J. Korean Soc. Ind. Appl. Math. Vol.28, No.1, 22–32, 2024 https://doi.org/10.12941/jksiam.2024.28.022

G^3 HEXIC Bézier CURVES APPROXIMATING CONIC SECTIONS

HYEONG MOON YOON[1] AND YOUNG JOON AHN[1,†]

[1]DEPARTMENT OF MATHEMATICS EDUCATION, CHOSUN UNIVERSITY, GWANGJU, 61452, SOUTH KOREA
Email address: ttygogo@naver.com, †ahn@chosun.ac.kr

ABSTRACT. In this paper we present a method of conic section approximation by hexic Bézier curves. The hexic Bézier approximants are G^3 Hermite interpolations of conic sections. We show that there exists at least one hexic Bézier approximant for each weight of the conic section. The hexic Bézier approximant depends one parameter and it can be obtained by solving a quartic polynomial, which is solvable algebraically. We present the explicit upper bound of the Hausdorff distance between the conic section and the hexic Bézier approximant. We also prove that our approximation method has the maximal order of approximation. The numerical examples for conic section approximation by hexic Bézier curves are given and illustrate our assertions.

1. INTRODUCTION

Conic section approximation by Bézier curves is an important problem in the fields of CAD/CAM and CAGD (Computer Aided Geometric Design). The researches for finding Bézier approximants of low degree and with high approximation order have been studied in recent forty years.

For planar curve, the maximal order of approximation by the n-th degree Bézier approximant is $2n$ [1]. The G^2 Hermite interpolation of planar curve by cubic Bézier curves with approximation order six has been presented by de Boor et al. [2]. The planar curve approximation by G^2 quadratic Bézier biarcs with approximation order four can be obtained [3]. A lot of approximation methods for circular arc by Bézier curves of degree $2 \leq n \leq 5$ with approximation order $2n$ were proposed [4, 5, 6, 7, 8, 9]. Using Chebyshev property, the approximation methods of circular arcs by polynomial curves of all degree n with very high precision have been presented [10, 11].

In case of conic approximations, Floater [12] presented the conic approximation by G^{n-1} polynomial curves of all odd degree $n \geq 3$ having approximation order $2n$. Many methods approximating conic sections by Bézier curves of degree $2 \leq n \leq 5$ have been obtained with

Received March 11 2024; Revised March 25 2024; Accepted in revised form March 25 2024; Published online March 25 2024.

2000 *Mathematics Subject Classification.* 41A10, 65D17.

Key words and phrases. conic approximation, Bézier curve, approximation order, error bound analysis, asymptotic analysis.

†Corresponding author.

at least G^2 continuity and with the maximal order of approximation [13, 14, 15]. Using the polynomial factorization over $\mathbb{C}$, the conic section approximation by polynomial curves of all degree n having $2n$ contacts with the conic sections was presented [16].

Recently, the approximation methods for circular arcs by hexic polynomial curves with approximation order twelve have been presented [17]. In this paper we aim to find the hexic Bézier curves approximating conic sections with at least G^2 continuity and with approximation order twelve. Our method yields the hexic Bézier approximant which is G^3 Hermite interpolation of the conic section. We show that there exists at least one G^3 hexic Bézier approximant for each weight of the conic section. The hexic Bézier approximant depends one parameter which can be obtained by solving a quartic polynomial. We present the explicit upper bound of the Hausdorff distance between the conic section and the hexic Bézier approximant, and thus our method yields the optimal approximation from four solutions. We also prove that the approximation order of our method is twelve. The numerical examples for conic section approximation by hexic Bézier curves are presented to illustrate our assertions.

The remainder of this paper is constructed as follows. In Section 2, the preliminaries for the conic sections and their approximation by Bézier curves are given. In Section 3, a method of conic section approximation by hexic Bézier curves are presented and its error bound analysis is obtained. The numerical examples are given in Section 4, and our study is summarized in Section 5.

2. PRELIMINARIES

In this section we present the preliminaries for conic sections and the geometric Hermite interpolation. Any conic section $\mathbf{c}$ can be represented by the standard quadratic rational Bézier form [13, 12, 18, 19],

$$\mathbf{c} : [0,1] \to \mathbb{R}^2, \quad \mathbf{c}(t) = \frac{B_0^2(t)\mathbf{c}_0 + wB_1^2(t)\mathbf{c}_1 + B_2^2(t)\mathbf{c}_2}{B_0^2(t) + wB_1^2(t) + B_2^2(t)},$$

