek the area of ΔABC."

Student 2: "In ΔABC, point E is midpoint of $\overline{AC}$, point D is on $\overline{BC}$, $\overline{BD} = 2\overline{DC}$, $\overline{AD}$ intersects $\overline{BE}$ with point G, $\overline{CG}$ intersects $\overline{AB}$ with point F, $S_{\Delta GCD} = 5$, seek the area of ΔABC."

Student 3: "In ΔABC, point E is midpoint of $\overline{AC}$, point D is on $\overline{BC}$, $\overline{BD} = 2\overline{DC}$, $\overline{AD}$ intersects $\overline{BE}$ with point G, $\overline{CG}$ intersects $\overline{AB}$ with point F,

$S_{\Delta GCE} = 4$, $S_{\Delta GCD} = \frac{16}{3}$, seek the area of ΔABC."

In the above whole process, the teacher mainly inspires the students by problems. Through teacher's guiding and inspiring the students to think step by step, students finally discover the nature of problems and experience the process of generating problem.

At this time, the teacher continues to guide students to pose new problems based on the original problem. Due to limitations of the ability of students' thinking, the students sometimes don't know how to do it. The teacher gives an example first (for example, variant problem below), then slowly guides students into the situations of posing new problems.

Variant problem. As shown in figure 4, in ΔABC, point E is midpoint of $\overline{AC}$, point D is on $\overline{BC}$, $\overline{BD} = 2\overline{DC}$, $\overline{AD}$ intersects $\overline{BE}$ with point G, $\overline{CG}$ intersects $\overline{AB}$ with point F. If we know the area of any one triangle among six small triangles in figure 3, Can we find out the area of ΔABC?

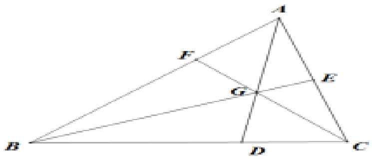

Figure 4

After the teacher puts forward this problem, students first want to try them one by one. After classification and induction, the results of students' inquiring are as follows.

Problem 1. If the area of ΔGCE (or ΔAGE) is s, seek the area of ΔABC.

Proof. We can get it quickly from the above solutions, that is, $S_{\Delta ABC} = 3S_{\Delta ADC} = 3(S_{\Delta AGC} + S_{\Delta GCD}) = 3(2S_{\Delta GCE} + \frac{4}{3}S_{\Delta GCE}) = 10s$. Or $S_{\Delta ABC} = 3S_{\Delta ADC} = 3(S_{\Delta AGC} + S_{\Delta GCD}) = 3(2S_{\Delta GAE} + \frac{4}{3}S_{\Delta GAE}) = 10s$.

Problem 2. If the area of ΔGCD (or ΔGBD) is s, seek the area of ΔABC.

Proof. We can get it quickly from the above solutions, that is, $S_{\Delta ABC} = 2S_{\Delta BCE} = 2(S_{\Delta BGC} + S_{\Delta GCE}) = 2(3S_{\Delta GCD} + \frac{4}{3}S_{\Delta GCD}) = \frac{15}{2}s$. Or $S_{\Delta ABC} = 2S_{\Delta BCE} = 2(S_{\Delta BGC} + S_{\Delta GCE}) = 2(\frac{3}{2}S_{\Delta GBD} + \frac{3}{8}S_{\Delta GBD}) = \frac{15}{4}s$.

Problem 3. If the area of ΔAGF is s, seek the area of ΔABC.

Proof. $\because$ E is midpoint of $\overline{AC}$, $\therefore$ $S_{\Delta ABE} = S_{\Delta CBE}$, $S_{\Delta GAE} = S_{\Delta GCE}$. $\therefore$ $S_{\Delta GBA} = S_{\Delta GBC}$. $\because$ $\overline{BD} = 2\overline{DC}$, $\therefore$ $S_{\Delta GBD} = \frac{2}{3}S_{\Delta GBC} = \frac{2}{3}S_{\Delta GBA}$, that is, $\frac{S_{\Delta GBA}}{S_{\Delta GBD}} = \frac{3}{2}$. $\therefore$ $\frac{\overline{AG}}{\overline{GB}} = \frac{3}{2}$, $\therefore$ $\frac{\overline{AG}}{\overline{AD}} = \frac{3}{5}$.

