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GENERALIZED CHEN INEQUALITY FOR CR-WARPED PRODUCTS OF LOCALLY CONFORMAL KÄHLER MANIFOLDS

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Abstract. The purpose of the Nash embedding theorem was to take extrinsic help for studying the intrinsic Riemannian geometry. To realize this aim in actual practice there is a need for optimal relationships between the known intrinsic invariants and the main extrinsic invariants for Riemannian submanifolds. This paper aims to provide an optimal relationship for CR-warped product submanifolds of locally conformal Kähler manifolds.

1. Introduction

The concept of the warped product is one of the most significant generalizations of the cartesian product, which appeared in the mathematical and physical literature long before [2] under the name semi-reducible spaces [12]. The notion of warped product is important to both differential geometry and mathematical physics, particularly general relativity [8, 16]. The study of warped product submanifolds has become an active research topic in the differential geometry of submanifolds. The CR-warped product manifolds are likewise of major interest [3, 4, 5, 17]. According to Nash's embedding theorem, every Riemannian manifold can be isometrically immersed in some Euclidean space with sufficiently high codimension. In particular, every warped product manifold can be isometrically embedded as a Riemannian submanifold in some Euclidean space with sufficiently high codimension. The main difficulty in applying Nash's theorem is the larger codimension. This challenge can be addressed by establishing the optimal potential relationships between the known intrinsic invariants and the main extrinsic invariants of a submanifold. A great amount of work has been done in this direction. Certain general inequalities between the intrinsic and extrinsic invariants have been established by many authors

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[8, 14, 7, 15, 6]. Continuing this sequel of inequalities, this paper aims to establish the generalized Chen inequality for warped product CR-submanifolds of locally conformal Kähler manifolds using the Gauss equation. This paper is organized as follows:
Section 2 recalls some preliminary notions related to our primary objective. Section 3 comprises some necessary lemmas and results for warped product CR-submanifolds of locally conformal Kähler manifolds, which are needed in the latter sections. Finally, in section 4, the generalized Chen inequality for warped product CR-submanifolds of locally conformal Kähler manifolds is established along with the characterization of the equality case followed by the applications.

2. Preliminaries

Suppose (M, g) be an n-dimensional submanifold of a Riemannian manifold $(\bar{M}, \bar{g})$ of dimension m, where g is the induced metric on M and $\bar{g}$ is the Riemannian metric on $\bar{M}$. Denote by ∇ and $\bar{\nabla}$ the Levi-Civita connections of M and $\bar{M}$, respectively. Then *Gauss* and *Weingarten formulae* are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$
$$\bar{\nabla}_X Z = -A_Z X + D_X Z,$$

respectively, for tangent vectors $X, Y \in \Gamma(TM)$ and a normal vector field $Z \in \Gamma(TM^{\perp})$, where h denotes the *second fundamental form*, D the *normal connection* and A the *shape operator* of M in $\bar{M}$. The shape operator and second fundamental form are related by

$$g(A_Z X, Y) = g(h(X, Y), Z)$$

for any vector fields X, Y tangent to M and Z normal to M.
The *curvature tensor* $\bar{R}$ is a tensor field of type $(1, 3)$, given by

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z$$

for $X, Y, Z \in \Gamma(T\bar{M})$.
The *Riemannian curvature tensor* is a $(0, 4)$ tensor field, defined by

$$\bar{R}(X, Y, Z, W) = \bar{g}(\bar{R}(X, Y)Z, W)$$

for any $X, Y, Z, W \in \Gamma(T\bar{M})$. The Riemann tensor provides an intrinsic way of describing the curvature of a surface.
The Gauss equation is given by

$$R(X, Y, Z, W) = \bar{R}(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)) \tag{2.2}$$

for $X, Y, Z, W \in \Gamma(TM)$, where R and $\bar{R}$ are the curvature tensors of M and $\bar{M}$, respectively.
For any $x \in \bar{M}$, if $X, Y \in T_x\bar{M}$ are two linearly independent tangent vectors,

then the *sectional curvature* of the 2-plane section π, spanned by X and Y, is given by

$$\bar{K}(X \wedge Y) = \frac{\bar{g}(\bar{R}(X,Y)Y,X)}{\bar{g}(X,X)\bar{g}(Y,Y) - (\bar{g}(X,Y))^2}.$$

