Honam Mathematical J. **46** (2024), No. 1, pp. 101–113
https://doi.org/10.5831/HMJ.2024.46.1.101

HOMOTHETIC MOTIONS WITH GENERALIZED TRICOMPLEX NUMBERS

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Abstract. In this paper, we define the generalized tricomplex numbers and give some algebraic properties of them. By using the matrix representation of generalized tricomplex numbers, we determine a motion on the hypersurface M in eight dimensional generalized linear space $\mathbb{R}^8_{\alpha\beta\gamma}$ and show that this is a homothetic motion. Also, for some special cases of the real numbers α, β and γ, we give some examples of homothetic motions in $\mathbb{R}^8$ and $\mathbb{R}^8_4$ and obtain some rotational matrices in these spaces, too.

1. Introduction

Corrado Segre was discovered multicomplex numbers in 1892 [14]. Let $\mathbb{C}_0$ be a real numbers and for every $n > 0$ let i_n be a imaginary number, that is, $i_n^2 = -1$. The multicomplex numbers denoted by $\mathbb{C}_{n+1}$ is given by:

$$\mathbb{C}_{n+1} = \left\{ z = x + i_{n+1}y : x,\ y \in \mathbb{C}_n \right\}.$$

In multicomplex numbers systems, different imaginer units are commutative, that is, $i_n i_m = i_m i_n$. For $n = 0$, $\mathbb{C}_1$ is the set of complex number, for $n = 1$, $\mathbb{C}_2$ is the set of bicomplex number, for $n = 2$, $\mathbb{C}_3$ is the set of tricomplex number and if it continues like this $\mathbb{C}_n$ is the set of multicomplex numbers order n.

Various studies have been done on bicomplex numbers and tricomplex numbers, which are a special case of multicomplex numbers. Price introduced the function theory of multicomplex numbers and gave some details about bicomplex numbers [13].

Also, the generalized bicomplex numbers and some algebraic properties of them were introduced in [10].

Number systems have a wide application area in motion geometry. Especially the relationship between number systems and homothetic motion was

Received May 27, 2023. Accepted July 16, 2023.
2020 Mathematics Subject Classification. 53A17, 53C50.
Key words and phrases. bicomplex numbers, tricomplex numbers, homothetic motion.
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first discussed by Yaylı in [15]. He proved that the motion described by matrix representations in terms of 4×4 of quaternions is a homothetic motion in $\mathbb{R}^4$ and then Yaylı and Bükcü defined homothetic motion in $\mathbb{R}^8$ with Cayley numbers (Octonions) [16]. Also, Jafari and Yaylı investigated homothetic motion with generalized quaternions [4]. Based on this idea, by means of bicomplex numbers the homothetic motions on a special hypersurfaces in $\mathbb{R}^4$ and $\mathbb{R}^4_2$ were defined in [5] and [2], respectively. And then Özkaldı Karakuş etc. by using generalized bicomplex numbers determined homothetic motions on some hypersurfaces in $\mathbb{R}^4_{\alpha\beta}$ [8] and this study is a generalization of the studies numbered by [5] and [2], too. In [1] Babadağ and others discribed a motion by using the matrix representation of tricomplex numbers and they showed that it is a homothetic motion in $\mathbb{R}^8$.

In this paper, we introduce the generalized tricomplex numbers and obtain their some algebraic properties. By means of the matrix representation of generalized tricomplex numbers, we determine a motion on the hypersurface M in eight dimensional generalized linear space $\mathbb{R}^8_{\alpha\beta\gamma}$ and show that it is a homothetic motion. Also, for some special cases of the real numbers α, β and γ, we obtain some examples of homothetic motions in $\mathbb{R}^8$ and $\mathbb{R}^8_4$.

2. Basic Concepts

In this section, some basic concepts which we need in the paper will be given.

2.1. Generalized Bicomplex Numbers

Generalized bicomplex numbers was introduced by Özkaldı Karakuş and Kahraman Aksoyak [10].

Any generalized bicomplex number x is as:

$$x = x_1 + x_2 i + x_3 j + x_4 ij,$$

such that $x_t \in \mathbb{R}$, for $1 \leq t \leq 4$ and imaginer units i and j hold $i^2 = -\alpha$, $j^2 = -\beta$, $ij = ji$ for $\alpha, \beta \in \mathbb{R}$. The set of generalized bicomplex numbers is showen by $\mathbb{C}_{\alpha\beta}$.

For x, $y \in \mathbb{C}_{\alpha\beta}$ the addition, multiplication, and scalar multiplication are given, respectively

$$x + y = (x_1 + y_1) + (x_2 + y_2)\, i + (x_3 + y_3)\, j + (x_4 + y_4)\, ij,$$

$$\begin{aligned} x \cdot y \;=\;& (x_1y_1 - \alpha x_2y_2 - \beta x_3y_3 + \alpha\beta x_4y_4) + (x_1y_2 + x_2y_1 - \beta x_3y_4 - \beta x_4y_3)\, i \\ & + (x_1y_3 + x_3y_1 - \alpha x_2y_4 - \alpha x_4y_2)\, j + (x_1y_4 + x_4y_1 + x_2y_3 + x_3y_2)\, ij, \end{aligned} \tag{1}$$

$$cx = cx_1 + cx_2 i + cx_3 j + cx_4 ij,\ c \in \mathbb{R}.$$

Hence, $\mathbb{C}_{\alpha\beta}$ is 4-dimensional vector space on $\mathbb{R}$ with respect to the addition and scalar multiplication are defined above and the base of $\mathbb{C}_{\alpha\beta}$ is $\{1, i, j, ij\}$. Also it has a commutative real algebra with generalized bicomplex number product given by (1).