Passing point G for $\overline{GH} \parallel \overline{BC}$, intersects $\overline{AB}$ with point H, as shown in figure 5, we can get $\frac{\overline{HG}}{\overline{BC}} = \frac{\overline{AG}}{\overline{AD}} = \frac{3}{5}$. $\therefore$ $\frac{\overline{HG}}{\overline{BC}} = \frac{\overline{HG}}{\overline{BD}} \times \frac{\overline{BD}}{\overline{BC}} = \frac{3}{5} \times \frac{2}{3} = \frac{2}{5}$.

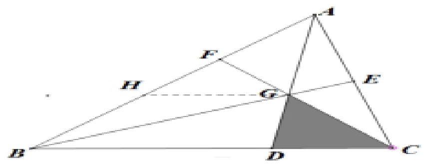

Figure 5

$\therefore$ $\frac{\overline{FG}}{\overline{FC}} = \frac{\overline{HG}}{\overline{BC}} = \frac{2}{5}$, $\therefore$ $\frac{\overline{FG}}{\overline{GC}} = \frac{2}{3}$. $\therefore$ $S_{\Delta GBF} = \frac{2}{3} S_{\Delta GBC} = S_{\Delta GBD}$, $\therefore$ $S_{\Delta GCD} = S_{\Delta GAF} = s$.

The problem is transformed into the above second situation (problem 2).

What we need to point out here is that students may not immediately think of constructing similar triangle by adding the parallel line. Students need good mathematical thinking in order to obtain the auxiliary line. The teacher prompts students to construct the auxiliary line appropriately.

Problem 4. If the area of ΔBGF is s, seek the area of ΔABC.

Proof. Based on the above solutions, we can easily get $S_{\Delta GCD} = S_{\Delta GAF} = \frac{1}{2} S_{\Delta GBF} = \frac{1}{2} s$.

The problem is also transformed into the above second situation (problem 2).

Teacher: “Summarize the above process of inquiring, what conclusions can we get?”

Student 4: “(Variant problem) In ΔABC, point E is midpoint of $\overline{AC}$, point D is on $\overline{BC}$, $\overline{BD} = 2\overline{DC}$, $\overline{AD}$ intersects $\overline{BE}$ with point G, $\overline{CG}$ intersects $\overline{AB}$ with point F. If we know the area of any one triangle among six small triangles in figure 3, we can find out the area of ΔABC.”

Student 5: “Because of $S_{\Delta GAF} = \frac{1}{2} S_{\Delta GBF}$, we obtain $\overline{AF} = \frac{1}{2}\overline{BF}$. Then we can also pose a new problem, that is, problem 5. In ΔABC, point E is midpoint of $\overline{AC}$. Point D is on $\overline{BC}$. $\overline{BD} = 2\overline{DC}$, $\overline{AD}$ intersects $\overline{BE}$ with point G. Link $\overline{CG}$ and extend, intersects $\overline{AB}$ with point F. Then point F must be the three bisection point of $\overline{AB}$.”

Based on the variant problem, teacher further inspires students' thinking and posing problem 6, which is generalization of variant problem.

Student 6: “Problem 6. As shown in figure 6, in ΔABC, points D, E, F are on the three sides respectively. $\overline{AE} = m\overline{EC}$, $\overline{BD} = n\overline{DC}$, $m, and\ n$ are constant and not equal to 0. $\overline{AD}, \overline{BE}, \overline{CF}$ intersect at point G. Then, what is the relationship between the area of △GCE and △GCD ?

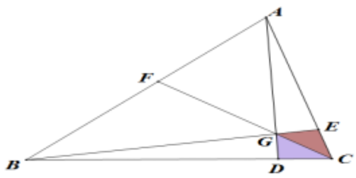

Figure 6

Similarly, we can easily obtain $\frac{S_{\Delta GCE}}{S_{\Delta GCD}} = \frac{m(n+1)}{n(m+1)}$. That is, we have problem 6.

