Honam Mathematical J. **46** (2024), No. 1, pp. 70–81
https://doi.org/10.5831/HMJ.2024.46.1.70

THE STUDY OF $*$-RICCI TENSOR ON LORENTZIAN PARA SASAKIAN MANIFOLDS

M. R. Bakshi*, T. Barman, and K. K. Baishya

Abstract. We consider the $*$-general critical equation on LP Sasakian manifolds, and show that such a manifold is generalized η-Einstein. After then, we consider LP Sasakian manifolds with $*$-conformally semisymmetric condition, and show that such manifolds are $*$-Einstein. Moreover, we show that the $*$-conformally semisymmetric LP Sasakian manifold is locally isometric to $E^{n+1}(0) \times S^n(4)$.

1. Introduction

The symbols $\hat{S}^*$, ρ^*, R and r^* stand for the $*$-Ricci operator, $*$-Ricci tensor, Riemann curvature tensor and $*$-scalar curvature respectively. The study of $*$-Ricci tensor was initiated by Tachibana [25] in 1959 in the context of almost Hermitian manifolds. The $*$-Ricci tensor of real hypersurfaces in a non-flat complex space form is defined [12] as

$$\rho^*(X_1, X_2) = g(\hat{S}^* X_1, X_2) = \frac{1}{2} Trace\{\phi \circ R(X_1, \phi X_2)\},$$

where ϕ is $(1,1)$ tensor field and X_1, X_2 are any vector fields.

The study of $*$-Ricci tensor has now become the topic of growing interest by many geometers and its characteristics were studied in the frame of different structures, namely, Kenmotsu manifolds [26], Sasakian manifolds and (κ, μ)-contact manifolds ([11], [27]), α-cosymplectic manifolds ([2]) and the references therein.

In 2019, Kaimakamis and Panagiotidou ([15]) introduced the $*$-Weyl conformal curvature tensor C^* of real hypersurfaces in non-flat complex

---

Received May 15, 2023. Accepted September 29, 2023.
2010 Mathematics Subject Classification. 53C15, 53C25
Key words and phrases. generalized η-Einstein, $*$-Einstein, $*$-general critical equation, Lorentzian para Sasakian structure.
*Corresponding author

space forms

$$
\begin{aligned}
&C^*(X_1, X_2) = R(X_1, X_2) \\
(1)\qquad &-\frac{1}{2n-1}\left[\frac{r^*}{2n}(X_1 \wedge_g X_2) + \left(X_1 \wedge_g \hat{S}^* X_2\right) + \left(\hat{S}^* X_1 \wedge_g X_2\right)\right],
\end{aligned}
$$

where $(X_1 \wedge_g X_2) X_3 = g(X_2, X_3)X_1 - g(X_1, X_3)X_2$. The $*$-Weyl conformal curvature tensor has also been studied by [27].

In 1989, Matsumoto [16] initiated the studies on Lorentzian Para-Sasakian manifolds (or in short $LPSM$) which had also been independently defined by Mihai and Rosca [19]. Matsumoto, Mihai and Rosca ([17]) gave a five dimensional example of $LPSM$. Thereafter, many research papers were published on this structure (see [20], [21], [7], [13], [3], [5], [14], [24]) and the references therein. In [13], authors studied $*$-Ricci tensor in the frame of $LPSM$ by finding the relation between the Ricci and the $*$-Ricci tensor.

Recently, the authors of [6] have claimed the existence of some critical metrics on GRW-spacetime by considering the general critical equation as

$$
(2)\qquad \lambda\rho + \sigma g = Hess(\lambda),\ \ \lambda, \sigma \text{ being smooth functions.}
$$

We note that the foregoing equation have the flavour of Fischer-Marsden critical equation ([9], [10]) for $\sigma = \Delta\lambda$ and Miao-Tam critical equation [18] for $\sigma = \Delta\lambda + 1$.