In addition, if X and Y are orthogonal unit vectors, then the preceding definition can be expressed as

$$\bar{K}(\pi) = \bar{K}(X \wedge Y) = \bar{g}(\bar{R}(X,Y)Y,X). \tag{2.3}$$

For a local orthonormal frame $\{e_1, e_2, ..., e_m\}$ of $\bar{M}$, the *scalar curvature* is another significant Riemannian intrinsic invariant given by

$$\begin{aligned} \bar{\tau}(T_x\bar{M}) &= \sum_{1\leq i<j\leq m} \bar{K}(e_i \wedge e_j) \\ &= \sum_{1\leq i<j\leq m} \bar{K}_{ij} \\ &= \frac{1}{2} \sum_{1\leq i\neq j\leq m} \bar{K}_{ij}. \end{aligned} \tag{2.4}$$

Definition 2.1. [8] A Riemannian metric $\check{g}$ on a complex manifold $\check{M}$ is called *Hermitian* if $\check{g}$ and the complex structure J such that $J^2 = -I$ are compatible, i.e.,

$$\check{g}(JU, JV) = \check{g}(U, V), \ \forall\, U, V \in \Gamma(T\check{M}).$$

The complex manifold equipped with Hermitian metric is called the *Hermitian manifold.*

The fundamental 2-form Ω of a Hermitian manifold $(\check{M}, \check{g}, J)$ is defined by

$$\Omega(U, V) = \check{g}(U, JV), \ U, V \in \Gamma(T\check{M}).$$

A Hermitian manifold is called *Kähler manifold* if its fundamental 2-form Ω is closed, i.e., $d\Omega = 0$. The corresponding metric is called Kähler metric. A Hermitian manifold is Kählerian if and only if J is parallel, i.e., $\nabla J = 0$.
A plane section of a Kähler manifold is called holomorphic if it is spanned by $\{U, JU\}$ for some non-null vector $U \in \Gamma(T\check{M})$. The sectional curvature $K(U \wedge JU)$ of a holomorphic section is called the *holomorphic sectional curvature* at U, it is denoted by $H(U)$. The holomorphic sectional curvature does not depend on the choice of U in the span $\{U, JU\}$.

Definition 2.2. [10] The Hermitian manifold $(\check{M}, J, \check{g})$ is called a *locally conformal Kähler (l.c.K.) manifold* if each point x of $\check{M}$ has an open neighborhood U with a positive differentiable function $\rho : U \to R$ such that the local metric

$$\check{g}_U = e^{-\rho}\check{g}_{|U}$$

is a Kählerian metric on U. If we take $U = M$, then the manifold M is said to be a *globally conformal Kähler manifold.*

Equivalently, $(\check{M}, J, \check{g})$ is l.c.K. manifold if and only if there exists a global closed 1-form α on $\check{M}$ such that

$$d\Omega = \alpha \wedge \Omega,$$

where α is the *Lee form* and its dual vector field λ is the *Lee vector field.* Further, $(\check{M}, J, \check{g})$ is l.c.K. manifold if and only if the following equation is satisfied

$$(\check{\nabla}_U J)V = \check{g}(\lambda, JV)U - \check{g}(\lambda, V)JU + \check{g}(JU, V)\lambda + \check{g}(U, V)J\lambda \tag{2.5}$$

$\forall\, U, V \in \Gamma(T\check{M})$ [13, 11].

Definition 2.3. [1] A submanifold M of a l.c.K. manifold $(\check{M}, J, \check{g})$ is called a *CR-submanifold* if there exists a differentiable distribution

$$\mathcal{H} : x \longmapsto \mathcal{H}_x \subset T_x M$$

on M satisfying the following conditions:

(i) $\mathcal{H}$ is holomorphic, i.e., $J\mathcal{H}_x = \mathcal{H}_x$, for each $x \in M$,
(ii) the complementary orthogonal distribution $\mathcal{H}^\perp : x \longmapsto \mathcal{H}_x^\perp \subset T_x M$ is totally real, i.e., $J\mathcal{H}_x^\perp \subset T_x^\perp M$ for each $x \in M$.

For a CR-submanifold M of a l.c.K. manifold $\check{M}$, the normal bundle $T^\perp M$ has direct sum decomposition as follows:

$$T^\perp M = J\mathcal{H}^\perp \oplus \mu,$$

where μ denotes the invariant normal subbundle of $T^\perp M$ under J.

Definition 2.4. [2] Let (N_1, g_{N_1}) and (N_2, g_{N_2}) be Riemannian manifolds of dimensions n_1 and n_2, respectively. Let $f : N_1 \to \mathbb{R}$ be a positive differentiable function. Consider the product manifold $N_1 \times N_2$ with projection maps $\pi_1 : N_1 \times N_2 \mapsto N_1$ and $\pi_2 : N_1 \times N_2 \mapsto N_2$. The *warped product* $M = N_1 \times_f N_2$ is the product manifold $N_1 \times N_2$ equipped with the Riemannian structure such that

$$\| Y \|^2 = \| d\pi_1 Y \|^2 + f^2(\pi_1(Y)) \| d\pi_2 Y \|^2$$

for any tangent vector $Y \in T_x M, x \in M$. Hence, we get $g = g_{N_1} + f^2 g_{N_2}$. The function f is called the warping function of the warped product manifold M. In case f is constant, the warped product is trivial.