Problem 6. In ΔABC, points D, E, F are on the three sides respectively. $\overline{AE} = m\overline{EC}$, $\overline{BD} = n\overline{DC}$, and m, n are constant and not equal to 0. $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ intersect at point G. Then $\frac{S_{\Delta GCE}}{S_{\Delta GCD}} = \frac{m\ (n+1\ )}{n\ (m+1)}$."

Student 7: "Based on problem 6, we can further generalize the result. We may change the condition of problem 6. That is, $\frac{\overline{\overline{AE}}}{\overline{\overline{EC}}} = \frac{m}{n}$, $\frac{\overline{\overline{BD}}}{\overline{\overline{DC}}} = \frac{a}{b}$, $m, n, a, and\ b$ are constant and not equal to 0. We can get problem 7.

Problem 7. In ΔABC, points D, E, F are on the three sides respectively. $\frac{\overline{\overline{AE}}}{\overline{\overline{EC}}} = \frac{m}{n}$, $\frac{\overline{\overline{BD}}}{\overline{\overline{DC}}} = \frac{a}{b}$, $m, n, a, and\ b$ are constant and not equal to 0. $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ intersect at point G. Then $\frac{S_{\Delta GCE}}{S_{\Delta GCD}} = \frac{m\ (a+b\ )}{a\ (m+n)}$."

Teacher: "Can we extend the above conclusions to the quadrilateral? Let's try it from a special quadrilateral."

Student 8: "I can pose a new problem. As shown in figure 7, in square ABCD, point E is midpoint of $\overline{CD}$. Point F is on $\overline{BC}$. $\overline{BF} = 2\overline{FC}$. $\overline{DF}$ intersects $\overline{BE}$ with point G. $\overline{CG}$ intersects $\overline{AD}$ with point H. If we know $S_{\Delta GCE} = s$, can we find area of square ABCD?

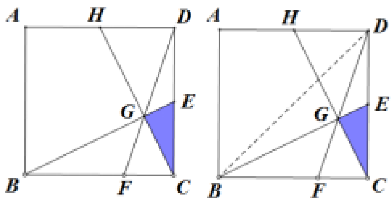

Figure 7

Proof. By connecting point B and D, we can get $S_{squareABCD} = 2S_{\Delta BCD} = 2 \times 10s = 20s$."

Student 9: "I can also pose a new problem. As shown in figure 8, in rectangle

ABCD, point E is midpoint of $\overline{CD}$. Point F is on $\overline{BC}$. $\overline{DF}$ intersects $\overline{BE}$ with point G. $\overline{CG}$ intersects $\overline{AD}$ with point H. If we know $S_{\Delta GCE} = s$, can we find area of rectangle ABCD?

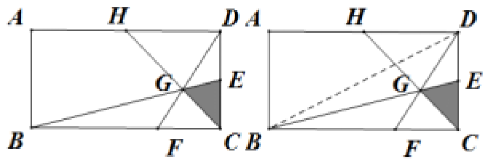

Figure 8

Proof. Similarly, by connecting point B and D, we can obtain $S_{rectangleABCD} = 2S_{\Delta BCD} = 2 \times 10s = 20s$."

Student 10: "I can also pose a new problem. As shown in figure 9, in parallelogram ABCD, point E is midpoint of $\overline{CD}$. Point F is on $\overline{BC}$. $\overline{DF}$ intersects $\overline{BE}$ with point G. $\overline{CG}$ intersects $\overline{AD}$ with point H. If we know $S_{\Delta GCE} = s$, can we find area of parallelogram ABCD?

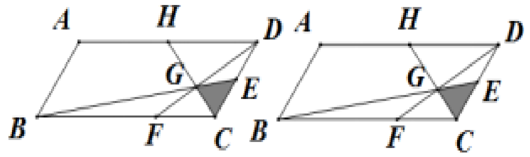

Figure 9

Proof. Similarly, we just need to connect with $\overline{BD}$, we can obtain $S_{parallelogramABCD} = 2S_{\Delta BCD} = 2 \times 10s = 20s$."