Any generalized bicomplex number can be rewritten as $x = (x_1 + x_2 i) + (x_3 + x_4 i)\, j$. There are three different kinds of conjugations for generalized bicomplex numbers. They are given as follows:

$$\begin{aligned} x^{t_1} &= (x_1 - x_2 i) + (x_3 - x_4 i)\, j, \\ x^{t_2} &= (x_1 + x_2 i) - (x_3 + x_4 i)\, j, \\ x^{t_3} &= (x_1 - x_2 i) - (x_3 - x_4 i)\, j, \end{aligned}$$

where x^{t_1}, x^{t_2} and x^{t_3} are the conjugations of x according to i, j and both i and j, respectively. If we product x and its conjugation, we calculate following equalities.

$$\begin{aligned} x \cdot x^{t_1} &= \left(x_1^2 + \alpha x_2^2 - \beta x_3^2 - \alpha\beta x_4^2\right) + 2\left(x_1 x_3 + \alpha x_2 x_4\right) j, \\ x \cdot x^{t_2} &= \left(x_1^2 - \alpha x_2^2 + \beta x_3^2 - \alpha\beta x_4^2\right) + 2\left(x_1 x_2 + \beta x_3 x_4\right) i, \\ x \cdot x^{t_3} &= \left(x_1^2 + \alpha x_2^2 + \beta x_3^2 + \alpha\beta x_4^2\right) + 2\left(x_1 x_4 - x_2 x_3\right) ij. \end{aligned}$$

If we take as $\alpha = 1$ and $\beta = 1$, we get bicomplex numbers.

2.2. Generalized Tricomplex Numbers

Now, the generalized tricomplex numbers was introduced by [9].

The set defined as:

$$T\mathbb{C}_{\alpha\beta} = \{z = x + ky : x, y \in \mathbb{C}_{\alpha\beta},\ k^2 = -\gamma,\ \gamma \in \mathbb{R}\},$$

is called generalized tricomplex numbers set. If we take

$$x = z_1 + z_2 i + z_3 j + z_4 ij \quad \text{and} \quad y = z_5 + z_6 i + z_7 j + z_8 ij,$$

any generalized tricomplex number z is defined as follows:

$$\begin{aligned} z &= (z_1 + z_2 i + z_3 j + z_4 ij) + k\,(z_5 + z_6 i + z_7 j + z_8 ij), \\ &= z_1 + z_2 i + z_3 j + z_4 ij + z_5 k + z_6 ik + z_7 jk + z_8 ijk. \end{aligned}$$

Specially here, if we take as $\alpha = \beta = \gamma = 1$, tricomplex numbers are obtained [1].

Let x_1, y_1, x_2, y_2 be generalized bicomplex numbers and $k^2 = -\gamma$, $k \in \mathbb{R}$. Addition of any two generalized tricomplex numbers $z = x_1 + ky_1$ and $w = x_2 + ky_2$ is as follows:

$$z + w = x_1 + x_2 + k\,(y_1 + y_2)\,.$$

The generalized tricomplex numbers set is closed according to the addition. That is, the sum of the two generalized tricomplex numbers is again a generalized tricomplex number. $(T\mathbb{C}_{\alpha\beta}, +)$ is an Abel group and identity element is $(0, 0, 0, 0, 0, 0, 0, 0)$.

The scalar multiplication of z in $T\mathbb{C}_{\alpha\beta}$ by a real number λ is defined as:

$$\lambda z = \lambda x + k\lambda y \in T\mathbb{C}_{\alpha\beta}.$$

The set of $T\mathbb{C}_{\alpha\beta}$ specifies $8-$dimensional vector space on $\mathbb{R}$ object based on addition and scalar multiplication. Also a base of $T\mathbb{C}_{\alpha\beta}$ is $\{1, i, j, ij, k, ik, jk, ijk\}$.

The product of any two generalized tricomplex numbers $z = x_1 + ky_1$ and $w = x_2 + ky_2$ is following:

$$\begin{aligned} zw &= (x_1 + ky_1)(x_2 + ky_2), \\ &= (x_1x_2 - \gamma y_1y_2) + k(x_1y_2 + x_2y_1). \end{aligned} \tag{2}$$

The Hamilton operator is isomorphic by multiplication in generalized tricomplex numbers as shown in the generalized bicomplex numbers. To show this we define a linear transformation as:

$$T : T\mathbb{C}_{\alpha\beta} \to T\mathbb{C}_{\alpha\beta}$$

$$z \to T(z) = T_z : T\mathbb{C}_{\alpha\beta} \to T\mathbb{C}_{\alpha\beta}$$

$$w \to T_z(w) = zw.$$

Using this linear transformation, the matrix representation T_z of generalized tricomplex number $z = z_1 + z_2i + z_3j + z_4ij + z_5k + z_6ik + z_7jk + z_8ijk$ based on the basis $\{1, i, j, ij, k, ik, jk, ijk\}$ on the real number set is obtained as:

$$T_z = \begin{pmatrix} z_1 & -\alpha z_2 & -\beta z_3 & \alpha\beta z_4 & -\gamma z_5 & \alpha\gamma z_6 & \beta\gamma z_7 & -\alpha\beta\gamma z_8 \\ z_2 & z_1 & -\beta z_4 & -\beta z_3 & -\gamma z_6 & -\gamma z_5 & \beta\gamma z_8 & \beta\gamma z_7 \\ z_3 & -\alpha z_4 & z_1 & -\alpha z_2 & -\gamma z_7 & \alpha\gamma z_8 & -\gamma z_5 & \alpha\gamma z_6 \\ z_4 & z_3 & z_2 & z_1 & -\gamma z_8 & -\gamma z_7 & -\gamma z_6 & -\gamma z_5 \\ z_5 & -\alpha z_6 & -\beta z_7 & \alpha\beta z_8 & z_1 & -\alpha z_2 & -\beta z_3 & \alpha\beta z_4 \\ z_6 & z_5 & -\beta z_8 & -\beta z_7 & z_2 & z_1 & -\beta z_4 & -\beta z_3 \\ z_7 & -\alpha z_8 & z_5 & -\alpha z_6 & z_3 & -\alpha z_4 & z_1 & -\alpha z_2 \\ z_8 & z_7 & z_6 & z_5 & z_4 & z_3 & z_2 & z_1 \end{pmatrix}. \tag{3}$$