Consider a local orthonormal frame $\{e_1, ..., e_{n_1}, e_{n_1+1}, ..., e_n\}$ of $\Gamma(TM)$ and let n_1, n_2 and n are dimensions of N_1, N_2 and M, respectively. The relationship between the warping function and the sectional curvature is given by [5, 9]

$$\sum_{\alpha=1}^{n_1} \sum_{\beta=n_1+1}^{m} K(e_\alpha \wedge e_\beta) = \frac{n_2 \Delta f}{f}. \tag{2.6}$$

Let $M = N_1 \times_f N_2$ be a warped product submanifold of $\bar{M}$. Choose a local orthonormal frame $\{e_1, ..., e_n, e_{n+1}, ..., e_m\}$ of $T\bar{M}$ such that $\{e_1, ..., e_n\}$ are tangent to M and $\{e_{n+1}, ..., e_m\}$ are normal to M. Then the *mean curvature vector* $\vec{H}$ is defined by

$$\vec{H} = \frac{1}{n}\ traceh = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).$$

In case $\vec{H} = 0$, M is called *minimal submanifold* and if the second fundamental form vanishes identically, then M is called a *totally geodesic* submanifold of $\bar{M}$. M is said to be *totally umbilical* in $\bar{M}$ if and only if $h(X,Y) = g(X,Y)\vec{H}$ for any $X, Y \in \Gamma(TM)$.
Let $h^r_{ij} = g(h(e_i, e_j), e_r)$ for $1 \leq i, j \leq n$ and $n+1 \leq r \leq m$. Hence, using (2.2) and (2.3), we get

$$K(e_i \wedge e_j) = \bar{K}(e_i \wedge e_j) + \sum_{r=n+1}^{m} (g(h^r_{ii}e_r, h^r_{jj}e_r) - g(h^r_{ij}e_r, h^r_{ij}e_r)).$$

Equivalently,

$$K(e_i \wedge e_j) = \bar{K}(e_i \wedge e_j) + \sum_{r=n+1}^{m} (h^r_{ii}h^r_{jj} - (h^r_{ij})^2). \tag{2.7}$$

By making use of (2.4) and (2.7), we obtain the following:

$$2\tau(T_xM) = 2\bar{\tau}(T_xM) + n^2 \parallel \vec{H} \parallel^2 - \parallel h \parallel^2, \tag{2.8}$$

where $\bar{\tau}$ denotes the scalar curvature of the n-plane T_xM in the ambient manifold $\bar{M}$.
Consider warped product manifold $M = N_1 \times_f N_2$. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ denote the distributions given by the vectors tangent to leaves and fibers, respectively. Thus, $\mathcal{D}_1$ is obtained from tangent vectors of N_1 via the horizontal lift and $\mathcal{D}_2$ is obtained by tangent vectors of N_2 via the vertical lift. Let $\psi : M \to \bar{M}$ be an isometric immersion of $N_1 \times_f N_2$ into an arbitrary Riemannian manifold $\bar{M}$. Then the immersion ψ is called *mixed totally geodesic* if $h(U, V) = 0$ for any U in $\mathcal{D}_1$ and V in $\mathcal{D}_2$ [5]. In particular, if h_1 and h_2 denote the restrictions of h to N_1 and N_2, respectively, then we call h_i the *partial second fundamental form* of ψ ; $i = 1, 2$. In a similar way, *partial mean curvature vectors* $\vec{H}_1$ and $\vec{H}_2$ are defined by partial traces as follows:

$$\vec{H}_1 = \frac{1}{n_1}\sum_{\alpha=1}^{n_1} h(e_\alpha, e_\alpha),\ \ \vec{H}_2 = \frac{1}{n_2}\sum_{\beta=n_1+1}^{n_1+n_2} h(e_\beta, e_\beta)$$

for some orthonormal frame fields $\{e_1, ..., e_{n_1}\}$ and $\{e_{n_1+1}, ..., e_{n_1+n_2}\}$ of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively.

Definition 2.5. [14, 15] An immersion $\psi : N_1 \times_f N_2 \to \bar{M}^m$ is called $\mathcal{D}_i$-*totally geodesic* if the partial second fundamental form h_i vanishes identically and $\mathcal{D}_i$-*totally umbilical* if, for all $U, V \in \mathcal{D}_i$, we have $h(U, V) = g(U, V)Z$ for some normal vector Z. It is called $\mathcal{D}_i$*-minimal* if the partial mean curvature vector $\vec{H}_i$ vanishes, for $i = 1, 2$.