Then the authors in [2] introduced and studied the $*$-general critical equation which is defined as

$$
(3)\qquad Hess(\lambda) = \lambda\rho^* + \sigma g.
$$

Motivated from the above studies, in the present article we consider the $*$-general critical equation and the $*$-conformally semisymmetric condition and obtained some interesting results.

Our present paper deals with the study of $*$-general critical equation on Lorentzian Para Sasakian manifolds and it is shown that such a manifold is generalized η-Einstein. We further consider the $*$-conformally semisymmetric Lorentzian Para Sasakian manifolds and established that such a manifold is locally isometric to $E^{n+1}(0) \times S^n(4)$.

2. Preliminaries

Let M^{2n+1} be a $(2n+1)$-dimensional differential manifold endowed with a $(1,1)$ tensor field ϕ, a vector field ξ, an 1-form η and a Lorentzian metric g of type $(0,2)$ such that for each point $a \in M$, the tensor g_a : $T_aM \times T_aM \to \mathbb{R}$ is a non-degenerate, symmetric and of signature $(-,+,+,\dots,+)$, where T_aM denotes the tangent space of M at a and $\mathbb{R}$ is the real number set which satisfies

$$\phi^2 = I + \eta \otimes \xi, \tag{4}$$

$$\eta(\xi) = -1, \tag{5}$$

$$g(X,\xi) = \eta(X), \tag{6}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{7}$$

for all vector fields X, Y on M^{2n+1}. Then the structure (ϕ,ξ,η,g) is called Lorentzian almost para contact structure and the manifold with the structure (ϕ,ξ,η,g) is called a Lorentzian almost para contact manifold. In the Lorentzian almost para contact manifold M, the following relations hold ([16])

$$\phi\xi = 0, \ \ \eta \circ \phi = 0, \tag{8}$$

$$g(\phi X, Y) = g(X, \phi Y). \tag{9}$$

If we put

$$\Omega(X,Y) = g(\phi X, Y) = g(X, \phi Y) \tag{10}$$

for any vector fields X and Y, then the tensor field $\Omega(X,Y)$ is a symmetric $(0,2)$ tensor field.

A Lorentzian almost para contact manifold M endowed with the structure (ϕ,ξ,η,g) is called an $LPSM$ if

$$(\nabla_X \phi)Y - g(\phi X, \phi Y)\xi = \eta(Y)\phi^2 X, \tag{11}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. In an $LPSM$ with the structure (ϕ,ξ,η,g), it is easily seen that ([16])

$$\nabla_X \xi = \phi X, \tag{12}$$

$$(\nabla_X \eta)Y = g(X, \phi Y) = \Omega(X,Y) = (\nabla_Y \eta)X, \tag{13}$$

$$\rho(X,\xi) = 2n\eta(X), \ \ \hat{S}\xi = 2n\xi, \tag{14}$$

$$R(Y,U)\xi = \eta(U)Y - \eta(Y)U, \tag{15}$$

$$\eta(R(Y,U)V) = \eta(Y)g(U,V) - \eta(U)g(Y,V), \tag{16}$$

$$\begin{aligned} & R(Y,U)\phi X \\ = \; & \phi R(Y,U)X + g(U,X)\phi Y - g(Y,X)\phi U + g(\phi Y,X)U \\ & -g(\phi U,X)Y + 2[g(\phi Y,X)\eta(U) - g(\phi U,X)\eta(Y)]\xi \\ & +2[\eta(U)\phi Y - \eta(Y)\phi U]\eta(X), \end{aligned} \tag{17}$$

$$\begin{aligned} & R(X,Y)Z \\ = \; & \phi R(X,Y)\phi Z + g(X,Z)Y - g(Y,Z)X + \Omega(Y,Z)\phi X \\ & -\Omega(X,Z)\phi Y + 2[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi \\ & +2[\eta(X)Y - \eta(Y)X]\eta(Z), \end{aligned} \tag{18}$$

$$\begin{aligned} & (\nabla_X R)(Y,U)\xi \\ = \; & 2[g(\phi U,X)Y - g(\phi Y,X)U] - \phi R(Y,U)X \\ & +g(Y,X)\phi U - g(U,X)\phi Y - 2[g(\phi Y,X)\eta(U) \\ & -g(\phi U,X)\eta(Y)]\xi - 2[\eta(U)\phi Y - \eta(Y)\phi U]\eta(X) \end{aligned} \tag{19}$$

for all vector fields X, Y, U and Z on M^{2n+1}.