Student 11: "I can also further extend it and obtain problem 8.

Problem 8: As shown in figure 10, in parallelogram ABCD, points E, F are on sides $\overline{CD}, \overline{BC}$ respectively. $\frac{\overline{DE}}{\overline{EC}} = \frac{a}{b}$, $\frac{\overline{BF}}{\overline{FC}} = \frac{c}{d}$, $\overline{DF}$ intersects $\overline{BE}$ with point G.

If we know $S_{\Delta GCE} = s$, then the area of parallelogram ABCD is $\frac{2(a+b)(\mathrm{ac+bc+ad})}{\mathrm{ab}(c+d)}$."

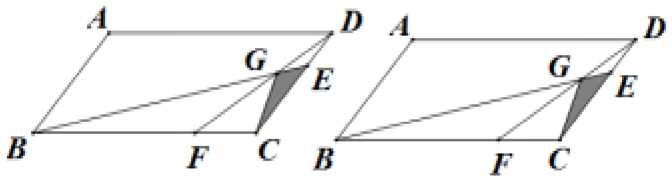

Figure 10

Guide Students to Reflect and Construct Mathematics Knowledge by Themselves

Teacher: "At last, do you learn anything from this lesson?"

Student 12: "The task begins with a wrong problem. The teacher transforms the wrong problem into a rich mathematical task by showing how it can be demonstrated with six different proofs and generalized to several different new problems. A variety of mathematical techniques address the aspect of connections by proving and generalizing the proposition to build new mathematics problems. A seemingly simple problem contains the mystery of mathematical discovery. Extending from triangle to quadrilateral, from experimental conjecture to reason demonstration, from special phenomenon to general rule, the students have set up a bridge to succeed by hands-on operation, deep thinking, analogy and generalization. The students think by studying, explore by thinking, and exercise in the inquiry learning. We have acquired and experienced mathematicians' methods of studying mathematics in practice, and learn how to pose mathematics problems. It is the real way of learning mathematics."

Student 13: "Teachers should pay attention to the process of training students' posing mathematics problems when teaching mathematics. The goal of mathematics teaching should be done (doing) instead of knowing. When the teachers pay attention to training students' posing mathematics problems, it is very important to guide the students to analyze mathematics problems by an active and orderly way, and guide students step by step to explore the nature of mathematical problems and discovery the relationships among mathematical problems."

Student 14: "Teachers should teach mathematics by inquiring, and not directly give the conclusion to the students. Teachers should guide students to find the solution step by step when solving problems, or guide students to create by doing a series of problems. Of course, It is not really invented for human beings, but it's true for students."

Student 15: "Teachers should give some hints when guiding students to experience and discover, just like GPS navigating the way to a fork. You can drive by yourself after entering the road. If students take an active part in developing mathematical thinking and procedures, students can learn to pose mathematics problems."

V. DISCUSSION AND CONCLUDING REMARKS

Teachers Need to have a Solid Mathematical Foundation when Implementing Deep Teaching in Problem-solving and Problem-posing Classroom

Deep learning cannot occur naturally, it requires triggering conditions. The prerequisite is teacher's deliberate guidance. For example, the contents that students learn must be structured teaching materials that have been carefully designed by teachers. Teaching process must have a plan which designs in advance to achieve rich and complex teaching goals in a planned and orderly manner within a limited time and space. It is very important for teachers to establish teaching goals for developing students' higher-order thinking in deep mathematics teaching. Teachers should put a special emphasis on meaning and connection of learning content, and guide students to construct mathematics knowledge. Teachers should create real situations that promote deep learning and guide students to actively experience discovering mathematics knowledge. Teachers should choose appropriate evaluation methods and guide students to reflect mathematics knowledge deeply. Teachers need to have problem awareness, including original problems that can reflect the basic elements of mathematics core knowledge, and inspiring problems that stimulate students' deep thinking. Teachers need to have the ability of solving problems by using multiple methods and posing new problems.