By using (3), we can express the generalized tricomplex numbers product as follows:

$$zw = \begin{pmatrix} z_1 & -\alpha z_2 & -\beta z_3 & \alpha\beta z_4 & -\gamma z_5 & \alpha\gamma z_6 & \beta\gamma z_7 & -\alpha\beta\gamma z_8 \\ z_2 & z_1 & -\beta z_4 & -\beta z_3 & -\gamma z_6 & -\gamma z_5 & \beta\gamma z_8 & \beta\gamma z_7 \\ z_3 & -\alpha z_4 & z_1 & -\alpha z_2 & -\gamma z_7 & \alpha\gamma z_8 & -\gamma z_5 & \alpha\gamma z_6 \\ z_4 & z_3 & z_2 & z_1 & -\gamma z_8 & -\gamma z_7 & -\gamma z_6 & -\gamma z_5 \\ z_5 & -\alpha z_6 & -\beta z_7 & \alpha\beta z_8 & z_1 & -\alpha z_2 & -\beta z_3 & \alpha\beta z_4 \\ z_6 & z_5 & -\beta z_8 & -\beta z_7 & z_2 & z_1 & -\beta z_4 & -\beta z_3 \\ z_7 & -\alpha z_8 & z_5 & -\alpha z_6 & z_3 & -\alpha z_4 & z_1 & -\alpha z_2 \\ z_8 & z_7 & z_6 & z_5 & z_4 & z_3 & z_2 & z_1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \\ w_8 \end{pmatrix}.$$

If generalized tricomplex number z is written as $z = x_1 + ky_1$ depending on base $\{1, k\}$, in that case the matrix notation of z is of type 2×2 as:

$$T_z = \begin{pmatrix} x_1 & -\gamma y_1 \\ y_1 & x_1 \end{pmatrix}.$$

The generalized tricomplex number product which is given by (2) can be expressed by following matrix product, too, that is,

$$zw = \begin{pmatrix} x_1 & -\gamma y_1 \\ y_1 & x_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

Let $x = z_1 + z_2 i + z_3 j + z_4 ij$ and $y = z_5 + z_6 i + z_7 j + z_8 ij$ be generalized bicomplex numbers. The conjugation of generalized tricomplex number $z = x + ky$ is defined by

$$\begin{aligned} z^t &= (x + ky)^{t_3}, \\ &= x^{t_3} - ky^{t_3}, \\ &= [(z_1 - z_2 i) - (z_3 - z_4 i)\, j] - [(z_5 - z_6 i) - (z_7 - z_8 i)\, j]\, k, \\ &= z_1 - z_2 i - z_3 j + z_4 ij - z_5 k + z_6 ik + z_7 jk - z_8 ijk, \end{aligned}$$

where x^{t_3} and y^{t_3} are the conjugations of x and y according to both i and j in generalized bicomplex numbers, respectively. So that, we can calculate the product of z and the conjugation of z as:

$$\begin{aligned} zz^t = \; & z_1^2 + \alpha z_2^2 + \beta z_3^2 + \alpha\beta z_4^2 + \gamma z_5^2 + \alpha\gamma z_6^2 + \beta\gamma z_7^2 + \alpha\beta\gamma z_8^2 \\ & + 2ij\,(z_1 z_4 - z_2 z_3 + \gamma z_5 z_8 - \gamma z_6 z_7) \\ & + 2ik\,(z_1 z_6 + \beta z_3 z_8 - z_2 z_5 - \beta z_4 z_7) \\ & + 2jk\,(z_1 z_7 + \alpha z_2 z_8 - z_3 z_5 - \alpha z_4 z_6)\,. \end{aligned}$$

In particular, if $\alpha = \beta = \gamma = 1$, we obtain the following equation which is given by Babadağ in 2009 [1].

$$\begin{aligned} zz^t = \; & z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^2 + z_7^2 + z_8^2 \\ & + 2ij\,(z_1 z_4 - z_2 z_3 + z_5 z_8 - z_6 z_7) \\ & + 2ik\,(z_1 z_6 + z_3 z_8 - z_2 z_5 - z_4 z_7) \\ & + 2jk\,(z_1 z_7 + z_2 z_8 - z_3 z_5 - z_4 z_6)\,. \end{aligned}$$

So, we can say that the algebraic properties of generalized tricomplex number include the algebraic properties of tricomplex number.

3. Homothetic Motions via Generalized Tricomplex Numbers

Now, we determine the homothetic motion on a hypersurface M at $\mathbb{R}^8_{\alpha\beta\gamma}$ with the help of generalized tricomplex numbers.

Let $z = (z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) \in \mathbb{R}^8_{\alpha\beta\gamma}$, for

$$M_1 = \left\{ \begin{array}{c} z = (z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) \in \mathbb{R}^8_{\alpha\beta\gamma} : \\ z_1 z_7 + \alpha z_2 z_8 - z_3 z_5 - \alpha z_4 z_6 = 0, \;\; z \neq 0 \end{array} \right\} \subset \mathbb{R}^8_{\alpha\beta\gamma},$$

$$M_2 = \left\{ \begin{array}{c} z = (z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) \in \mathbb{R}^8_{\alpha\beta\gamma} : \\ z_1 z_4 + \gamma z_5 z_8 - z_2 z_3 - \gamma z_6 z_7 = 0, \;\; z \neq 0 \end{array} \right\} \subset \mathbb{R}^8_{\alpha\beta\gamma},$$

$$M_3 = \left\{ \begin{array}{c} z = (z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) \in \mathbb{R}^8_{\alpha\beta\gamma} : \\ z_1 z_6 + \beta z_3 z_8 - z_2 z_5 - \beta z_4 z_7 = 0, \ z \neq 0 \end{array} \right\} \subset \mathbb{R}^8_{\alpha\beta\gamma}.$$