Lemma 2.1. ([13]) *In a $(2n+1)$-dimensional LPSM the followings hold*

$$\nabla \hat{S}\xi = 2n\phi - \hat{S}\phi, \tag{20}$$

$$\nabla_\xi \hat{S} = 2aI - 2\hat{S}\phi + 2a\eta \otimes \xi, \tag{21}$$

$$\hat{S}^* = \hat{S} - a\phi + (2n-1)I + (4n-1)\eta \otimes \xi, \tag{22}$$

where a is traceϕ.

A Lorentzian Para-Sasakian manifold is said to be a generalized η-Einstein manifold [28] if its Ricci tensor satisfies

$$\rho = xg + y\eta \otimes \eta + z\Omega,$$

where x, y, z are smooth functions. For $z = 0$, the manifold reduces to an η-Einstein manifold.

3. $*$-general critical equations on $LPSM$

Lemma 3.1. *An $LPSM$ with $*$-general critical equations satisfies the followings*

$$\begin{aligned} & R(X,Y)D\lambda \\ = \; & \lambda\{(\nabla_X\hat{S})Y - (\nabla_Y\hat{S})X\} - a\lambda\{\eta(Y)X - \eta(X)Y\} \\ & +(X\lambda)\hat{S}Y - (Y\lambda)\hat{S}X - a\{(X\lambda)\phi Y - (Y\lambda)\phi X\} \\ & +\{(2n-1)(X\lambda) + (X\sigma)\}Y - \{(2n-1)(Y\lambda) + (Y\sigma)\}X \\ & +(4n-1)\{(X\lambda)\eta(Y) - (Y\lambda)\eta(X)\}\xi \\ & +(4n-1)\lambda\{\eta(Y)(\phi X) - \eta(X)(\phi Y)\}, \end{aligned} \tag{23}$$

$$\begin{aligned} & \frac{\lambda}{2}Dr + r(Dr) + a(\phi D\lambda) + 2nD\sigma \\ & -\{a^2 - 2n(2n-1) + (4n-1)\}D\lambda \\ = \; & \{(2n-1)a\lambda + (4n-1)(\xi\lambda)\}\xi. \end{aligned} \tag{24}$$

Proof. Let an $LPSM$ admit the $*$-general critical equation (3). In view of (22), we have

$$\begin{aligned} & Hess(\lambda)(X,Y) \\ = \; & \lambda\rho(X,Y) + \{(2n-1)\lambda + \sigma\}g(X,Y) \\ & -a\lambda g(X,\phi Y) + (4n-1)\lambda\eta(X)\eta(Y), \end{aligned} \tag{25}$$

which leads to

$$\begin{aligned} & \nabla_X D\lambda \\ = \; & \lambda\hat{S}X + \{(2n-1)\lambda + \sigma\}X - a\lambda\phi X + (4n-1)\lambda\eta(X)\xi \end{aligned} \tag{26}$$

and

$$\begin{aligned} & \nabla_Y\nabla_X D\lambda \\ = \; & \lambda\nabla_Y\hat{S}(X) + (Y\lambda)\hat{S}X - a\lambda\nabla_Y\phi(X) - (Y\lambda)\phi X \\ & +\{(2n-1)\lambda + \sigma\}\nabla_Y X + \{(2n-1)(Y\lambda) + (Y\sigma)\}X \\ & +(4n-1)\{(Y\lambda)\eta(X)\xi + \lambda\nabla_Y\eta(X)\xi + \lambda\eta(X)\nabla_Y\xi\} \end{aligned} \tag{27}$$

after taking the covariant differentiation. In view of (26) and (27), we obtain (23) and then taking the contraction of (23), we obtain (24). □