where $B_i^2(t) = \binom{2}{i} t^i (1-t)^{2-i}$, $i = 0, 1, 2$, are quadratic Bernstein polynomials, $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2$ are the control points in $\mathbb{R}^2$, and $w \in \mathbb{R}$ is the weight associated with $\mathbf{c}_1$. If the conic section $\mathbf{c}$ is subdivided at the shoulder point

$$\mathbf{c}(1/2) = \frac{\mathbf{m} + w\mathbf{c}_1}{1+w},$$

where $\mathbf{m} = (\mathbf{c}_0 + \mathbf{c}_2)/2$, then two subdivided conic sections $\mathbf{c}^1, \mathbf{c}^2$ are obtained and they have the same weight $w = \sqrt{(w+1)/2}$ and different control points

$$\mathbf{c}_0, \frac{\mathbf{c}_0 + w\mathbf{c}_1}{1+w}, \mathbf{c}(1/2), \text{ and } \mathbf{c}(1/2), \frac{\mathbf{c}_2 + w\mathbf{c}_1}{1+w}, \mathbf{c}_2,$$

respectively, [15]. Any point $\mathbf{p} \in \mathbb{R}^2$ can be written uniquely in terms of barycentric coordinates τ_0, τ_1, τ_2, where $\tau_0 + \tau_1 + \tau_2 = 1$, with respect to the triangle $\triangle \mathbf{c}_0\mathbf{c}_1\mathbf{c}_2 : (x, y) = \tau_0\mathbf{c}_0 + \tau_1\mathbf{c}_1 + \tau_2\mathbf{c}_2$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(\mathbf{p}) = \tau_1^2 - 4w^2\tau_0\tau_2.$$

The conic section $\mathbf{c}$ satisfies the equation $f(\mathbf{c}(t)) = 0$ for all $t \in [0,1]$, [13, 12].

Let $\mathbf{x} : [a,b] \to \mathbb{R}^2$ be a parametric curve. If there exists a regular bijective reparameterization $\rho : [a,b] \to [0,1]$ with $\rho' > 0$, such that

$$\frac{d^j\mathbf{x}}{dt^j}(t) = \frac{d^j(\mathbf{c}\circ\rho)}{dt^j}(t),\ t = a, b,\ \ j = 0, 1, \ldots, k,$$

then we say that $\mathbf{x}$ is a G^k Hermite interpolation of $\mathbf{c}$ [17]. It is well known [12] that $\mathbf{x}$ is a G^k Hermite interpolation of the conic section $\mathbf{c}$ if and only if the function $f(\mathbf{x}(t))$ has zeros of multiplicity $k+1$ at $t = a, b$ with $\mathbf{c}'(0)\cdot\mathbf{x}'(a) > 0$ and $\mathbf{c}'(1)\cdot\mathbf{x}'(b) > 0$.

The Hausdorff distance between two curves $\mathbf{x} : [a,b] \to \mathbb{R}^2$ and $\mathbf{y} : [c,d] \to \mathbb{R}^2$ is defined by

$$d_H(\mathbf{x},\mathbf{y}) = \max\{\max_{t\in[a,b]}\min_{s\in[c,d]}||\mathbf{x}(t) - \mathbf{y}(s)||, \max_{s\in[c,d]}\min_{t\in[a,b]}||\mathbf{x}(t) - \mathbf{y}(s)||\},$$

[12, 7].

3. GEOMETRIC HERMITE INTERPOLATION OF CONIC SECTIONS BY HEXIC BÉZIER CURVES

In this section, we present the geometric Hermite interpolation of the conic section by hexic Bézier curves. Let $\mathbf{b} : [0,1] \to \mathbb{R}^2$ be the hexic Bézier curve approximating the conic section $\mathbf{c}$, represented by

$$\mathbf{b}(t) = \sum_{i=0}^{6} B_i^6(t)\mathbf{b}_i,$$

where $B_i^6(t) = \binom{6}{i}t^i(1-t)^{6-i}$, $i = 0, 1, \ldots, 6$, are the hexic Bernstein polynomials and $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_6$ are the control points of $\mathbf{b}$ in $\mathbb{R}^2$. The control points can be expressed in barycentric coordinates with respect to $\triangle\mathbf{c}_0\mathbf{c}_1\mathbf{c}_2$,

$$\mathbf{b}_i = \sum_{j=0}^{2} b_{i,j}\mathbf{c}_j \text{ with } b_{i,0} + b_{i,1} + b_{i,2} = 1,$$

