Honam Mathematical J. **46** (2024), No. 1, pp. 38–46
https://doi.org/10.5831/HMJ.2024.46.1.38

A NOTE ON THE EXTENSION OF ε-ISOMETRIES ON THE UNIT SPHERE OF BANACH SPACES

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Abstract. Let X, Y be Banach spaces, S_X and S_Y be the unit sphere of X and Y, respectively. Let $f_0 : S_X \to S_Y$ be ε-isometry for some $\varepsilon \geq 0$. In this paper, we show that there is an extension $f : X \to Y$ of f_0 such that f is linear.

1. Introduction

Isometry mappings are very important because it has properties that preserve continuity and injectivity. Isometric mapping is increasingly becoming special since every isometry is affine [15]. In other words, an isometry is just a translation of linear mapping. With this fact, comes the term ε-isometry $f : X \to Y$ which is defined for all $\eta, \xi \in X$ as

$$|\|f(\eta) - f(\xi)\| - \|\eta - \xi\|| \leq \varepsilon.$$

This map f is *standard* if $f(0)$=0. With this definition, it is clear that 0-isometry is nothing but an isometry, so the problem with ε-isometry mapping becomes interesting for $\varepsilon > 0$.

Assume that $U{:}X{\to}Y$ is an isometry. With the definition of ε-isometry above, it is natural that the question arises, "If there is an ε-isometry f, is there an isometry U and a constant k such that the furthest distance from the mapping f and U is $k\varepsilon$, or mathematically

$$\|f(\eta) - U(\eta)\| \leq k\varepsilon$$

for all $\eta \in X$?". Hyers and Ulam [14] first posed this problem and found that for a given standard surjective ε-isometry f there is a surjective isometry mapping U and k=10, where X and Y are Euclidean spaces. If $X = Y = \mathrm{L}_p$ (0,1), 1<p<∞, then the value of k was equal 12 [2]. After some time, finally, Gruber [13] first generalized for any Banach space, and Gevirtz [12] found the value of k=5. This constant was sharpened by Omladič and Šemrl to 2 [17]. In the studies mentioned just now, f is assumed to be a surjective mapping.

Received May 04, 2023. Accepted September 13, 2023.
2020 Mathematics Subject Classification. 46B04, 47A20.
Key words and phrases. Banach space, extension, ε-isometry, linearity, unit sphere.
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In contrast to Mazur-Ulam who provides the surjectivity condition of U, Figiel [11] showed that for every (non-surjective) isometry U, there is $F : \overline{span}U(\eta) \to X$ which is a bounded linear operator such that $FU = Id_X$. Similar to the previous one, a new problem arises, namely "is there a positive constant k and a linear mapping F such that $\|Ff(\eta) - \eta\| \leq k\varepsilon$ is true for any (non-surjective) ε-isometry f between two Banach spaces? ". However, by a simple counterexample Qian showed that if the ε-isometry f is a mapping from an uncomplemented subspace X of a separable Banach space Y into Y, then no F can be found [19]. Furthermore, he also concluded that if $X = Y = L_p$ spaces, $1<p<\infty$, then there is a linear mapping of F with $\|F\|=1$ such that $\|Ff(\eta) - \eta\| \leq 6\varepsilon$. Qian's conclusion provides an opportunity to conduct research related to (non-surjective) ε-isometries for any Banach spaces.

Cheng, et. al.[5] first showed that the stability of any ε-isometry mapping can be weakened by using a weak topology. Instead of norm, weak topology uses the members of dual space to build the basis topology [16]. Research related to the weak stability of ε-isometry can be found in [3, 6, 21, 22, 27].

Tingley [25] asked the question of whether a surjective isometry on a unit sphere is a restriction of the whole space? Many results have been given to answer Tingley's question, one of which is Ding and Li showing that any surjective isometry on unit spheres of l_∞- sum of strictly convex normed spaces can be extended to linear isometry on the space. Recently, Vestfrid showed that the ε-isometry in the unit sphere of l_2^n and l_∞^n can be approximated by linear isometry [26].

Based on the findings of Ding-Li and Vestfrid, this paper will provide important properties of ε-isometry mapping on the unit sphere of Banach spaces. We denote S_X as a unit sphere of Banach space X.

2. Preliminaries

This section will contain some of the basic concepts that will be used in the following discussion. Note that the Mazur-Ulam Theorem holds only for real Banach spaces. Indeed, the mapping $U(\eta) = \bar{\eta}$ is an isometry which is not affine. Therefore, X and Y always denote real Banach spaces.