Again, the relation (23) yields

$$\begin{aligned} & R(X,Y,Z,D\lambda)+\lambda\, div(R(X,Y)Z) \\ = \ & a\lambda\{\eta(Y)g(X,Z)-\eta(X)g(Y,Z)\}-(X\lambda)\rho(Y,Z)+(Y\lambda)\rho(X,Z) \\ & +a\{(X\lambda)g(\phi Y,Z)-(Y\lambda)g(\phi X,Z)\}-\{(2n-1)(X\lambda)+(X\sigma)\}g(Y,Z) \\ & +\{(2n-1)(Y\lambda)+(Y\sigma)\}g(X,Z)-(4n-1)\{(X\lambda)\eta(Y)-(Y\lambda)\eta(X)\}\eta(Z) \\ & -(4n-1)\lambda\{\eta(Y)g(\phi X,Z)-\eta(X)g(\phi Y,Z)\}Z. \end{aligned}$$

Thus, we can state that:

Proposition 3.2. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an $LPSM$ satisfying the ∗-general critical equations. For the harmonic and radial Riemannian curvature tensor, we obtain*

$$\begin{aligned} & \lambda\eta\otimes\{aI-(4n-1)\phi\}+d\sigma\otimes I \\ & +d\lambda\otimes\{\hat{S}-a\phi+(2n-1)I+(4n-1)\eta\otimes\xi\} \\ = \ & \lambda\{aI-(4n-1)\phi\}\otimes\eta+I\otimes d\sigma \\ & +\{\hat{S}-a\phi+(2n-1)I+(4n-1)\eta\otimes\xi\}\otimes d\lambda. \end{aligned}$$

Lemma 3.3. *If $(M^{2n+1},\phi,\xi,\eta,g)$ is an $LPSM$ satisfying the ∗-general critical equation, then $\nabla_\xi D\lambda=\sigma\xi$.*

Proof. Introducing $Y=\xi$ in (25) and using (14), we obtain

$$\nabla_\xi D\lambda=\sigma\xi.$$

□

Introducing $Y=\xi$ in (23) and then taking the help of (20) and (21), we get

$$\begin{aligned} & R(X,\xi)D\lambda \\ = \ & \lambda\{\hat{S}\phi X-(2n-1)\phi X-aX-a\eta(X)\xi\}+(X\sigma)\xi-(\xi\sigma)X \\ & -(\xi\lambda)\{\hat{S}X-a\phi X+(2n-1)X+(4n-1)\eta(X)\xi\}. \end{aligned} \tag{28}$$

Next using (15) in (28), we get

$$\begin{aligned} & (\xi\lambda)g(X,Z)-(X\lambda)\eta(Z) \\ = \ & \lambda\{g(\hat{S}\phi X,Z)-(2n-1)g(\phi X,Z)-ag(X,Z) \\ & -a\eta(X)\eta(Z)\}+(X\sigma)\eta(Z)-(\xi\sigma)g(X,Z) \\ & -(\xi\lambda)\{g(\hat{S}X,Z)-ag(\phi X,Z) \\ & +(2n-1)g(X,Z)+(4n-1)\eta(X)\eta(Z)\} \end{aligned} \tag{29}$$

which yields

$$X(\lambda+\sigma)=-\xi(\lambda+\sigma)\eta(X).$$

for $Z=\xi$. By taking the help of the above equation in (29), we have

$$\begin{aligned} & \{(\xi\lambda)+(\xi\sigma)\}\{g(X,Z)+\eta(X)\eta(Z)\} \\ = & -(2n-1)\lambda g(\phi X,Z)+\lambda\{\rho(\phi X,Z)-ag(X,Z)-a\eta(X)\eta(Z)\} \\ & -(\xi\lambda)\{\rho(X,Z)-ag(\phi X,Z) \\ & +(2n-1)g(X,Z)+(4n-1)\eta(X)\eta(Z)\}. \end{aligned} \tag{30}$$