for $i = 0, 1, \ldots, 6$. We consider the symmetry of $b_{i,j}$ with respect to $i = 3$ and $j = 1$, i.e., $b_{i,j} = b_{6-i,2-j}$, for $0 \le i \le 6$, $0 \le j \le 2$, since the symmetry of $b_{i,j}$ yields that when $\mathbf{c}$ is symmetric with respect to the line passing $\mathbf{c}_1$ and $\mathbf{m}$, so is its approximant $\mathbf{b}$. The curve $\mathbf{b}$ is G^1 Hermite interpolation of $\mathbf{c}$ if and only if $b_{0,1} = b_{0,2} = 0$, $b_{1,2} = 0$, $b_{1,0} > 0$. Then, all control points of $\mathbf{b}$ can be expressed by only four parameters $b_{1,0}$, $b_{2,0}$, $b_{2,2}$, $b_{3,0}$, as follows:

$$\begin{aligned}
&\mathbf{b}_0 = \mathbf{c}_0 \qquad \mathbf{b}_1 = b_{1,0}\mathbf{c}_0 + (1 - b_{1,0})\mathbf{c}_1\\
&\mathbf{b}_2 = b_{2,0}\mathbf{c}_0 + (1 - b_{2,0} - b_{2,2})\mathbf{c}_1 + b_{2,2}\mathbf{c}_2\\
&\mathbf{b}_3 = b_{3,0}\mathbf{c}_0 + (1 - 2b_{3,0})\mathbf{c}_1 + b_{3,0}\mathbf{c}_2\\
&\mathbf{b}_4 = b_{2,2}\mathbf{c}_0 + (1 - b_{2,0} - b_{2,2})\mathbf{c}_1 + b_{2,0}\mathbf{c}_2\\
&\mathbf{b}_5 = (1 - b_{1,0})\mathbf{c}_1 + b_{1,0}\mathbf{c}_2 \qquad \mathbf{b}_6 = \mathbf{c}_2.
\end{aligned} \tag{3.1}$$

The error function $f \circ \mathbf{b} : [0,1] \to \mathbb{R}$ is defined by

$$f(\mathbf{b}(t)) = \tau_1^2 - 4w^2\tau_0\tau_2,$$

where τ_0, τ_1, τ_2 are the barycentric coordinates of $\mathbf{b}(t)$, $t \in [0,1]$. The G^1 Hermite interpolation $\mathbf{b}$ of $\mathbf{c}$ implies that $f(\mathbf{b}(t))$ has a factor $t^2(1-t)^2$. Let $f_1(t) = f(\mathbf{b}(t))/t^2(1-t)^2$. Since

$$f_1(0) = 36 - 72b_{1,0} + 36b_{1,0}^2 - 60w^2 b_{2,2},$$

$\mathbf{b}$ is G^2 Hermite interpolation of $\mathbf{c}$ if and only if $f_1(0) = 0$ or equivalently

$$b_{2,2} = \frac{3}{5}\left(\frac{1-b_{1,0}}{w}\right)^2. \tag{3.2}$$

Then, $f_1(t)$ has a factor $t(1-t)$. Let $f_2(t) = f_1(t)/t(1-t)$. Since

$$f_2(0) = \frac{4\{27(b_{1,0}-1)^3 + 9(1-b_{1,0})(5-6b_{1,0}+6b_{1,0}^2-5b_{2,0})w^2 - 20b_{3,0}w^4\}}{w^2},$$

$\mathbf{b}$ is G^3 Hermite interpolation of $\mathbf{c}$ if and only if $f_2(0) = 0$ or equivalently

$$b_{3,0} = \frac{9}{20}\frac{3(b_{1,0}-1)^3 + (1-b_{1,0})(5-6b_{1,0}+6b_{1,0}^2-5b_{2,0})w^2}{w^4}. \tag{3.3}$$

Then, $f_2(t)$ also has a factor $t(1-t)$. Let $f_3(t) = f_2(t)/t(1-t)$. Now, since $f_3(0)$ is of degree at least two with respect to $b_{2,0}$ and $b_{1,0}$, their roots for $f_3(0) = 0$ are complicated. Thus we use the interpolation of the shoulder point instead of the endpoint. Since

$$\mathbf{b}(1/2) = \frac{\mathbf{m}+31\mathbf{c}}{32} + \frac{\mathbf{m}-\mathbf{c}_1}{32}\{6b_{1,0} + 15b_{2,0} + 20b_{3,0} + 15b_{2,2}\},$$

$\mathbf{b}$ interpolates the shoulder point if and only if $\mathbf{b}(1/2) = (\mathbf{m} + w\mathbf{c}_1)/(1+w)$, which is equivalent to

$$\begin{aligned} b_{2,0} \quad &= \quad \frac{9(1-b_{1,0})^3 + 3(b_{1,0}-1)(6-7b_{1,0}+2b_{1,0}^2)w^2 - 2b_{1,0}w^4}{5w^2(w^2+3b_{1,0}-3)} \\ &+ \quad \frac{w^2(31-w)}{15(w+1)(w^2+3b_{1,0}-3)}. \end{aligned} \tag{3.4}$$