Definition 2.1. *Let X and Y be real Banach spaces. A mapping $f : X \to Y$ is called an ε-isometry if*

$$|\|f(\eta) - f(\xi)\| - \|\eta - \xi\|| \leq \varepsilon.$$

for all $\eta, \xi \in X$

Definition 2.2. *([7] Definition 1.1) Let X be real normed space. The angle $A(\eta,\xi)$ between $\eta,\xi\in X$ is defined as*

$$A(\eta,\xi)=arccos\left[\frac{2-\left\|\frac{\eta}{\|\eta\|}-\frac{\xi}{\|\xi\|}\right\|^2}{2}\right]$$

Nabavi Sales provided a generalization of Defintion 2.2 which is used to determine the characteristics of Hilbert spaces [23]. The interesting thing is that the angle between two points in Banach spaces is determined by the restriction of those points on the unit sphere. Thus $A(\eta,\xi)$ is only depends on $\ell(\eta,\xi)=\left\|\frac{\eta}{\|\eta\|}-\frac{\xi}{\|\xi\|}\right\|$. By this definition, it is clear that 1) $\ell(\eta,\xi)=0$ if and only if $\eta=0$ or $\xi=0$; 2) $\ell(\alpha\eta,\beta\xi)=\ell(\eta,\xi)$ for all $\alpha,\beta>0$; and 3) $\ell:(X-\{0\})^2\to[0,2]$ is a continuous mapping.

The following lemma can be found in Dimminnie, et al. (Theorem 2.4) [7] and Freese, et al. (Theorem 2.1) [8].

Lemma 2.3. *Let X be a normed space and $\eta,\xi\in X$. If $\zeta\in X$, $\zeta=\alpha\eta+\beta\xi$ for $\alpha,\beta>0$ and $\|\eta\|=\|\xi\|>0$, then $\|\zeta-\eta\|\le\|\xi-\eta\|$. If $\zeta\in S_X$, then either $\|\zeta-\eta\|<\|\xi-\eta\|$ or $\|\xi+\eta\|<\|\zeta+\eta\|$.*

Lemma 2.3 says that the length of a vector obtained from a linear combination of two vectors, when subtracted by one of its constituent vectors, will always be smaller than the length of subtraction between the two constituent vectors. Let $\zeta=\alpha\frac{\eta}{\|\eta\|}+\beta\frac{\xi}{\|\xi\|}$ such that $\zeta\in S_X$, then Lemma 2.3 says that

$$\left\|\zeta-\frac{\eta}{\|\eta\|}\right\|\le\left\|\frac{\xi}{\|\xi\|}-\frac{\eta}{\|\eta\|}\right\|.$$

Since $\zeta,\frac{\eta}{\|\eta\|},\frac{\xi}{\|\xi\|}\in S_X$, we have $\ell\left(\zeta,\frac{\eta}{\|\eta\|}\right)\le\ell\left(\frac{\xi}{\|\xi\|},\frac{\eta}{\|\eta\|}\right)$. This simple inspection will be used in the proof of Lemma 3.2.

3. Main Results

In this section, we will show that an ε-isometry on the unit sphere of real Banach spaces can be extended to whole spaces by deploying norm topology.

As in Definition 2.2, instead of $A(\eta,\xi)$ we will use $\ell(\eta,\xi)$ which is appeared in the following lemma.

Lemma 3.1. *Let X and Y be real Banach spaces and $f_0:S_X\to S_Y$ be ε-isometry. Then the positive homogeneous extension $f:X\to Y$ of f_0 satisfies*

$$\|f(\eta)-f(\xi)\|\le(3+3\varepsilon)\|\eta-\xi\|$$

and

$$\|f(\eta)+f(\xi)-f(\eta+\xi)\| \leq (4+4\varepsilon)\min\{\|\eta\|,\|\xi\|\}\,\ell(\eta,\xi)$$

for all $\eta, \xi \in X$.

Proof. By classical rule and definition of ε-isometry we have

$$\begin{aligned}
\|f(\eta)-f(\xi)\| &\leq \left\|f(\eta)-\|\eta\|\,f\left(\frac{\xi}{\|\xi\|}\right)\right\| + \left\|\|\eta\|\,f\left(\frac{\xi}{\|\xi\|}\right)-f(\xi)\right\| \\
&= \|\eta\|\left\|f\left(\frac{\eta}{\|\eta\|}\right)-f\left(\frac{\xi}{\|\xi\|}\right)\right\| + |\|\eta\|-\|\xi\||\left\|f\left(\frac{\xi}{\|\xi\|}\right)\right\| \\
&\leq \|\eta\|\left(\left\|\frac{\eta}{\|\eta\|}-\frac{\xi}{\|\xi\|}\right\|+\varepsilon\right) + |\|\eta\|-\|\xi\||\,(1+\varepsilon) \\
&\leq \|\eta-\xi\| + \left\|\xi-\|\eta\|\,\frac{\xi}{\|\xi\|}\right\| + \|\eta\|\,\varepsilon + \|\eta-\xi\|\,(1+\varepsilon) \\
&\leq (3+3\varepsilon)\,\|\eta-\xi\|
\end{aligned}$$

.