Next, replacing X by ϕX in the foregoing equation we get

$$\begin{aligned} & (\xi\lambda)\rho(\phi X,Z)+\{(\xi\lambda)+(\xi\sigma)+a\lambda+(2n-1)(\xi\lambda)\}g(\phi X,Z) \\ = & \{a(\xi\lambda)+(2n-1)\lambda\}\{g(X,Z)+\eta(X)\eta(Z)\} \\ & +\lambda\{\rho(X,Z)+2n\eta(X)\eta(Z)\}. \end{aligned} \tag{31}$$

In view of (31) and (30), we obtain

$$\begin{aligned} & \{(\xi\lambda)^2-\lambda^2\}\rho(\phi X,\phi Z) \\ & +\{(4n-1)\lambda(\xi\lambda)+\lambda(\xi\sigma)-a(\xi\lambda)^2+a\lambda^2\}g(\phi X,Z) \\ = & \{(2n-1)\lambda^2-(\xi\lambda)(\xi\sigma)-2n(\xi\lambda)^2\}g(\phi X,\phi Z). \end{aligned} \tag{32}$$

Therefore, we can state the following:

Theorem 3.4. *Every LPSM admitting the $*$-general critical equations reduces to generalized η-Einstein manifolds .*

Example 3.5. *Let $M^3(\phi,\xi,\eta,g)$ be a Lorentzian Para Sasakian manifold with $\{e_1,e_2,e_3\}$ linearly independent vector fields.*

$$e_1=e^z\frac{\partial}{\partial x},\ e_2=e^{z-\alpha x}\frac{\partial}{\partial y},\ e_3=\xi=\frac{\partial}{\partial z},$$

where α is non-zero constant. Let us take the Lorentzian metric g as

$$\begin{aligned} g(e_1,e_1) &= g(e_2,e_2)=1,\ g(e_3,e_3)=-1 \\ g(e_i,e_j) &= 0 \ \text{ for } i\neq j. \end{aligned}$$

Let η be the one form defined by

$$g(X,e_3)=\eta(X),$$

for all X in M. Let ϕ be the $(1,1)$ tensor field defined by

$$\phi e_1=e_1,\ \phi e_2=e_2,\ \phi e_3=0.$$

Then from the Koszul's formula for Lorentzian metric g, we can obtain the Levi-Civita connection as follows:

$$\begin{aligned} \nabla_{e_1}e_3 &= -e_1, \qquad \nabla_{e_1}e_2 = 0, \qquad \nabla_{e_1}e_1 = -e_3, \\ \nabla_{e_2}e_3 &= -e_2, \qquad \nabla_{e_2}e_2 = -\alpha e^z e_1 - e_3, \qquad \nabla_{e_2}e_1 = \alpha e^z e_2, \\ \nabla_{e_3}e_3 &= 0, \qquad \nabla_{e_3}e_2 = 0, \qquad \nabla_{e_3}e_1 = 0. \end{aligned}$$

Using the above relations, we can easily calculate the non-vanishing components of the Riemann curvature tensor R (up to symmetry and skew-symmetry) and the Ricci curvature tensor ρ as following

$$\begin{aligned} R(e_1,e_2)e_1 &= -(1-\alpha^2e^{2z})e_2, \quad R(e_1,e_2)e_2 = (1-\alpha^2e^{2z})e_1, \\ R(e_1,e_3)e_1 &= -e_3, \qquad R(e_1,e_3)e_3 = -e_1, \\ R(e_2,e_3)e_2 &= -\alpha e^z e_1 - e_3, \quad R(e_2,e_3)e_3 = -e_2, \end{aligned}$$