By Eqs. (3.1)-(3.4), $\mathbf{b}$ has unique parameter $b_{1,0}$. Let $b = 1 - b_{1,0}$ for convenience. Now, we construct the approximant $\mathbf{b}$ by determining the parameter b. Since $f(\mathbf{b}(t))$ is symmetric with respect to $t = 1/2$ and has a zero at $t = 1/2$, $f(\mathbf{b}(t))$ has double zeros at $t = 1/2$ and so is $f_2(t)$. Let $f_3(t) = f_2(t)/(t-1/2)^2$. Then f_3 is a quadratic polynomial and

$$f_3(t) = \frac{16(w-1)f_4(t)}{w^2(w+1)^2(w^2-3b)^2},$$

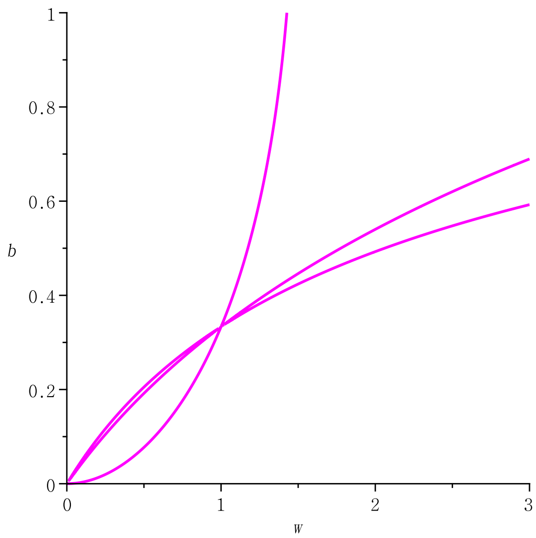


FIGURE 1. Graphs of implicit curves of the equation $g_1(b) = 0$ in the wb-plane. All curves pass through the point $(w, b) = (1, \frac{1}{3})$.

where $f_4(t) = g_0(b)g_1(b) - 4w^2(w+1)(w-3b)^2 g_2(b)^2(t-1/2)^2$ and

$$\begin{aligned} g_0(b) &= w^2(5-2w+w^2) - 12w(w+1)b + 9(w+1)^2b^2, \\ g_1(b) &= 16w^5 - 24w^3(w+2)(w+1)b + 9w^2(w+1)(w^2+4w+19)b^2 \\ &\quad -108w(w+1)^2b^3 + 81(w-1)(w+1)^2b^4, \\ g_2(b) &= -w(3w-5) + 3(w+1)(w-3)b + 9(w+1)b^2. \end{aligned} \tag{3.5}$$

The quadratic polynomial g_0 with respect to b has only two imaginary zeros

$$b = \frac{w}{3}\frac{2 \pm (1-w)\mathrm{i}}{1+w}$$

and no real zero. Thus if b is a zero of quartic polynomial g_1, then $f(\mathbf{b}(t))$ has twelve zeros in the t-interval $[0, 1]$, and $\mathbf{b}$ has twelve contacts with the conic section $\mathbf{c}$.

Proposition 3.1. *For any conic section* $\mathbf{c}$ *with the weight* $w \in (0, w_0)$*, there is at least one* G^3 *Hermite interpolation of* $\mathbf{c}$ *by the hexic Bézier curve having twelve contacts with* $\mathbf{c}$*, where* $w_0 \approx 1.427$ *is the zero of cubic polynomial* $h_1(w) = w^3 - 21w^2 + 9w + 27$ *contained in the open interval* $(7-\sqrt{46}, 7+\sqrt{46})$.

Proof. For all $w > 0$,

$$g_1(0) = 16w^5 > 0, \qquad g_1(1) = h_0(w)h_1(w),$$

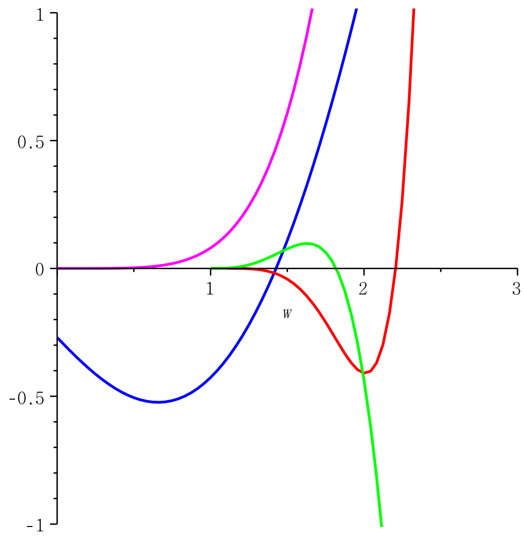


FIGURE 2. Graphs of functions $g_1(0)/200$ (magenta color), $g_1(1)/300$ (blue), $g_1(\frac{w-1}{5}+\frac{1}{3})$ (red), $g_1(\frac{w-1}{6}+\frac{1}{3})$ (green), on the w-interval $[0,3]$.