But our mathematics teaching is often driven by solving problems. Teachers or textbooks give problems, and students try their best to find solutions of the problems. Once the students have solved the problem, they will stop further exploring and studying this problem. Students are rarely asked to pose new problems based on the problem that has been solved. In fact, such exploration should become the core of mathematics learning activities. By these activities of exploring, students can often find the internal relations among different knowledge and methods, and obtain important generalization or deeper understanding of the original problem. If students lack the process of exploring in mathematics learning, students will surely lose an excellent opportunity of improving their mathematical ability. This will make it impossible to carry out deep learning. As Mason, Burton and Stacey (1982) pointed out, there is no mistake of further studying and discussing the progress of a problem. It is undoubtedly of great value to think, change, repeatedly refine a problem in different directions. Deep learning mathematics classroom teaching aims to engage students in such activities.

In order to teach students to learn mathematics deeply, teachers should be asked to design exercises and guide students to explore. This requires that teachers should have some experiences of mathematics discovery or deep learning mathematics. As Polya (1962) pointed out, if a teacher has never had any practical experience of creative work, how can he evoke, guide, help, or even appreciate his students' creative activities? If a teacher's knowledge in mathematics is passively accepted, it's hard for him to promote his students' active learning. If a teacher has never been born with a fancy, he probably rebukes a student who comes up with a clever idea, and can't encourage the student to do so. Cuoco (2001) also pointed out there are very few absolutes in education, but there's one thing of which I am absolutely certain: The best high school teachers are those who have a research-like experience in mathematics. Teachers who have done this type of research are much less likely to think of mathematics as an established body of facts than are teachers who have simply taken a set of courses. They are more likely to stay engaged in mathematics after they start teaching. They are used to looking for connections that don't live on the surface. They are much more likely to organize their classes around large investigations rather than low-level exercises. What mathematics teachers should do is to ask students to learn mathematics through their own doing mathematics. Teachers should guide students to learn mathematics by the way of doing mathematics. It is the most important thing. All the activities carried out and organized by Teacher A in this case demonstrate that Teacher A has a solid mathematical foundation. This is the mathematical quality that teachers must possess to carry out deep teaching in problem-solving and problem-posing classroom. If teachers lack this mathematical quality, they can't guide students to engage in deep learning.

Teachers Need to have an Ability of Organizing and Managing Classroom Teaching Effectively when Implementing Deep Teaching in Problem-solving and Problem-posing Classroom

Mathematics deep learning emphasizes communicating and expressing among teachers and students. On the one hand, teachers should create time and opportunities to provide conditions for cultivating students to think deeply about problems, and learn to listen to students' ideas and give timely evaluations. On the other hand, students should learn to listen to others' ideas, compare their own viewpoints, provide correct responses, and be good at expressing their own opinions, thinking about the content learned from multiple perspectives. Teachers need to have the ability of managing the classroom, including creating an atmosphere of communication and exploring, and providing timely and appropriate guidance. Students need to have an attitude of actively learning mathematics knowledge, including an attitude of self-adjusting the mathematics content of learning, and an attitude of not giving them up when facing difficulties. Students need to have good cooperative and communication skills, including the ability of expressing mathematics language and mathematical dialogue ability among peers. Students need to be able to think and reflect on mathematics, which includes thinking about mathematics problems independently, as well as self-reflecting and summarizing mathematics core content.

In this case, there are multiple times when students have no ideas, Teacher A constantly stimulates and guides them to try their best to lure out their ideas as much as possible. This can reflect that Teacher A has the ability of organizing and managing classroom teaching as required for implementing deep teaching. For example, when students can't recognize the errors of solutions of the original problem, Teacher A first guides students to see where the problem lies by dynamically dragging Geometric sketchpad, then guides students to propose conjectures and proofs. When the students sometimes don't know how to pose new mathematics problem. Teacher A gives an example first (for example, variant problem below), then slowly guides students into the situations of posing new problems. When students have ideas, Teacher A actively encourages them to express their ideas, making the classroom atmosphere very lively, which is easy to induce students to engage in deep learning. Implementing deep teaching in problem-solving and problem-posing classroom, it is necessary to organize and manage classroom teaching like Teacher A.