3. Warped product CR-submanifolds of l.c.K. manifolds

In this section, we derive some important results and lemmas for Warped product CR-submanifolds of l.c.K. manifolds that are required in the later section to prove the main results. Our main aim is to prove the existence of $\mathcal{D}_i$-minimal warped product submanifolds in l.c.K. manifolds.

Lemma 3.1. *Let $M = N_T \times_f N_\perp$ be an n-dimensional CR-warped product submanifold of an arbitrary l.c.K. manifold $\check{M}$ of dimension m. Then the following hold:*

(i) $g(h(U,U), JW) = 0$,
(ii) $g(h(U,U), \eta) = -g(h(JU, JU), \eta)$

for all vector fields U, W and η tangent to N_T, $N_\perp$ and μ, respectively.

Proof. For $U, V \in TN_T, W \in TN_\perp$, we have

$$J\nabla_U W + Jh(U,W) = -A_{JW}U + D_U JW. \tag{3.1}$$

Taking the inner product of (3.1) with JV, we have

$$g(\nabla_U W, V) = -g(A_{JW}U, JV) = -g(h(U, JV), JW). \tag{3.2}$$

We know that for a warped product $M = N_T \times_f N_\perp$, N_T is totally geodesic in M. Hence, we have $g(\nabla_U W, V) = 0$. Combining this with (3.2), we get (i). By making use of (2.1) and (2.5), we obtain

$$\begin{aligned}\nabla_U JV + h(U, JV) - J\nabla_U V - Jh(U,V) =& g(\lambda, JV)U - g(\lambda, V)JU + g(JU, V)\lambda \\ &+ g(U,V)J\lambda.\end{aligned}$$

Taking the inner product with $J\eta$; $\eta \in \Gamma(\mu)$ in the above equation, we get

$$g(h(U, JV), J\eta) - g(Jh(U,V), J\eta) = 0,$$

which implies

$$g(h(U, JV), J\eta) = g(h(U,V), \eta).$$

Substituting $V = U$ in the above equation, we get

$$g(h(U, JU), J\eta) = g(h(U,U), \eta). \tag{3.3}$$

Interchanging U with JU, (3.3) becomes

$$-g(h(JU, U), J\eta) = g(h(JU, JU), \eta),$$

$$g(h(U, JU), J\eta) = -g(h(JU, JU), \eta). \tag{3.4}$$

Using (3.3) and (3.4), we obtain (ii). ☐

Theorem 3.2. *Let $\psi : M = N_1 \times_f N_2 \to \check{M}$ be an isometric immersion of the warped product submanifold M into a l.c.K. manifold $\check{M}$. Then, we have the following:*

$$(i)\ \tau(T_xM) = \frac{n_2\Delta f}{f} + \sum_{r=n+1}^{m}\{\sum_{1\leq\alpha<\beta\leq n_1}(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2) + \sum_{n_1+1\leq\gamma<\delta\leq n}(h^r_{\gamma\gamma}h^r_{\delta\delta} - (h^r_{\gamma\delta})^2)\} + \check{\tau}(T_xN_1) + \check{\tau}(T_xN_2)$$

$$(ii)\ \sum_{1\leq\alpha\neq\beta\leq k}(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2) = (\sum_{\alpha=1}^{k}h^r_{\alpha\alpha})^2 - \sum_{\alpha,\beta=1}^{k}(h^r_{\alpha\beta})^2,$$

where n_1, n_2, n and m are the dimensions of N_1, N_2, M and $\check{M}$, respectively.

Proof. We know that the scalar curvature of the manifold M is given by

$$\begin{aligned}\tau(T_xM) &= \sum_{1\leq a,b\leq n} K_{ab}\\ &= \sum_{\alpha=1}^{n_1}\sum_{\gamma=n_1+1}^{n} K_{\alpha\gamma} + \sum_{1\leq\alpha<\beta\leq n_1} K_{\alpha\beta} + \sum_{n_1+1\leq\gamma<\delta\leq n} K_{\gamma\delta}.\end{aligned}$$