where $h_0(w) = w^2 - 6w - 3$. Since $h_1'(w)$ has two extremum at $w = w_j$, where $w_j = 7 + (-1)^j\sqrt{46}$, $j = 1, 2$, and

$$h_1(w_1) > 0 > h_1(w_2),$$

h_1 has a unique zero w_0 in the open interval (w_1, w_2). It follows from $h_0(w) < 0$ and $h_1(w) > 0$ for all $w \in (0, w_0)$ that $g_1(1) < 0$. By the intermediate value theorem, for all $w \in (0, w_0)$, g_1 has at least one zero in the b-interval $(0, 1)$, and thus there is at least one G^3 Hermite interpolation of $\mathbf{c}$ by the hexic Bézier curve having twelve contacts with $\mathbf{c}$. □

Even if Proposition 3.1 is proved for $w \in (0, w_0)$, we can extend the interval to $(0, 3]$. For all $w \in (0, 3]$, $g_1(0)$ is positive and at least one of three values $g_1(1)$, $g_1(\frac{w-1}{5}+\frac{1}{3})$, $g_1(\frac{w-1}{6}+\frac{1}{3})$ is negative, as shown in Fig. 2. Thus g_1 has at least one zero in the b-interval $(0, 1)$ for all $w \in (0, 3]$ by the intermediate value theorem. We omit here the rather complicated proof.

Since $b_{1,0} > 0$ is equivalent to $b < 1$, the hexic Bézier curve $\mathbf{b}$ with $b < 1$ satisfying $g_1(b) = 0$, is the G^3 Hermite interpolation of $\mathbf{c}$ and has twelve contacts with $\mathbf{c}$. The error bound analysis for the G^3 Hermite interpolation $\mathbf{b}$ of $\mathbf{c}$ is as follows.

Proposition 3.2. *If the control points of hexic Bézier curve* $\mathbf{b}$ *with* b *satisfying* $g_1(b) = 0$ *are contained in* $\triangle \mathbf{c}_0\mathbf{c}_1\mathbf{c}_2$, *then*

$$d_H(\mathbf{c}, \mathbf{b}) \le \max(\frac{1}{w^2}, 1)\frac{g_3(b)^2}{2^4 \cdot 3^6}\frac{|w-1|}{w+1}|\mathbf{c}_0 - 2\mathbf{c}_1 + \mathbf{c}_2|,$$

where $g_3(b) = g_2(b)(w - 3b)/(w^2 - 3b)$.

Proof. If all control points are contained in $\triangle \mathbf{c}_0\mathbf{c}_1\mathbf{c}_2$, then so is $\mathbf{b}$ by the convex hull property. Thus we have

$$d_H(\mathbf{c}, \mathbf{b}) \le \frac{1}{4}\max(\frac{1}{w^2}, 1)\max_{t\in[0,1]}|f(\mathbf{b}(t))||\mathbf{c}_0 - 2\mathbf{c}_1 + \mathbf{c}_2|,$$

from Lemma 3.2 in [12]. If b is the zero of g_1 in Eq. (3.5), then

$$f(\mathbf{b}(t)) = \frac{64(w-1)(w-3b)^2 g_2(b)^2}{(w+1)(w^2-3b)^2} t^4(t-1/2)^4(1-t)^4,$$

which has the extremum at $t = 1/2 \pm \sqrt{3}/6$. It follows from

$$\max_{t\in[0,1]}|f(\mathbf{b}(t))| = \frac{g_2(b)^2}{2^2\cdot 3^6}\frac{|w-1|(w-3b)^2}{(w+1)(w^2-3b)^2}$$

that

$$d_H(\mathbf{c}, \mathbf{b}) \le \max(\frac{1}{w^2}, 1)\frac{g_3(b)^2}{2^4\cdot 3^6}\frac{|w-1|}{w+1}|\mathbf{c}_0 - 2\mathbf{c}_1 + \mathbf{c}_2|.$$

□

For the given conic section, our approximation curve depends only on b which are zeros of g_1. Since g_1 is a quartic polynomial, it is solvable algebraically. As stated in Proposition 3.2, the upper bound depends on the value $|g_3(b)|$. The best approximation from the four solutions of $g_1(b) = 0$ can be obtained when $|g_3(b)|$ is minimized.