REFERENCES

Bransford, J. D, Brown, A. L, & Cocking, R. R. (2000/2004) (Expanded Edition). *How people learn: Brain, mind, experience, and school*. National Academic Press.

Brown, S. I., & Walter, M. I. (1983). *The art of problem posing*. Franklin Institute Press.

Brown, S. I., & Walter, M. I. (1993). Problem posing in mathematics education. In I. Brown Stephen, & I. Walter Marion (Eds.), *Problem posing: Reflection and applications* (pp. 16–27). Lawrence Erlbaum Associates.

Bruner, J. (1977). *The process of education.* Harvard University Press.

Cuoco, Al. (2001). Mathematics for teaching. *Notices of the American Mathematical Society, 48*(2), 168-174.

Da Ponte, J. P. (2007). Investigations and explorations in the mathematics classroom. *ZDM—The International Journal on Mathematics Education, 39* (5-6), 419–430. https://doi.org/10.1007/s11858-007-0054-z

Davis, P. J. (1985). What do I know? A study of mathematical self-awareness. *The College Mathematics Journal, 16* (1), 22-41. https://doi.org/10.1080/07468342.1985.11972847

English, L. D. (1996). Children's problem posing and problem solving preferences. In J. Mulligan, & M. Mitchelmore (Eds.), *Research in early number learning* (pp. 227–242). Australian Association of Mathematics Teachers.

English, L. (1999). Reasoning by analogy: A fundamental process in children's mathematical learning. In L. V. Stiff, & F. R. Curcio (Eds.), *Developing mathematical reasoning in grades K-12* (pp. 22-36). National Council of Teachers of Mathematics.

Fullan, M., & Langworthy, M. (2014). *A rich seam: How new pedagogies find deep learning*. Pearson.

Fullan, M., Quinn, J., & McEachen, J. (2018). *Deep learning: Engage the world, change the world*. SAGE.

Hanna, G., & De Villiers, M. (Eds.). (2012). *Proofs and proving in mathematics education. The 19th ICMI Study*. Springer.

Hashimoto, Y. (1987). Classroom practice of problem solving in Japanese elementary schools. In Becker, J., & Miwa, T. (Eds.), *Proceedings of the U.S.-Japan seminar on mathematical problem solving* (pp. 94-119). Southern Illinois University.

Hattie, J., & Donoghue, G. (2016). Learning strategies: A synthesis and conceptual model. *npj Science of Learning, 1*, 16013. https://doi.org/10.1038/npjscilearn.2016.13

Healy, C. C. (1993). *Creating miracles: A story of student discovery*. Key Curriculum Press.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning: A project of the National Council of Teachers of Mathematics* (pp. 65–97). Macmillan Publishing Co, Inc.

Kilpatrick, J. (1987). Problem formulating: Where do good problems come from? In A. H. Schoenfeld (Ed.), *Cognitive science and mathematics education* (pp. 123-147). Lawrence Erlbaum Associates.

Klamkin, M. S. (1986). Problem posing and mathematical creativity. *Crux*

Mathematicorum, 12(10), 23-42.

Kontorovicha, I., Koichua, B., Leikinb,R., & Berman, A. (2012). An exploratory framework for handling the complexity of mathematical problem posing in small groups. *Journal of Mathematical Behavior, 31*(1), 149-161. https://doi.org/10.1016/j.jmathb.2011.11.002

Lavy, I., & Shriki, A. (2009). Small change-big difference. *The Mathematics Enthusiast, 6*(3), 395–410. https://doi.org/10.54870/1551-3440.1160

Lavy, I., & Shriki, A. (2010). Engaging in problem posing activities in a dynamic geometry setting and the development of prospective teachers' mathematical knowledge. *Journal of Mathematical Behavior, 29*(1), 11-24. https://doi.org/10.1016/j.jmathb.2009.12.002