$$\rho(e_1,e_1) = \rho(e_2,e_2) = -\alpha^2e^{2z}, \quad \rho(e_3,e_3) = -2,$$

and the non-vanishing components of $\rho^\star$ are

$$\rho^\star(e_1,e_1) = \rho^\star(e_2,e_2) = -(1+\alpha^2e^{2z}).$$

Let $\Omega(X,Y) = g(\phi X,Y)$, then the non zero components are

$$\begin{aligned} \Omega(e_1,e_1) &= g(e_1,\phi e_1) = 1, \\ \Omega(e_2,e_2) &= g(e_2,\phi e_2) = 1. \end{aligned}$$

Assuming

$$\begin{aligned} A &= (1-\alpha^2e^{2z}), \\ B &= -(1+\alpha^2e^{2z}), \\ C &= -1, \end{aligned}$$

we have

$$\rho(X,Y) = Ag(X,Y) + B\eta(X)\eta(Y) + C\Omega(X,Y).$$

This implies that the manifold is a generalized η-Einstein manifold under the above considerations.

Next we choose smooth functions λ and σ such that

$$g(\nabla_{e_1}D\lambda, e_1) = g(\nabla_{e_2}D\lambda, e_2) = -(1+\alpha^2e^{2z})\lambda + \sigma.$$

Suppose $\lambda = z$, so that $D\lambda = e_3$ and therefore $Hess(z)(e_i,e_i) = -1$ for $i = 1,2$. Therefore, $(g, z, (1+\alpha^2e^{2z})t - 1)$ is a solution of the $*$-general critical equation.

4. $LPSM$ admitting the semisymmetric condition $R \cdot C^* = 0$

In view of (1) and (16), we get

$$\begin{aligned} & \eta(C^*(X,Y)Z) \\ = & \left(\frac{r^*}{2n(2n-1)} - 1\right)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \\ & -\frac{1}{2n-1}[\rho^*(Y,Z)\eta(X) - \rho^*(X,Z)\eta(Y)]. \end{aligned} \tag{33}$$

Using (22) we obtain

$$\begin{aligned} & \eta(C^*(\xi,Y)Z) \\ = & \left(1 - \frac{r^*}{2n(2n-1)}\right)[g(Y,Z) + \eta(X)\eta(Z)] + \frac{\rho^*(Y,Z)}{2n-1}, \end{aligned}$$

and

$$\eta(C^*(X,Y)\xi) = 0.$$

Suppose the $LPSM$ is $*$-conformally semisymmetric. Then

$$R \cdot C^* = 0,$$

which yields after taking the inner product with ξ

$$\begin{aligned} & \eta(R(\xi,Y)C^*(U,V)W) - \eta(C^*(R(\xi,Y)U,V)W) \\ & -\eta(C^*(U,R(\xi,Y)V)W) - \eta(C^*(U,V)R(\xi,Y)W) = 0. \end{aligned} \tag{34}$$

In view of (16), the relation (34) becomes

$$\begin{aligned} & -g(C^*(U,V)W,Y) - \eta(Y)\eta(C^*(U,V)W) \\ & -g(Y,U)\eta(C^*(\xi,V)W) + \eta(U)\eta(C^*(Y,V)W) \\ & -g(Y,V)\eta(C^*(U,\xi)W) + \eta(V)\eta(C^*(U,Y)W) \\ & -g(Y,W)\eta(C^*(U,V)\xi) + \eta(W)\eta(C^*(U,V)Y) = 0. \end{aligned} \tag{35}$$

Next, using (33) the foregoing equation reduces to

$$\begin{aligned} & C^*(U,V,W,Y) \\ & + \left(\frac{r^*}{2n(2n-1)} - 1\right)[g(Y,U)g(V,W) - g(Y,V)g(U,W)] \\ & -\frac{1}{(2n-1)}[\rho^*(Y,V)\eta(U)\eta(W) - \rho^*(Y,U)\eta(V)\eta(W)] \\ & +\rho^*(V,W)g(Y,U) - \rho^*(U,W)g(Y,V) = 0. \end{aligned} \tag{36}$$