The asymptotic analysis of our method is as follows. If the length h of the conic section goes to zero, then $w - 1 = \mathcal{O}(h^2)$ [12], and the quartic polynomial g_1 has three analytic zeros and one divergent zero, since

$$\alpha_3 = -432 + \mathcal{O}(h^2) \text{ and } \alpha_4 = \mathcal{O}(h^2),$$

where α_3, α_4 are the coefficients of the third, fourth order term of the quartic polynomial g_1, respectively. The three analytic solutions of $g_1(b) = 0$ are

$$\frac{1}{3} + \frac{w-1}{4} + \mathcal{O}((w-1)^2), \quad \frac{1}{3} + \frac{(3\pm\sqrt{3})}{6}(w-1) + \mathcal{O}((w-1)^2),$$

and their asymptotic behaviors of $g_3(b)$ are

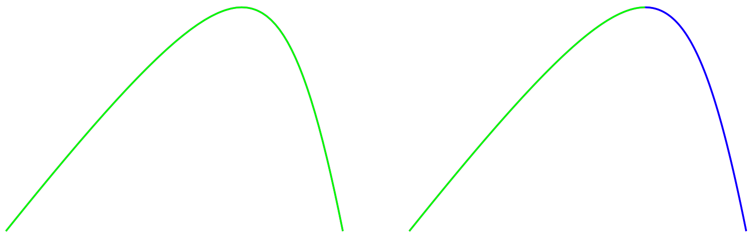
$$\frac{1}{8}(w-1)^2 + \mathcal{O}((w-1)^3), \quad (26 \pm 15\sqrt{3})^2(w-1)^2 + \mathcal{O}((w-1)^3), \tag{3.6}$$

respectively.

Proposition 3.3. *The approximation order of the approximant* $\mathbf{b}$ *with any analytic zero* b *of* g_1 *is twelve.*

Proof. If b is any of three analytic zeros of g_1, then

$$b_{1,0} = \frac{2}{3} + \mathcal{O}(h^2),\ b_{2,0} = \frac{2}{5} + \mathcal{O}(h^2),\ b_{3,0} = \frac{1}{5} + \mathcal{O}(h^2),\ b_{2,2} = \frac{1}{15} + \mathcal{O}(h^2)$$



(a) (b)

FIGURE 3. (a) Conic section $\mathbf{c}$ (magenta color) with the control points $(0, 0)$, $(8, 10)$, $(10, 0)$ and the weight $w = 2$, and the hexic Bézier approximant $\mathbf{b}$ (green). (b) After the shoulder point subdivision, $\mathbf{c}$ (magenta) and two hexic Bézier approximations $\mathbf{b}^1$ (green) and $\mathbf{b}^2$ (blue).

and thus all control points of $\mathbf{b}$ are contained in the triangle $\triangle \mathbf{c}_0\mathbf{c}_1\mathbf{c}_2$ for sufficiently small $h > 0$. By Proposition 3.2, we have

$$d_H(\mathbf{c}, \mathbf{b}) \leq \max(\frac{1}{w^2}, 1)\frac{g_3(b)^2}{2^4 \cdot 3^6}\frac{|w-1|}{w+1}|\mathbf{c}_0 - 2\mathbf{c}_1 + \mathbf{c}_2|.$$

Since the quantities $w - 1$ and $|\mathbf{c}_0 - 2\mathbf{c}_1 + \mathbf{c}_2|$ are $\mathcal{O}(h^2)$ [12], and by Eq. (3.6),

$$g_3(b)^2 = \mathcal{O}(h^8),$$

the approximation order is twelve. □

4. NUMERICAL EXAMPLES

We present two numerical examples in this section. The first example is hyperbola approximation by hexic Bézier curves. The hyperbola segment $\mathbf{c}$ in quadratic rational Bézier form has the control points $(0, 0)$, $(8, 10)$, $(10, 0)$ and the weight $w = 2$, as shown in Fig. 3. The quartic polynomial g_1 in Eq. (3.5) is solvable explicitly and has two real zeros $b \approx 0.493, 0.540$, and we have

$$g_3(0.493) \approx 2.510 \times 10^{-2}, \quad g_3(0.540) \approx 0.162.$$

Thus our method yields the hexic Bézier approximation $\mathbf{b}$ with $b \approx 0.493$ and by Proposition 2, the upper bound is