$M = M_1 \cap M_2 \cap M_3$ be a hypersurface in $\mathbb{R}^8_{\alpha\beta\gamma}$. Then the norm of generalized tricomplex number z on the hypersurface M is defined by

$$\begin{aligned} \|z\| &= \sqrt{|g(z,z)|}, \\ &= \sqrt{|zz^t|}, \\ &= \sqrt{|z_1^2 + \alpha z_2^2 + \beta z_3^2 + \alpha\beta z_4^2 + \gamma z_5^2 + \alpha\gamma z_6^2 + \beta\gamma z_7^2 + \alpha\beta\gamma z_8^2|}. \end{aligned}$$

In that case, a unit sphere in $\mathbb{R}^8_{\alpha\beta\gamma}$ is given by

$$S^7_{\alpha\beta\gamma} = \left\{ \begin{array}{c} (z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) \in \mathbb{R}^8_{\alpha\beta\gamma} : \\ z_1^2 + \alpha z_2^2 + \beta z_3^2 + \alpha\beta z_4^2 + \gamma z_5^2 + \alpha\gamma z_6^2 + \beta\gamma z_7^2 + \alpha\beta\gamma z_8^2 = 1 \end{array} \right\}.$$

Let us consider the following curve

$$\eta : I \subset \mathbb{R} \to M \subset \mathbb{R}^8_{\alpha\beta\gamma},$$

$$s \to \eta(s) = (\eta_1(s), \eta_2(s), \eta_3(s), \eta_4(s), \eta_5(s), \eta_6(s), \eta_7(s), \eta_8(s)),$$

for every $s \in I$. We suppose that the curve $\eta(s)$ is smooth regular curve of order r. By using (3), the matrix representation of the curve $\eta \in \mathbb{R}^8_{\alpha\beta\gamma}$ is given by

$$(4) \quad B = \begin{pmatrix} \eta_1 & -\alpha\eta_2 & -\beta\eta_3 & \alpha\beta\eta_4 & -\gamma\eta_5 & \alpha\gamma\eta_6 & \beta\gamma\eta_7 & -\alpha\beta\gamma\eta_8 \\ \eta_2 & \eta_1 & -\beta\eta_4 & -\beta\eta_3 & -\gamma\eta_6 & -\gamma\eta_5 & \beta\gamma\eta_8 & \beta\gamma\eta_7 \\ \eta_3 & -\alpha\eta_4 & \eta_1 & -\alpha\eta_2 & -\gamma\eta_7 & \alpha\gamma\eta_8 & -\gamma\eta_5 & \alpha\gamma\eta_6 \\ \eta_4 & \eta_3 & \eta_2 & \eta_1 & -\gamma\eta_8 & -\gamma\eta_7 & -\gamma\eta_6 & -\gamma\eta_5 \\ \eta_5 & -\alpha\eta_6 & -\beta\eta_7 & \alpha\beta\eta_8 & \eta_1 & -\alpha\eta_2 & -\beta\eta_3 & \alpha\beta\eta_4 \\ \eta_6 & \eta_5 & -\beta\eta_8 & -\beta\eta_7 & \eta_2 & \eta_1 & -\beta\eta_4 & -\beta\eta_3 \\ \eta_7 & -\alpha\eta_8 & \eta_5 & -\alpha\eta_6 & \eta_3 & -\alpha\eta_4 & \eta_1 & -\alpha\eta_2 \\ \eta_8 & \eta_7 & \eta_6 & \eta_5 & \eta_4 & \eta_3 & \eta_2 & \eta_1 \end{pmatrix}.$$

Now we will describe the one parameter motion on hypersurface M at $\mathbb{R}^8_{\alpha\beta\gamma}$ by means of the matrix representation of the curve η given by (4).

Definition 3.1. *Let B be the matrix representation of the curve $\eta(s)$ on M and C be the 8×1 real matrix depends on a real parameter s at $\mathbb{R}^8_{\alpha\beta\gamma}$. Then the one-parameter motion on M is defined by*

$$\begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} B & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix},$$

or it can be expressed as

$$(5) \qquad Y = BX + C.$$

By differentiating of (5) with respect to s, we get following equality

$$\dot{Y} = \dot{B}X + \dot{C} + B\dot{X},$$

where $\dot{Y}$, $\dot{B}X+\dot{C}$ and $B\dot{X}$ are the absolute velocity, the sliding velocity and the relative velocity of the point X, respectively. When the sliding velocity is equal to zero for all s, we find the pole points of the motion. That is, we find the pole points of the motion by the solition of the equation (6)

$$\dot{B}X+\dot{C}=0. \tag{6}$$

See for more details [3].

Theorem 3.2. *The equation (5) is a homothetic motion on M.*

Proof. Let the curve η be on M. In that case it does not pass through the origin. So the matrix given by (4) can be expressed as:

$$B=h\begin{bmatrix} \frac{\eta_1}{h} & \frac{-\alpha\eta_2}{h} & \frac{-\beta\eta_3}{h} & \frac{\alpha\beta\eta_4}{h} & \frac{-\gamma\eta_5}{h} & \frac{\alpha\gamma\eta_6}{h} & \frac{\beta\gamma\eta_7}{h} & \frac{-\alpha\beta\gamma\eta_8}{h} \\ \frac{\eta_2}{h} & \frac{\eta_1}{h} & \frac{-\beta\eta_4}{h} & \frac{-\beta\eta_3}{h} & \frac{-\gamma\eta_6}{h} & \frac{-\gamma\eta_5}{h} & \frac{\beta\gamma\eta_8}{h} & \frac{\beta\gamma\eta_7}{h} \\ \frac{\eta_3}{h} & \frac{-\alpha\eta_4}{h} & \frac{\eta_1}{h} & \frac{-\alpha\eta_2}{h} & \frac{-\gamma\eta_7}{h} & \frac{\alpha\gamma\eta_8}{h} & \frac{-\gamma\eta_5}{h} & \frac{\alpha\gamma\eta_6}{h} \\ \frac{\eta_4}{h} & \frac{\eta_3}{h} & \frac{\eta_2}{h} & \frac{\eta_1}{h} & \frac{-\gamma\eta_8}{h} & \frac{-\gamma\eta_7}{h} & \frac{-\gamma\eta_6}{h} & \frac{-\gamma\eta_5}{h} \\ \frac{\eta_5}{h} & \frac{-\alpha\eta_6}{h} & \frac{-\beta\eta_7}{h} & \frac{\alpha\beta\eta_8}{h} & \frac{\eta_1}{h} & \frac{-\alpha\eta_2}{h} & \frac{-\beta\eta_3}{h} & \frac{\alpha\beta\eta_4}{h} \\ \frac{\eta_6}{h} & \frac{\eta_5}{h} & \frac{-\beta\eta_8}{h} & \frac{-\beta\eta_7}{h} & \frac{\eta_2}{h} & \frac{\eta_1}{h} & \frac{-\beta\eta_4}{h} & \frac{-\beta\eta_3}{h} \\ \frac{\eta_7}{h} & \frac{-\alpha\eta_8}{h} & \frac{\eta_5}{h} & \frac{-\alpha\eta_6}{h} & \frac{\eta_3}{h} & \frac{-\alpha\eta_4}{h} & \frac{\eta_1}{h} & \frac{-\alpha\eta_2}{h} \\ \frac{\eta_8}{h} & \frac{\eta_7}{h} & \frac{\eta_6}{h} & \frac{\eta_5}{h} & \frac{\eta_4}{h} & \frac{\eta_3}{h} & \frac{\eta_2}{h} & \frac{\eta_1}{h} \end{bmatrix}=hA, \tag{7}$$

where

$$\begin{aligned} h &: I\subset\mathbb{R}\to\mathbb{R} \\ s &\to h(s)=\|\eta(s)\|=\sqrt{\eta_1^2+\alpha\eta_2^2+\beta\eta_3^2+\alpha\beta\eta_4^2+\gamma\eta_5^2+\alpha\gamma\eta_6^2+\beta\gamma\eta_7^2+\alpha\beta\gamma\eta_8^2}\neq 0. \end{aligned}$$

Since $\eta(s)\in M$, it satisfies

$$\begin{aligned} \eta_1\eta_7+\alpha\eta_2\eta_8-\eta_3\eta_5-\alpha\eta_4\eta_6&=0, \\ \eta_1\eta_4+\gamma\eta_5\eta_8-\eta_2\eta_3-\gamma\eta_6\eta_7&=0, \\ \eta_1\eta_6+\beta\eta_3\eta_8-\eta_2\eta_5-\beta\eta_4\eta_7&=0. \end{aligned}$$

By using these equalities, we see that the matrix A in (7) is a semi-orthogonal matrix. Thus it holds

$$A^T\varepsilon A=\varepsilon \text{ and } \det A=1,$$

in here ε is the signature matrix associated with the metric g and it is as:

$$\varepsilon=\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&\alpha&0&0&0&0&0&0\\ 0&0&\beta&0&0&0&0&0\\ 0&0&0&\alpha\beta&0&0&0&0\\ 0&0&0&0&\gamma&0&0&0\\ 0&0&0&0&0&\alpha\gamma&0&0\\ 0&0&0&0&0&0&\beta\gamma&0\\ 0&0&0&0&0&0&0&\alpha\beta\gamma \end{pmatrix}.$$

Hence A is a semi-orthogonal matrix, h is the homothetic scale and C is the translation matrix. Thus the equation (5) becomes a homothetic motion. □

Remark 3.3. *In this paper, we suppose that he norm of the curve $\eta \in \mathbb{R}^8_{\alpha\beta\gamma}$ is positive, that is, $\eta_1^2 + \alpha\eta_2^2 + \beta\eta_3^2 + \alpha\beta\eta_4^2 + \gamma\eta_5^2 + \alpha\gamma\eta_6^2 + \beta\gamma\eta_7^2 + \alpha\beta\gamma\eta_8^2 > 0$.*

Corollary 3.4. *Let $\eta(s)$ be a curve on $S^7_{\alpha\beta\gamma} \cap M$. In that case one-parameter motion on M defined by (5) is a general motion forms of a rotation and a translation.*

Proof. Let $\eta(s)$ be a curve lying on both $S^7_{\alpha\beta\gamma}$ and M. So we have

$$\eta_1^2 + \alpha\eta_2^2 + \beta\eta_3^2 + \alpha\beta\eta_4^2 + \gamma\eta_5^2 + \alpha\gamma\eta_6^2 + \beta\gamma\eta_7^2 + \alpha\beta\gamma\eta_8^2 = 1.$$

Then the matrix B given by (4) determines a semi orthogonal matrix. So the motion defined by (5) becomes a general motion. □

Theorem 3.5. *Let $\eta(s)$ be a unit speed curve and $\dot{\eta}(s)$ be on M, then $\dot{B}$ is a semi-orthogonal matrix in $\mathbb{R}^8_{\alpha\beta\gamma}$.*

Proof. Since η is a unit speed curve

$$\dot{\eta}_1^2 + \alpha\dot{\eta}_2^2 + \beta\dot{\eta}_3^2 + \alpha\beta\dot{\eta}_4^2 + \gamma\dot{\eta}_5^2 + \alpha\gamma\dot{\eta}_6^2 + \beta\gamma\dot{\eta}_7^2 + \alpha\beta\gamma\dot{\eta}_8^2 = 1,$$

and $\dot{\eta}(s) \in M$, it occurs

$$\begin{aligned}
\dot{\eta}_1\dot{\eta}_7 + \alpha\dot{\eta}_2\dot{\eta}_8 - \dot{\eta}_3\dot{\eta}_5 - \alpha\dot{\eta}_4\dot{\eta}_6 &= 0,\\
\dot{\eta}_1\dot{\eta}_4 + \gamma\dot{\eta}_5\dot{\eta}_8 - \dot{\eta}_2\dot{\eta}_3 - \gamma\dot{\eta}_6\dot{\eta}_7 &= 0,\\
\dot{\eta}_1\dot{\eta}_6 + \beta\dot{\eta}_3\dot{\eta}_8 - \dot{\eta}_2\dot{\eta}_5 - \beta\dot{\eta}_4\dot{\eta}_7 &= 0.
\end{aligned}$$