Also, from (2.6), we have

$$\sum_{\alpha=1}^{n_1}\sum_{\gamma=n_1+1}^{n} K(e_\alpha\wedge e_\gamma) = n_2\frac{\Delta f}{f},$$

where $\{e_1, ..., e_{n_1}, e_{n_1+1}, ..., e_n\}$ is the local orthonormal frame of $\Gamma(TM)$ and n_1, n_2 and n are the dimensions of N_1, N_2 and M, respectively. Combining the above two equations and using (2.7), we get,

$$\begin{aligned}\tau(T_xM) &= n_2\frac{\Delta f}{f} + \tau(T_xN_1) + \tau(T_xN_2)\\ &= n_2\frac{\Delta f}{f} + \sum_{r=n+1}^{m}\sum_{1\leq\alpha<\beta\leq n_1}(h^r_{\alpha\alpha}h^r_{\beta\beta} - (h^r_{\alpha\beta})^2) + \check{\tau}(T_xN_1) +\\ &\quad \sum_{r=n+1}^{m}\sum_{n_1+1\leq\gamma<\delta\leq n}(h^r_{\gamma\gamma}h^r_{\delta\delta} - (h^r_{\gamma\delta})^2) + \check{\tau}(T_xN_2).\end{aligned}$$

Hence, we get (i).
Next, we have

$$\begin{aligned}\sum_{1\leq\alpha\neq\beta\leq k}(h^r_{\alpha\alpha}h^r_{\beta\beta}-(h^r_{\alpha\beta})^2) &= \sum_{1\leq\alpha\neq\beta\leq k}h^r_{\alpha\alpha}h^r_{\beta\beta}-\sum_{1\leq\alpha\neq\beta\leq k}(h^r_{\alpha\beta})^2\\ &=\Big\{\sum_{1\leq\alpha\neq\beta\leq k}h^r_{\alpha\alpha}h^r_{\beta\beta}+\sum_{\alpha=1}^{k}(h^r_{\alpha\alpha})^2\Big\}\\ &-\Big\{\sum_{1\leq\alpha\neq\beta\leq k}(h^r_{\alpha\beta})^2+\sum_{\alpha=1}^{k}(h^r_{\alpha\alpha})^2\Big\}.\end{aligned}\tag{3.5}$$

Using binomial theorem, we deduce that

$$\sum_{1\leq\alpha\neq\beta\leq k}h^r_{\alpha\alpha}h^r_{\beta\beta}+\sum_{\alpha=1}^{k}(h^r_{\alpha\alpha})^2=\Big(\sum_{\alpha=1}^{k}h^r_{\alpha\alpha}\Big)^2\tag{3.6}$$

and

$$\sum_{1\leq\alpha\neq\beta\leq k}(h^r_{\alpha\beta})^2+\sum_{\alpha=1}^{k}(h^r_{\alpha\alpha})^2=\sum_{\alpha,\beta=1}^{k}(h^r_{\alpha\beta})^2.\tag{3.7}$$

Using (3.5), (3.6) and (3.7), we get (ii).

□

Theorem 3.3. *Let $\psi : M = N_1 \times_f N_2 \to \check{M}$ be an isometric immersion of an n-dimensional warped product submanifold M into a l.c.K. manifold $\check{M}$ of dimension m. Then, the following relation holds:*

$$\begin{aligned}\parallel h\parallel^2 &= 2\check{\tau}(T_xM)-2\check{\tau}(T_xN_1)-2\check{\tau}(T_xN_2)-2\frac{n_2\Delta f}{f}+n^2\parallel\vec{H}\parallel^2+\\ &\sum_{r=n+1}^{m}\Big[\Big\{\sum_{\alpha,\beta=1}^{n_1}(h^r_{\alpha\beta})^2+\sum_{\gamma,\delta=n_1+1}^{n}(h^r_{\gamma\delta})^2\Big\}-\Big\{(\sum_{\alpha=1}^{n_1}h^r_{\alpha\alpha})^2+(\sum_{\gamma=n_1+1}^{n}h^r_{\gamma\gamma})^2\Big\}\Big],\end{aligned}$$

where n_1 and n_2 are the dimensions of N_1 and N_2 respectively.

Proof. Using (2.8) and part (i) of theorem (3.2), we obtain

$$
\begin{aligned}
\| h \|^2 = & \, 2\check{\tau}(T_xM) - \frac{2n_2\Delta f}{f} - 2\sum_{r=n+1}^{m}\Big\{\sum_{1\le\alpha<\beta\le n_1}(h^r_{\alpha\alpha}h^r_{\beta\beta}-(h^r_{\alpha\beta})^2)+ \\
& \sum_{n_1+1\le\gamma<\delta\le n}(h^r_{\gamma\gamma}h^r_{\delta\delta}-(h^r_{\gamma\delta})^2)\Big\} - 2\check{\tau}(T_xN_1) - 2\check{\tau}(T_xN_2) + n^2 \| \vec{H} \|^2 \\
= & \, 2\check{\tau}(T_xM) - \frac{2n_2\Delta f}{f} - \sum_{r=n+1}^{m}\Big\{\sum_{1\le\alpha\ne\beta\le n_1}(h^r_{\alpha\alpha}h^r_{\beta\beta}-(h^r_{\alpha\beta})^2)+ \\
& \sum_{n_1+1\le\gamma\ne\delta\le n}(h^r_{\gamma\gamma}h^r_{\delta\delta}-(h^r_{\gamma\delta})^2)\Big\} - 2\check{\tau}(T_xN_1) - 2\check{\tau}(T_xN_2) + n^2 \| \vec{H} \|^2 .
\end{aligned} \tag{3.8}
$$