$$d_H(\mathbf{b}, \mathbf{c}) \leq 3.761 \times 10^{-7}.$$

If the upper bound is larger than the error tolerance, then a subdivision is required. Subdividing at the shoulder point, two subdivided conic segments $\mathbf{c}^1$ and $\mathbf{c}^2$ are obtained. They have the same weight $w = \sqrt{6}/2$ and the control points

$$(0,0), \left(\frac{16}{3}, \frac{20}{3}\right), \left(7, \frac{20}{3}\right), \text{ and } \left(7, \frac{20}{3}\right), \left(\frac{26}{3}, \frac{20}{3}\right), (10,0),$$

respectively. Two conic sections have the same polynomial g_1 in Eq. (3.5), which has four real zeros $b \approx 0.377, 0.387, 0.564, 5.937$, and their values of $g_3(b)$ are

$$1.068 \times 10^{-3}, 6.938 \times 10^{-3}, 3.188, 27.66$$

in order. Thus our approximation method yields two hexic Bézier curves $\mathbf{b}^1$ and $\mathbf{b}^2$ with $b \approx 0.377$ for approximating $\mathbf{c}^1$ and $\mathbf{c}^2$, respectively, as shown in Fig. 3(b), and the upper bounds are

$$d_H(\mathbf{b}^1, \mathbf{c}^1) \leq 7.520 \times 10^{-11} \text{ and } d_H(\mathbf{b}^2, \mathbf{c}^2) \leq 6.598 \times 10^{-11},$$

respectively. The upper bound of global error is 7.520×10^{-11}, and if it is larger than error tolerance, then subdivisions and approximations are repeated. The numerical approximation orders are 12.29 and 12.48, respectively, which can be computed by

$$\log_2 \left(\frac{d_H(\mathbf{b}, \mathbf{c})}{d_H(\mathbf{b}^j, \mathbf{c}^j)} \right),$$

$j = 1, 2$ [8, 20].

The second example is ellipse approximation by hexic Bézier curves. As shown in Fig. 4, the implicit equation of the ellipse is given by $(x/3)^2 + y^2 = 1$. The conic section $\mathbf{c}$ which is a quarter ellipse has the control points $(3,0)$, $(3,1)$, $(0,1)$ and the weight $w = \cos(\pi/4)$. The quartic equation $g_1(b) = 0$ has four real roots $b \approx 0.156, 0.255, 0.264, -3.894$ and their values of $g_3(b)$ are

$$4.276, 8.847 \times 10^{-3}, 1.363 \times 10^{-3}, 2.855 \times 10^{2}$$

in order. Thus our approximation method yields the hexic Bézier curve $\mathbf{b}$ with $b \approx 0.264$ and its upper bound is

$$d_H(\mathbf{b}, \mathbf{c}) \leq 1.728 \times 10^{-10}$$

Subdividing the conic section at the shoulder point, the subdivided conic sections $\mathbf{c}^1$ and $\mathbf{c}^2$ have the same weight $w = \sqrt{2 + \sqrt{2}}/2$ and the control points

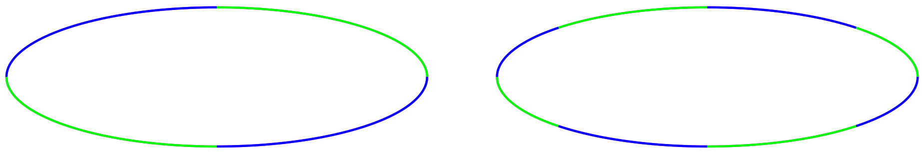
$$(3,0), (3, \sqrt{2} - 1), \frac{(3,1)}{\sqrt{2}} \text{ and } \frac{(3,1)}{\sqrt{2}}, (3\sqrt{2} - 3, 1), (0,1),$$

respectively. Two conic sections have the same polynomial g_1, which has four real zeros $b \approx 0.277, 0.313, 0.316, -17.09$ and their values of $g_3(b)$ are

$$0.292, 6.955 \times 10^{-4}, 1.070 \times 10^{-4}, 5.272 \times 10^{3},$$

in order. Thus our method yields two hexic Bézier curves $\mathbf{b}^1$ and $\mathbf{b}^2$ with $b \approx 0.316$ for approximating $\mathbf{c}^1$ and $\mathbf{c}^2$, respectively, as shown in Fig. 4(b), and the upper bounds of the approximation error for two subdivided conic sections are

$$d_H(\mathbf{c}^1, \mathbf{b}^1) \leq 4.039 \times 10^{-14} \text{ and } d_H(\mathbf{c}^2, \mathbf{b}^2) \leq 2.127 \times 10^{-14},$$



(a) (b)

FIGURE 4. (a) Ellipse $(x/3)^2 + y^2 = 1$ (magenta color) and four hexic Bézier approximants (green and blue). The consecutive approximating segments are plotted by different colors. (b) After the shoulder point subdivision, ellipse (magenta) and eight hexic Bézier approximants (green and blue).

respectively. The numerical approximation orders are 12.06 and 12.99, respectively. In case of the full ellipse approximation using four hexic Bézier segments the upper bound is 1.728×10^{-10}, as shown in Fig. 4(a), and using eight hexic Bézier segments the upper bound is 4.039×10^{-14}, as shown in Fig. 4(b).