Then the matrix $\dot{B}$ holds $\dot{B}^T\varepsilon\dot{B} = \varepsilon$ and $\det\dot{B} = 1$. So it becomes a semi orthogonal matrix in $\mathbb{R}^8_{\alpha\beta\gamma}$. □

Theorem 3.6. *If the curve η is a unit velocity curve and $\dot{\eta}(s) \in M$, then the motion defined by the matrix $\dot{B}$ is a regular motion, and it does not depend on h.*

Proof. From Theorem (3.5), we know that $\dot{B}$ is a semi-orthogonal matrix in $\mathbb{R}^8_{\alpha\beta\gamma}$. So the motion determined by the matrix $\dot{B}$ becomes a regular motion. Since $\det\dot{B} = 1$, it does not depend on h. □

Theorem 3.7. *Let the curve η be a unit speed curve on M whose the tangent vector $\dot{\eta}(s)$ are on M. Then the pole point of the motion defined by (5) is $X = -\dot{B}^{-1}C$.*

Proof. If the curve η is on M, from Theorem (3.2), we know that the equation (5) is a homothetic motion. Also, since the curve η is a unit speed curve and its tangent vector belongs to M, from Theorem (3.5) $\det\dot{B} = 1$ and it means that there is inverse of the matrix $\dot{B}$ and only one solution of the equation (6). Then the pole point of the motion is found as $X = -\dot{B}^{-1}C$. □

4. Examples of Homothetic Motions on Hypersurface M at $\mathbb{R}^8_{\alpha\beta\gamma}$

In this paper, we support the theory in the paper with some examples.

4.1. Case I $\alpha = \beta = \gamma = 1$

If we take as $\alpha = \beta = \gamma = 1$, the hypersurface M becomes at eight dimensional Euclidean space $\mathbb{R}^8$ and it is given by

$$M = \left\{ \begin{array}{c} z = (z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) \in \mathbb{R}^8 : z_1 z_7 + z_2 z_8 - z_3 z_5 - z_4 z_6 = 0, \\ z_1 z_4 + z_5 z_8 - z_2 z_3 - z_6 z_7 = 0,\ z_1 z_6 + z_3 z_8 - z_2 z_5 - z_4 z_7 = 0,\ z \neq 0. \end{array} \right\}.$$

Example 4.1. *Let $\eta : I \subset \mathbb{R} \to M \subset \mathbb{R}^8$ be a curve given by*

$$\eta(s) = \frac{1}{\sqrt{2}} h(s) \left(\begin{array}{c} \cos\theta(s)\cos\delta(s) + i\cos\theta(s)\sin\delta(s) \\ +j\cos\theta(s)\cos\delta(s) + ij\cos\theta(s)\sin\delta(s) \\ +k\sin\theta(s)\cos\delta(s) + ik\sin\theta(s)\sin\delta(s) \\ +jk\sin\theta(s)\cos\delta(s) + ijk\sin\theta(s)\sin\delta(s) \end{array} \right), \tag{8}$$

where θ, $\delta : I \subset \mathbb{R} \to \mathbb{R}$ are differentiable functions. By using (4) and (8) the matrix B is a homothetic matrix in here h is a homothetic scale. Also, if $h(s) = 1$ in (8), then the curve η is on unit sphere S^7 and the matrix B becomes a rotation matrix in $\mathbb{R}^8$. Now let find some special examples by using the example given by (8).

If we get as $\theta(s) = as$ and $\delta(s) = bs$, a, b are real numbers

$$\eta(s) = \frac{1}{\sqrt{2}} h(s) \left(\begin{array}{c} \cos(as)\cos(bs), \cos(as)\sin(bs), \cos(as)\cos(bs), \cos(as)\sin(bs), \\ \sin(as)\cos(bs), \sin(as)\sin(bs), \sin(as)\cos(bs), \sin(as)\sin(bs) \end{array} \right).$$

If we have as $\theta(s) = \frac{\pi}{4}$ and $\delta(s) = s$, we obtain the following curve

$$\eta(s) = \frac{1}{2} h(s) (\cos s, \sin s, \cos s, \sin s, \cos s, \sin s, \cos s, \sin s).$$

If we get as $\theta(s) = s$ and $\delta(s) = \frac{\pi}{4}$,

$$\eta(s) = \frac{1}{2} h(s) (\cos s, \cos s, \cos s, \cos s, \sin s, \sin s, \sin s, \sin s).$$

If we take as $\theta(s) = 0$ and $\delta(s) = s$,

$$\eta(s) = \frac{1}{\sqrt{2}} h(s) (\cos s, \sin s, \cos s, \sin s, 0, 0, 0, 0).$$

If we take as $\theta(s) = s$ and $\delta(s) = 0$,

$$\eta(s) = \frac{1}{\sqrt{2}} h(s) (\cos s, 0, \cos s, 0, \sin s, 0, \sin s, 0).$$

Example 4.2. *Let $\eta : I \subset \mathbb{R} \to M \subset \mathbb{R}^8$ be a curve given by*

$$\eta(s) = h(s)(\cos s + i\sin s). \tag{9}$$

By using (4) and (9), the matrix B becomes the matrix of the homothetic motion. If we take as $h(s) = 1$, then we get

$$\eta(s) = \cos s + i \sin s. \tag{10}$$

By using (4) and (10), we obtain the matrix as :

$$B = \begin{pmatrix} \cos s & -\sin s & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin s & \cos s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos s & -\sin s & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin s & \cos s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos s & -\sin s & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin s & \cos s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos s & -\sin s \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin s & \cos s \end{pmatrix}.$$

This matrix is a rotational matrix in $\mathbb{R}^8$ which leaves the planes Ox_1x_2, Ox_3x_4, Ox_5x_6, Ox_7x_8 invariant. Since the curve given by (10) is unit speed and its tangent vector is on M, the derivative of the above matrix is orthogonal matrix, too.