Leikin, R., & Grossman, D. (2013). Teachers modify geometry problems: from proof to investigation. *Educational Studies in Mathematics, 82*(3), 515-531. https://doi.org/10.1007/s10649-012-9460-4

Li, Chunhui & Wen, Fengtong (2023). Promoting deep learning in mathematics education—Based on understanding by design theory. *Academic Journal of Mathematical Sciences, 4*(4), 57-61. https://doi.org/10.25236/AJMS.2023.040409

Mackie, D. (2002). Using computer algebra to encourage a deep learning approach to calculus. In Deborah, H.H., & Constantinos, T. (Eds.), *Proceedings from 2nd International Conference on the Teaching of Mathematics at the Undergraduate Level* (pp. 1872-1880). Crete, Greece.

Marton, F., & Säljö, R. (1976). On qualitative differences in learning: I—Outcome and process. *British Journal of Educational Psychology, 46*(1), 4–11. https://doi.org/10.1111/j.2044-8279.1976.tb02980.x

Mason, J., Burton, L., & Stacey, K. (1982). Thinking mathematically. Addison Wesley Longman.

Maxwell, J. (1992). Understanding and validity in qualitative research. *Harvard Educational Review, 62*(3), 279–301. https://doi.org/10.17763/haer.62.3.8323320856251826

McPhail, G. (2021). The search for deep learning: A curriculum coherence model. *Journal of Curriculum Studies, 53*(4), 420-434. https://doi.org/10.1080/00220272.2020.1748231

Ministry of Education of the People's Republic of China (2022). *Mathematics curriculum standards for ordinary high schools*. Beijing Normal University Press. (In Chinese)

Ministry of Education of the People's Republic of China (2022). *Mathematics curriculum standards for compulsory education*. Beijing Normal University Press. (In Chinese)

National Council of Teachers of Mathematics. (1991). *Professional standards for teaching mathematics*. Author.

Parker, W., Mosborg, S., Bransford, J., Vye, N., Wilkerson, J., & Abbott, R. (2011). Rethinking high school coursework: Tackling the depth/breadth tension in the AP US government and politics course. *Journal of Curriculum Studies, 43*(4), 533–559. https://doi.org/10.1080/00220272.2011.584561

Polya, G. (1957). *How to solve it* (2nd ed.). Doubleday.

Polya, G. (1962). *Mathematical discovery on understanding, learning, and teaching problem solving* (Volume 2). John Wiley.

Sawyer, R. K. (Eds.) (2014). *The Cambridge handbook of the learning sciences*.

Cambridge University Press. https://doi.org/10.1017/CBO9781139519526

Shriki, A. (2006). The area problem. *Mathematical Spectrum, 38*(1), 27–33.

Shriki, A., & Lavy, I. (2004). Exploring mathematical patterns using dynamic geometry software. *The Australian Mathematics Teacher, 60*(3), 36–40.

Silver, E. A. (1994). On mathematical problem posing. *For the Learning of Mathematics, 14*(1), 19–28.

Silver, E. A., & Cai, J. (2005). Assessing students' mathematical: problem posing. *Teaching Children Mathematics, 12*(3), 129-135. https://doi.org/10.5951/TCM.12.3.0129

Silverman, F. L., Ken, W., and Donna, S. (1992). Student-generated story problems. *Arithmetic Teacher, 39*(8), 6-12. https://doi.org/10.5951/AT.39.8.0006

Stein, M., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2009). *Implementing standards-based mathematics instruction: A casebook for professional development* (2nd ed.). NCTM.

Whitin, D. (2004). Building a mathematical community through problem posing. In R, Rubenstein & G. W. Bright (Ed.), *Perspectives on the teaching of mathematics* (pp. 129-140). National Council of Teachers of Mathematics.

Winch, C. (2017). *Teachers' know-how: A philosophical investigation*. Wiley Blackwell.

Yerushalmy, M., Chazan, D., & Gordon, M. (1990). Mathematical problem posing: Implications for facilitating student inquiry in classrooms. *Instructional Science, 19*(3), 219-245. https://doi.org/10.1007/BF00120197