5. CONCLUSION

In this paper we presented a method of conic section approximation by hexic Bézier curves. The hexic Bézier approximant is G^3 Hermite interpolation of the conic section. We showed that there exists at least one hexic Bézier approximant for each weight of the conic section. One of the merits of our method is that the hexic Bézier approximant can be obtained by solving a quartic polynomial, which can be solved algebraically. Our method yields the best approximation from the four approximation curves. We presented the explicit upper bound of the Hausdorff distance between the conic section and the hexic Bézier approximant. We also proved that our approximation method has the maximal order of approximation.

ACKNOWLEDGEMENT

This study was supported by research funds from Chosun University, 2023, and by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. NRF-2021R1F1A1045830). The authors are very grateful to two anonymous reviewers for their valuable comments and constructive suggestions.

REFERENCES

[1] K. Höllig and J. Koch. *Geometric Hermite interpolation with maximal order and smoothness*, Comput. Aided Geom. Design, **13(8)** (1996), 681–695.

[2] C. de Boor, K. Höllig, and M. Sabin. *High accuracy geometric Hermite interpolation*, Comput. Aided Geom. Design, **4** (1987), 269–278.

[3] K. Höllig and J. Koch. *Geometric Hermite interpolation* Comput. Aided Geom. Design, **12(6)** (1995), 567–580.

[4] T. Dokken, M. Dæhlen, T. Lyche, and K. Mørken. *Good approximation of circles by curvature-continuous Bézier curves*, Comput. Aided Geom. Design, **7** (1990), 33–41.

[5] K. Mørken. *Best approximation of circle segments by quadratic Bézier curves*, Curves and Surfaces, pages 387–396, Academic Press, 1991.

[6] M. Goldapp. *Approximation of circular arcs by cubic polynomials*, Comput. Aided Geom. Design, **8** (1991), 227–238.

[7] Y. J. Ahn and H. O. Kim. *Approximation of circular arcs by Bézier curves*, J. Comput. Appl. Math., **81** (1997), 145–163.

[8] L. Fang. *Circular arc approximation by quintic polynomial curves*, Comput. Aided Geom. Design, **15** (1998), 843–861.

[9] S. H. Kim and Y. J. Ahn. *Approximation of circular arcs by quartic Bézier curves*, Comput.-Aided Design, **39(6)** (2007), 490–493.

[10] G. Jaklič. *Uniform approximation of a circle by a parametric polynomial curve*, Comput. Aided Geom. Design, **41** (2016), 36–46.

[11] G. Jaklič and J. Kozak. *On parametric polynomial circle approximation*, Numerical Algorithms, **77** (2018), 433–450.

[12] M. Floater. *An $O(h^{2n})$ Hermite approximation for conic sections*, Comput. Aided Geom. Design, **14** (1997), 135–151.

[13] M. Floater. *High-order approximation of conic sections by quadratic splines*, Comput. Aided Geom. Design, **12(6)** (1995), 617–637.

[14] L. Fang. *G^3 approximation of conic sections by quintic polynomial*, Comput. Aided Geom. Design, **16** (1999), 755–766.

[15] Y. J. Ahn. *Approximation of conic sections by curvature continuous quartic Bézier curves*, Comput. Math. Appl., **60** (2010), 1986–1993.

[16] G. Jaklič, J. Kozak, M. Krajnc, V. Vitrih, and E. Žagar. *High-order parametric polynomial approximation of conic sections*, Constr. Approx., **38** (2013), 1–18.

[17] H. M. Yoon and Y. J. Ahn. *Circular arc approximation by hexic polynomial curves*, Comput. Appl. Math., **42** (2023).

[18] Y. J. Ahn. *Conic approximation of planar curves*, Comput.-Aided Design, **33** (2001), 867–872.

[19] I. Juhász. *Gardener's spline curve*, Annales Mathematicae et Informaticae, **47** (2017), 109–118.

[20] A. Sestini, L. Landolfi, and C. Manni. *On the approximation order of a space data-dependent PH quintic Hermite interpolation scheme*, Comput. Aided Geom. Design, **30** (2013), 148–158.