In view of part (ii) of theorem (3.2), (3.8) reduces to

$$
\begin{aligned}
\| h \|^2 = & \, 2\check{\tau}(T_xM) - \frac{2n_2\Delta f}{f} - \sum_{r=n+1}^{m}\Big\{\Big(\sum_{\alpha=1}^{n_1}h^r_{\alpha\alpha}\Big)^2 - \sum_{\alpha,\beta=1}^{n_1}(h^r_{\alpha\beta})^2\Big\} - \\
& \sum_{r=n+1}^{m}\Big\{\Big(\sum_{\gamma=n_1+1}^{n}h^r_{\gamma\gamma}\Big)^2 - \sum_{\gamma,\delta=n_1+1}^{n}(h^r_{\gamma\delta})^2\Big\} - 2\check{\tau}(T_xN_1) - 2\check{\tau}(T_xN_2) \\
& + n^2 \| \vec{H} \|^2 \\
= & \, 2\check{\tau}(T_xM) - 2\check{\tau}(T_xN_1) - 2\check{\tau}(T_xN_2) - 2\frac{n_2\Delta f}{f} + n^2 \| \vec{H} \|^2 + \\
& \sum_{r=n+1}^{m}\Big[\Big\{\sum_{\alpha,\beta=1}^{n_1}(h^r_{\alpha\beta})^2 + \sum_{\gamma,\delta=n_1+1}^{n}(h^r_{\gamma\delta})^2\Big\} - \Big\{(\sum_{\alpha=1}^{n_1}h^r_{\alpha\alpha})^2 + (\sum_{\gamma=n_1+1}^{n}h^r_{\gamma\gamma})^2\Big\}\Big].
\end{aligned}
$$

Hence, we get the assertion. □

Lemma 3.4. [14] *Let ψ be a $\mathcal{D}_2$-minimal isometric immersion of a warped product submanifold $M = N_1 \times_f N_2$ of dimension n into a l.c.K manifold $\check{M}$ of dimension m. If N_2 is totally umbilical in $\check{M}$, then ψ is $\mathcal{D}_2$-totally geodesic.*

Some authors established that a large class of warped product submanifolds possess $\mathcal{D}_T$ minimality. Some of these warped product submanifolds are proven to have this geometric property [15, 14]. In our study, warped product CR-submanifolds of l.c.K. manifolds are also found to possess the property of $\mathcal{D}_T$ minimality. Here, we prove the existence of $\mathcal{D}_T$-minimal warped product CR-submanifolds in l.c.K. manifolds.

Theorem 3.5. *Every CR-warped product submanifold of the type $M = N_T \times_f N_\perp$ is $\mathcal{D}_T$-minimal in l.c.K. manifold $\check{M}$ when the Lee vector field λ is tangent to M, where the dimensions of N_1, N_2, M and $\check{M}$ are n_1, n_2, n and m, respectively.*

Proof. We can decompose $T\check{M}$ as

$$T\check{M} = TN_T \oplus TN_\perp \oplus JTN_\perp \oplus \mu.$$

Based on the above decomposition, we can construct the local orthonormal frames as follows:

$$TN_T : \{e_1, ..., e_l, e_{l+1} = Je_1, ..., e_{n_1} = e_{2l} = Je_l\},$$
$$TN_\perp : \{e_{n_1+1} = e_1^\star, ..., e_{n_1+n_2} = e_n = e_p^\star\},$$
$$JTN_\perp : \{e_{n+1} = Je_1^\star, ..., e_{n+n_2} = Je_p^\star\},$$
$$\mu : \{e_{n+n_2+1} = \bar{e_1}, ..., e_{2m} = \bar{e}_{2d=\kappa}\}.$$

Hence, the following is the local orthonormal frame of the l.c.K. manifold $\check{M}$ $\{e_1, ..., e_l, e_{l+1} = Je_1, ..., e_{2l} = Je_l = e_{n_1}, e_{n_1+1} = e_1^\star, ..., e_{n_1+n_2} = e_p^\star = e_n, e_{n+1} = Je_1^\star, ..., e_{n+n_2} = Je_p^\star, e_{n+n_2+1} = \bar{e_1}, ..., e_{2m} = \bar{e}_{2d=\kappa}\}$.
Next, we have

$$\sum_{r=n+1}^{2m} \sum_{\alpha=1}^{n_1} h^r_{\alpha\alpha} = \sum_{r=n+1}^{2m-\kappa} \sum_{\alpha=1}^{n_1} h^r_{\alpha\alpha} + \sum_{r=n+p+1}^{2m} \sum_{\alpha=1}^{n_1} h^r_{\alpha\alpha}.$$