Executing the contraction over U and W, the above equation gives

$$\rho^*(Y,V) = -2n(2n-1)g(Y,V), \tag{37}$$

and

$$r^* = -2n(2n-1)(2n+1). \tag{38}$$

Therefore we can state

Theorem 4.1. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be an LPSM admitting the semisymmetric condition $R \cdot C^* = 0$. Then the manifold M^{2n+1} is $*$-Einstein and of constant $*$-scalar curvature $-2n(2n-1)(2n+1)$.*

By virtue of (37) and (38), the equation (36) becomes

$$R(U,V,Y,W) + (2n+1)[g(Y,V)g(U,W) - g(Y,U)g(V,W)]$$
$$+2n[g(Y,V)\eta(U)\eta(W) - g(Y,U)\eta(V)\eta(W)] = 0.$$

Using (16) in the above equation, we obtain

$$\eta(R(U,V)Y) = 0.$$

Hence

$$R(U,V)\xi = 0,$$

for all Y in M^{2n+1}.

Thus we can conclude the following:

Theorem 4.2. [8] *Suppose $M^{2n+1}(\phi,\xi,\eta,g)$ be an LPSM admitting the semisymmetric condition $R \cdot C^* = 0$. Then the manifold M^{2n+1} is locally isometric to the Riemannian product of a flat $(n+1)$-dimensional Riemannian manifold and an n-dimensional manifold of positive curvature 4, i.e., $E^{n+1}(0) \times S^n(4)$.*

Acknowledgements

Authors would like to thank the referees for his/her suggestions to improve the manuscript. The second named author gratefully acknowledges UGC (NTA Ref. No. 211610054686), for financial assistance.

References

[1] K. K. Baishya and P. R. Chowdhury, *Semi-symmetry type LP-Saskian manifolds*, Acta Math. Acad. Paedagog. Nyiregyhaziensis. **33** (2017), no. 1, 67–83.

[2] K. K. Baishya and M. R. Bakshi, *Critical metric equation on α-cosymplectic manifolds*, J. Anal. **31** (2023), 871–880.

[3] K. K. Baishya, M. R. Bakshi and A. Biswas, On generalized weakly (Ricci) phi-symmetric LP Sasakian manifolds, Novi Sad J Math. **51** (2021), no. 2, 89–105.

[4] K. K. Baishya and P. R. Chowdhury, *On almost generalized weakly symmetric LP-Sasakian manifolds*, Annals of West University of Timisoara - Mathematics and Computer Science **55** (2017), no. 2, 51–64.

[5] M. R. Bakshi, A. Das and K. K. Baishya, *Existence of hyper generalized weakly symmetric Lorentzian para-Sasakian manifold*, Acta Universitatis Apulensis **65** (2021), 51–66.

[6] M. R. Bakshi, K. K. Baishya, M. Pain and H. Kundu, *Existence of some critical metrics on certain type of GRW-spacetime*, submitted.

[7] A. M. Blaga, *η-Ricci solitons on LP Sasakian manifolds*, Filomat **30** (2016), no. 2, 489–496.

[8] D. E. Blair, *Two remarks on contact metric structures,* Tohoku Math. J. **29** (1977), 319–324.

[9] A. E. Fischer and J. E. Marsden, *Manifolds of Riemannian metrics with prescribed scalar curvature,* Bull. Amer. Math. Soc. **80** (1974), 479–484.

[10] A. E. Fischer and J. E. Marsden, *Deformations of the scalar curvature,* Duke Math. J. **42** (1975), no. 3, 519–547.

[11] A. Ghosh and D. S. Patra, *$*$-Ricci soliton within the frame-work of Sasakian and (κ,μ)-contact manifold*, Int. J. Geom. Methods Mod. Phys. **15** (2018), no. 7, 1850120.