4.2. Case II $\alpha = \beta = 1$, $\gamma = -1$

For $\alpha = \beta = 1$ and $\gamma = -1$, M is a hypersurface in eight dimensional pseudo-Euclidean space with index 4 $\mathbb{R}^8_4$ and it is given by

$$M = \left\{ \begin{array}{c} z = (z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) \in \mathbb{R}^8_4 : z_1z_7 + z_2z_8 - z_3z_5 - z_4z_6 = 0, \\ z_1z_4 - z_5z_8 - z_2z_3 + z_6z_7 = 0,\ z_1z_6 + z_3z_8 - z_2z_5 - z_4z_7 = 0,\ z \neq 0, \end{array} \right\}.$$

Example 4.3. *Let η be a curve on M at $\mathbb{R}^8_4$.*

$$\eta(s) = \frac{1}{\sqrt{2}} h(s) \begin{pmatrix} \cosh\theta(s)\cos\delta(s) + i\cosh\theta(s)\sin\delta(s) \\ +j\cosh\theta(s)\cos\delta(s) + ij\cosh\theta(s)\sin\delta(s) \\ +k\sinh\theta(s)\cos\delta(s) + ik\sinh\theta(s)\sin\delta(s) \\ +jk\sinh\theta(s)\cos\delta(s) + ijk\sinh\theta(s)\sin\delta(s) \end{pmatrix}, \tag{11}$$

where θ, $\delta : I \subset \mathbb{R} \to \mathbb{R}$ are smooth functions. By using (4) and (11), the matrix B determines a homothetic motion, in here h is a homothetic scale. If $h(s) = 1$, then the curve η is on unit sphere S^7_4 at $\mathbb{R}^8_4$ and B becomes a rotational matrix in $\mathbb{R}^8_4$. Now let investigate some special examples by using the example given by (11).

If we get as $\theta(s) = as$ and $\delta(s) = bs$, a, b are real numbers,

$$\eta(s) = \frac{1}{\sqrt{2}} h(s) \left(\begin{array}{c} \cosh(as)\cos(bs), \cosh(as)\sin(bs), \cosh(as)\cos(bs), \cosh(as)\sin(bs), \\ \sinh(as)\cos(bs), \sinh(as)\sin(bs), \sinh(as)\cos(bs), \sinh(as)\sin(bs) \end{array} \right).$$

If $\theta(s) = s$ and $\delta(s) = \frac{\pi}{4}$,

$$\eta(s) = \frac{1}{2} h(s) (\cosh s, \cosh s, \cosh s, \cosh s, \sinh s, \sinh s, \sinh s, \sinh s).$$

If $\theta(s) = 0$ *and* $\delta(s) = s$,

$$\eta(s) = \frac{1}{\sqrt{2}}h(s)(\cos s, \sin s, \cos s, \sin s, 0, 0, 0, 0).$$

If $\theta(s) = s$ *and* $\delta(s) = 0$,

$$\eta(s) = \frac{1}{\sqrt{2}}h(s)(\cosh s, 0, \cosh s, 0, \sinh s, 0, \sinh s, 0).$$

Example 4.4. *Let* η *be a curve on* M *as:*

$$\eta(s) = h(s)(\cosh s + ij\sinh s). \tag{12}$$

The matrix representation of (12) describes a homothetic motion. If we get as $h(s) = 1$, *we have the following curve*

$$\eta(s) = \cosh s + ij\sinh s, \tag{13}$$

the matrix B *associated with the curve given by (13) is a real semi-orhogonal matrix, that is, it becomes a rotational matrix as:*

$$B = \begin{pmatrix} \cosh s & 0 & 0 & 0 & 0 & 0 & 0 & \sinh s \\ 0 & \cosh s & 0 & 0 & 0 & 0 & -\sinh s & 0 \\ 0 & 0 & \cosh s & 0 & 0 & -\sinh s & 0 & 0 \\ 0 & 0 & 0 & \cosh s & \sinh s & 0 & 0 & 0 \\ 0 & 0 & 0 & \sinh s & \cosh s & 0 & 0 & 0 \\ 0 & 0 & -\sinh s & 0 & 0 & \cosh s & 0 & 0 \\ 0 & -\sinh s & 0 & 0 & 0 & 0 & \cosh s & 0 \\ \sinh s & 0 & 0 & 0 & 0 & 0 & 0 & \cosh s \end{pmatrix}.$$

The above matrix is a rotational matrix in $\mathbb{R}^8_4$ *which leaves the planes* Ox_1x_8, Ox_2x_7, Ox_3x_6, Ox_4x_5 *invariant. Also, since the curve given by (13) is unit speed and its tangent vector is on* M, *the derivative of the above matrix* $\dot{B}$ *is a real semi-orthogonal matrix, too.*

4.3. Case III $\alpha = -1$, $\beta = \gamma = 1$

If we choose as $\alpha = -1$, $\beta = \gamma = 1$, M is a hypersurface in eight dimensional pseudo Euclidean space with index 4 $\mathbb{R}^8_4$ and it is given by

$$M = \left\{ \begin{array}{c} z = (z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) \in \mathbb{R}^8_{\alpha\beta\gamma} : z_1z_7 - z_2z_8 - z_3z_5 + z_4z_6 = 0, \\ z_1z_4 + z_5z_8 - z_2z_3 - z_6z_7 = 0,\ z_1z_6 + z_3z_8 - z_2z_5 - z_4z_7 = 0,\ z \neq 0 \end{array} \right\}.$$