Using statement (i) of lemma 3.1, the first summation on R.H.S. of the above equation vanishes. Hence we get

$$\begin{aligned}\sum_{r=n+1}^{2m} \sum_{\alpha=1}^{n_1} h^r_{\alpha\alpha} &= \sum_{r=n+p+1}^{2m} \sum_{\alpha=1}^{n_1} h^r_{\alpha\alpha} \\ &= \sum_{r=n+p+1}^{2m} \sum_{\alpha=1}^{n_1} g(h(e_\alpha, e_\alpha), e_r).\end{aligned}$$

Expanding the summation on R.H.S. using the orthonormal frames given above, we obtain

$$\begin{aligned}\sum_{r=n+1}^{2m} \sum_{\alpha=1}^{n_1} h^r_{\alpha\alpha} &= \sum_{r=n+p+1}^{2m} \Big[\sum_{\alpha=1}^{l} g(h(e_\alpha, e_\alpha), e_r) + \sum_{\alpha=l+1}^{2l} g(h(e_\alpha, e_\alpha), e_r)\Big] \\ &= \sum_{r=n+p+1}^{2m} \Big[\sum_{\alpha=1}^{l} g(h(e_\alpha, e_\alpha), e_r) + \sum_{\alpha=1}^{l} g(h(Je_\alpha, Je_\alpha), e_r)\Big] \\ &= \sum_{r=n+p+1}^{2m} \sum_{\alpha=1}^{l} \Big[g(h(e_\alpha, e_\alpha), e_r) + g(h(Je_\alpha, Je_\alpha), e_r)\Big].\end{aligned}$$

From part (ii) of lemma 3.1, we get the following

$$\sum_{r=n+1}^{2m} \sum_{\alpha=1}^{n_1} h^r_{\alpha\alpha} = \sum_{r=n+p+1}^{2m} \sum_{\alpha=1}^{l} \Big[g(h(e_\alpha, e_\alpha), e_r) - g(h(e_\alpha, e_\alpha), e_r)\Big] = 0.$$

Hence, the coefficients of the second fundamental form $h^r_{\alpha\alpha}$ vanish under summation for $\alpha = \{1, ..., n_1\}, r = \{n+1, ..., 2m\}$. This implies the vanishing of the partial mean curvature vector $\vec{H}$. Therefore, we get the assertion. $\square$

4. Generalized Chen inequality for CR-warped products in locally conformal Kähler manifolds

In this section, we derive the main result of this paper. We establish generalized Chen inequality for CR-warped products in l.c.K. manifolds using the Gauss equation and characterize the equality case of the inequality along with applications.

Theorem 4.1. *Let $\psi : M = N_T \times_f N_\perp \to \check{M}$ be an isometric immersion of an n-dimensional warped product CR-submanifold M into a l.c.K. manifold $\check{M}$ of dimension m. Then, we have*

$$\| h \|^2 \geq 2(\check{\tau}(T_xM) - \check{\tau}(T_xN_T) - \check{\tau}(T_xN_\perp)) - \frac{2n_2\Delta f}{f}. \tag{4.1}$$

The equality holds identically if and only if N_T is a totally geodesic submanifold of $\check{M}$, $N_\perp$ is totally umbilical submanifold of $\check{M}$ and M is minimal in $\check{M}$.

Proof. Using theorem (3.3) and theorem (3.5), we have

$$\begin{aligned}\| h \|^2 = & 2\check{\tau}(T_xM) - \frac{2n_2\Delta f}{f} - \check{\tau}(T_xN_T) - \check{\tau}(T_xN_\perp) + n^2 \| \vec{H} \|^2 + \\ & \sum_{r=n+1}^{m} \Big\{ \sum_{\alpha,\beta=1}^{n_1} (h^r_{\alpha\beta})^2 + \sum_{\gamma,\delta=n_1+1}^{n} (h^r_{\gamma\delta})^2 \Big\} - \sum_{r=n+1}^{m} \Big(\sum_{\gamma=n_1+1}^{n} h^r_{\gamma\gamma} \Big)^2 \\ & \geq 2\check{\tau}(T_xM) - \frac{2n_2\Delta f}{f} - \check{\tau}(T_xN_T) - \check{\tau}(T_xN_\perp) + n^2 \| \vec{H} \|^2 - \\ & \sum_{r=n+1}^{m} \Big(\sum_{\gamma=n_1+1}^{n} h^r_{\gamma\gamma} \Big)^2.\end{aligned}$$