[12] T. Hamada, *Real hypersurfaces of complex space forms in terms of Ricci $*$-tensor*, Tokyo J. Math. **25** (2002), no. 2, 473–483.

[13] A. Haseeb and S. K. Chaubey, *Lorentzian Para Sasakian Manifolds and $*$-Ricci Solitons*, Kragujev. J. Math. **48** (2024), no. 2, 167–179.

[14] S. K. Hui, S. Uddin, C. Ozel and A. A. Mustafa, *Warped product submanifolds of LP-Sasakian manifolds,* Discrete Dynamics in Nature and Society Vol. 2012, Article ID 868549, doi: 10.1155/2012/868549.

[15] G. Kaimakamis and K. Panagiotidou, *On a new type of tensor on real hypersurfaces in nonflat complex space forms*, Symmetry **11** 2019, no. 4, 559; https://doi.org/10.3390/sym11040559.

[16] K. Matsumoto, *On Lorentzian paracontact manifolds*, Bull. Yamagata Univ. Natur. Sci. **12** (1989), no. 2, 151–156.

[17] K. Matsumoto, I. Mihai and R. Rosca, *ξ-null geodesic gradient vector fields on a Lorentzian para-Sasakian manifold*, J. Korean Math. Soc. **32** (1995), no. 1, 17–31.

[18] P. Miao and L. F. Tam, *On the volume functional of compact manifolds with boundary with constant scalar curvature,* Calc. Var. Partial Differential Equations **36** (2009), no. 2, 141–171.

[19] I. Mihai and R. Rosca, *On Lorentzian P-Sasakian manifolds*, Classical analysis (Kazimierz Dolny, 1991), 155–169, World Sci. Publ. River Edge, NJ, 1992.

[20] A. A. Shaikh and K. K. Baishya, *Some results on LP-Sasakian manifolds*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **49** (2006), no. 2, 193–205.

[21] A. A. Shaikh and K. K. Baishya, *On ϕ-symmetric LP-Sasakian manifolds*, Yokohama Math. J. **52** (2006), no. 2, 97–112.

[22] A. A. Shaikh, D. W. Yoon, and S. K. Hui, *On quasi-Einstein spacetimes*, Tsukuba J. Math. **33** (2009), no. 2, 305–326.

[23] A. A. Shaikh, Y. H. Kim, and S. K. Hui, *On Lorentzian quasi-Einstein manifolds*, J. Korean Math. Soc. **48** (2011), no. 4, 669–689.

[24] T. Pal, S. S. Shukla, and S. K. Hui, *v-pointwise bi-slant Lorentzian submersions from LP-Sasakian manifolds*, Afrika Mathematika **32** (2021), 1341–1354.

[25] S. Tachibana, *On almost-analytic vectors in almost-Kahlerian manifolds*, Tohoku Math. J. **11** (1959), no. 2, 247–265.

[26] V. Venkatesha, D. M. Naik, and H. A. Kumara, *$*$-Ricci solitons and gradient almost $*$-Ricci solitons on Kenmotsu manifolds*, Math. Slovaca **69** (2019), no. 6, 1447–1458.

[27] V. Venkatesha and H. A. Kumara, *$*$-Weyl curvature tensor within the framework of Sasakian and (κ, μ)-contact manifolds*, Tamkang J. Math. **52** (2021), no. 3, 383–395.

[28] A. Yildiz, U. C. De, and E. Ata, *On a type of Lorentzian Para-Sasakian manifolds,* Math. Reports **16** (2014), no. 66, 61–67.

M. R. Bakshi
Department of Mathematics, Salesian College,
West Bengal, BHARAT.
E-mail: raybakshimanoj@gmail.com

T. Barman
Department of Mathematics, Raiganj University,
West Bengal, BHARAT.
E-mail: barmantanushri98@gmail.com

K. K. Baishya
Department of Mathematics, Kurseong College,
West Bengal, BHARAT.
E-mail: kanakkanti.kc@gmail.com