Example 4.5. *Let* η *be a curve on* M *at* $\mathbb{R}^8_4$.

$$\eta(s) = \frac{1}{\sqrt{2}}h(s) \begin{pmatrix} \cosh\theta(s)\cosh\delta(s) + i\cosh\theta(s)\sinh\delta(s) \\ +j\cosh\theta(s)\cosh\delta(s) + ij\cosh\theta(s)\sinh\delta(s) \\ -k\sinh\theta(s)\sinh\delta(s) - ik\sinh\theta(s)\cosh\delta(s) \\ +jk\sinh\theta(s)\sinh\delta(s) + ijk\sinh\theta(s)\cosh\delta(s) \end{pmatrix}, \tag{14}$$

where θ, $\delta : I \subset \mathbb{R} \to \mathbb{R}$ *are smooth functions. By using (4) and (14), the matrix* B *associated with the curve* η *is a homothetic motion matrix and* h *is a homothetic scale. If* $h(s) = 1$, *then the curve* η *is on unit sphere* S^7_4 *at* $\mathbb{R}^8_4$

and the matrix B determines a rotational matrix in $\mathbb{R}^8_4$. Now let research some special examples by using the example given by (14).

If we get as $\theta(s) = as$ and $\delta(s) = bs$, a, b are real numbers,

$$\eta(s) = \frac{1}{\sqrt{2}} h(s) \begin{pmatrix} \cosh(as)\cosh(bs), \cosh(as)\sinh(bs), \cosh(as)\cosh(bs), \cosh(as)\sinh(bs), \\ -\sinh(as)\sinh(bs), -\sinh(as)\cosh(bs), \sinh(as)\sinh(bs), \sinh(as)\cosh(bs) \end{pmatrix}.$$

If $\theta(s) = s$ and $\delta(s) = 0$,

$$\eta(s) = \frac{1}{\sqrt{2}} h(s) (\cosh s, 0, \cosh s, 0, 0 - \sinh s, 0, \sinh s).$$

If $\theta(s) = 0$ and $\delta(s) = s$,

$$\eta(s) = \frac{1}{\sqrt{2}} h(s) (\cosh s, \sinh s, \cosh s, \sinh s, 0, 0, 0, 0).$$

5. Conclusion

In this paper, using the generalized tricomplex numbers, we determine a motion on the hypersurface M in eight dimensional generalized linear space $\mathbb{R}^8_{\alpha\beta\gamma}$ and prove that this is a homothetic motion. For some special cases of the real numbers α, β and γ, we support the theory in this paper with some examples of homothetic motions in $\mathbb{R}^8$ and $\mathbb{R}^8_4$. Also, we give some algebraic properties of the generalized tricomplex numbers.

References

[1] F. Babadag, Y. Yaylı, and N. Ekmekci, *Homothetic motions at E^8 with bicomplex numbers C_3,* Int. J. Contemp. Math. Sciences **4** (2009), no. 33–36, 1619–1626.

[2] F. Babadag, Y. Yaylı, and N. Ekmekci, *Homothetic motion and bicomplex numbers*, Commun. Fac. Sci. Univ. Ank. Series A1 Math. Stat. **58** (2009), no. 1, 23–28.

[3] H. H. Hacısalihoğlu, *On the rolling of one curve or surface upon another*, Proc. Roy. Irish Acad. Sect. **71** (1971), 13–17.

[4] M. Jafari and Y. Yaylı, *Homothetic motion at $\mathbb{E}^4_{\alpha\beta}$*, Int. J. Contemp. Math. Sci. **5** (2010), no. 45–48, 2319–2326.

[5] H. Kabadayı and Y. Yaylı, *Homothetic motions at E^4 with bicomplex numbers*, Adv. Appl. Clifford Algebra, **21** (2011), no. 3, 541–546.

[6] F. Kahraman Aksoyak and Y. Yaylı *Homothetic motions and Lie groups in $\mathbb{R}^4_2$*, J. Dyn. Syst. Geom. Theor. **11** (2013), no. 1–2, 23–38.

[7] F. Kahraman Aksoyak and Y. Yaylı *Homothetic motions and surfaces in $\mathbb{E}^4$*, Bull. Malays. Math. Sci. Soc. **38** (2015), no. 1, 259–269.

[8] F. Kahraman Aksoyak and S. Özkaldı Karakuş, *Homotetic motions via generalized bicomplex numbers*, Facta Univ. Ser. Math. Inform **36** (2021), no. 2, 275–291.

[9] G. Özaydın, *Homothetic Motion with Generalized Bicomplex Numbers*, Master Thesis, Bilecik Şeyh Edebali University, Institute of Science, Bilecik, Turkey, 2019.

[10] S. Özkaldı Karakuş and F. Kahraman Aksoyak, *Generalized bicomplex numbers and Lie groups*, Adv. Appl. Clifford Algebra **25** (2015), no. 4, 943–963.

[11] S. Özkaldi and Y. Yaylı *Tensor product surfaces in $\mathbb{R}^4$ and Lie groups*, Bull. Malays. Math. Sci. Soc. **33** (2010), no. 1, 69–77.

[12] S. Özkaldi and Y. Yaylı *Bicomplex number and tensor product surfaces in* $\mathbb{R}^4_2$, Ukrainian Math. J. **64** (2012) no. 3, 344–355.

[13] G. B. Price, *An Introduction to Multicomplex Spaces and Functions*, Marcel Deccer, Inc., New York, 1990.

[14] C. Segre, *Le rappresentazioni reali delle forme complesse e gli enti iperalgebrici*, Math. Ann. **40** (1892), no. 3, 413–467.

[15] Y. Yaylı, *Homothetic motion at* E^4, Mech. Mach. Theory **27** (1992), 303–305.

[16] Y. Yaylı and B. Bükcü, *Homothetic motions at* E^8 *with Cayley numbers*, Mech. Mach. Theory **30** (1995), 417–420.

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