We know that for $\mathcal{D}_T$-minimal warped product CR-submanifolds, the last term in the R.H.S. of the above inequality equals n^2 times the squared norm of the mean curvature vector. Hence, we obtain the desired inequality.
It is clear from (4.1) that the equality holds if and only if $h(\mathcal{D}_T, \mathcal{D}_T) = 0$ and $h(\mathcal{D}_\perp, \mathcal{D}_\perp) = 0$. For a warped product manifold $M = N_T \times_f N_\perp$, N_T is a totally geodesic submanifold of M and $N_\perp$ is a totally umbilical submanifold of M. Therefore, from the above discussion and by lemma (3.4), it is clear that the equality in (4.1) holds if and only if N_T is totally geodesic submanifold of $\check{M}$, $N_\perp$ is totally umbilical submanifold of $\check{M}$, and M is a minimal submanifold of $\check{M}$. $\square$

4.1. Applications in Physics

It is well-known that

$$\Delta f = \Delta(lnf) + \parallel \nabla(lnf) \parallel^2 .$$

Therefore, the preceding inequality has a significant role in physics, as it gives a direct application of the law of conservation of energy which is a fundamental concept. From the wave equation, we have

$$0 = (u_{tt} - c^2 \Delta u) u_t = \Big(\frac{1}{2} u_t^2 + \frac{1}{2} c^2 |\nabla u|^2 \Big)_t - c^2 \nabla . (u_t \nabla u).$$

On integrating the above identity, the integral of the last term will vanish if the derivative of $u(x, t)$ tends to zero as $|x| \to \infty$. We obtain

$$0 = \iiint \frac{\partial}{\partial t} \Big(\frac{1}{2} u_t^2 + \frac{1}{2} c^2 |\nabla u|^2 \Big) dx.$$

Therefore, the total energy

$$E = \frac{1}{2} \iiint (u_t^2 + c^2 |\nabla u|^2) dx$$

is a constant. Here the first term represents the kinetic energy and the second term gives the potential energy.

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References

[1] A. Bejancu, *Geometry of CR-submanifolds (Vol. 23)*, Springer Science and Business Media, 2012.

[2] R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Am. Math. Soc. **145** (1969), 1–49.

[3] B. Y. Chen, *Twisted Product CR-submanifolds in Kähler manifolds*, Tamsui Oxf. J. Math. Sci **16** (2000), no. 2, 105–121.

[4] B. Y. Chen, *Geometry of warped product CR-submanifolds in Kähler manifolds*, Monatsh. Math. **133** (2001), no. 3, 177–195.

[5] B. Y. Chen, *Geometry of warped products as Riemannian submanifolds and related problems*, Soochow J. Math. **28** (2002), no. 2, 125–156.

[6] B. Y. Chen, *On isometric minimal immersions from warped products into real space forms*, Proc. Edinburgh Math. Soc. **45** (2002), no. 3, 579–587.

[7] B. Y. Chen, *Another general inequality for CR-warped products in complex space forms*, Hokkaido Math. J. **32** (2003), no. 2, 415–444.

[8] B. Y. Chen, *Pseudo-Riemannian geometry, δ-invariants and applications*, World Scientific, 2011.
[9] B. Y. Chen, *Geometry of warped product submanifolds: a survey*, J. Adv. Math. Stud. **6** (2013), no. 2, 1–8.
[10] S. Dragomir and L. Ornea, *Locally conformal Kähler geometry* (No. 155), Springer Science and Business Media, 1998.
[11] T. Kashiwada, *Some properties of locally conformal Kähler manifolds*, Hokkaido Math. J. **8** (1979), no. 2, 191–198.
[12] G. Kruckovic, *On semi-reducible Riemannian spaces*, In Dokl. Acad. Nauk. SSSR **115** (1957), 862–865.
[13] K. Matsumoto, *Locally conformal Kähler manifolds and their submanifolds*, Memoriile Sectiilor Stiintifice **14** (1991), 1–49.
[14] A. Mustafa, C. Ozel, P. Linker, M. Sati and A. Pigazzini, *A General inequality for warped product CR-Submanifolds of Kähler Manifolds*, Hacet. J. Math. Stat. **52** (2021), no. 1, 1–16.
[15] A. Mustafa, S. Uddin and B. R. Wong, *Generalized inequalities on warped product submanifolds in nearly trans-Sasakian manifolds*, J Inequal Appl. **2014** (2014), no. 1, 1–16.
[16] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.
[17] G. Shanker, R. Kaur, and A. Yadav, *Study on warped product of screen real lightlike submanifolds of a Golden semi-Riemannian manifold*, Balk. J. Geom. its Appl. **27** (2022), no. 1, 152–161.

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