Furthermore,

$$\begin{aligned}
\|f(\eta)+f(\xi)-f(\eta+\xi)\| &\leq \left\|f(\eta)-f\left(\frac{\|\eta\|}{\|\xi\|}\xi\right)\right\| + \left\|f\left(\frac{\|\eta\|}{\|\xi\|}\xi+\xi\right)-f(\eta+\xi)\right\| \\
&\leq \left\|\eta-\frac{\|\eta\|}{\|\xi\|}\xi\right\| + \varepsilon + (3+3\varepsilon)\left\|\frac{\|\eta\|}{\|\xi\|}\xi-\eta\right\| \\
&= (4+4\varepsilon)\left\|\frac{\|\eta\|}{\|\xi\|}\xi-\eta\right\|.
\end{aligned}$$

Similarly, we get

$$\|f(\eta)+f(\xi)-f(\eta+\xi)\| \leq (4+4\varepsilon)\left\|\xi-\frac{\|\xi\|}{\|\eta\|}\eta\right\|.$$

Therefore

$$\|f(\eta)+f(\xi)-f(\eta+\xi)\| \leq (4+4\varepsilon)\min\{\|\eta\|,\|\xi\|\}\,\ell(\eta,\xi)$$

which completes the proof. □

Lemma 3.2. *Let* $\eta, \xi \in X$ *with* $\|\eta\| \leq \|\xi\|$ *and* $\angle(\eta,\xi)$ *be the angle between* η *and* ξ. *If* $0 < \angle(\eta,\xi) \leq \frac{\pi}{2}$, *there is a pair sequence* η_n *and* ξ_n *such that* $\lim_{n\to\infty} \ell(\eta_n,\xi_n) = 0$.

Proof. Put $\eta_1 = \eta + \frac{\|\eta\|}{\|\xi\|}\xi$ and $\xi_1 = \xi - \frac{\|\eta\|}{\|\xi\|}\xi$ with $\|\eta_1\| \leq \|\xi_1\|$. Then $\eta_1 + \xi_1 = \eta + \xi$ and $\|\eta_1\| + \|\xi_1\| \leq \|\eta\| + \|\xi\|$. Repeating this process will give

$$\eta_{n+1} = \eta_n + \frac{\|\eta_n\|}{\|\xi_n\|}\xi_n$$

and

$$\xi_{n+1} = \xi_n - \frac{\|\eta_n\|}{\|\xi_n\|}\xi_n$$

with $\|\eta_{n+1}\| \leq \|\xi_{n+1}\|$, $\eta_{n+1} + \xi_{n+1} = \eta + \xi$ and $\|\eta_{n+1}\| + \|\xi_{n+1}\| \leq \|\eta\| + \|\xi\|$. Equivalently,

$$\eta_{n+1} = \|\eta_n\| \left(\frac{\eta_n}{\|\eta_n\|} + \frac{\xi_n}{\|\xi_n\|} \right) \tag{1}$$

and

$$\xi_{n+1} = (\|\xi_n\| - \|\eta_n\|) \frac{\xi_n}{\|\xi_n\|} \tag{2}$$

Since the scalar product does not change the size of the angle between the two vectors, i.e. $\|\eta_n\|$ and $\|\xi_n\| - \|\eta_n\|$, respectively, in Equation 1 and Equation 2, we get

$$\angle(\eta_{n+1}, \xi_{n+1}) = \angle\left(\frac{\eta_n}{\|\eta_n\|} + \frac{\xi_n}{\|\xi_n\|}, \frac{\xi_n}{\|\xi_n\|} \right) \leq \angle(\eta_n, \xi_n).$$

Let $A = span(\eta, \xi)$ which is a subspace of X. By definition of η_n, ξ_n, it is clear that $\eta_n, \xi_n \in A$. Put $p = min\{\|\eta\| : \eta \in S_A\}$, $q = max\{\|\eta\| : \eta \in S_A\}$, and $\angle_n = \angle(\eta_n, \xi_n)$. These assumptions show that there exist $a_n, b_n \in S_A$ such that $p = \|a_n\|$, $q = \|b_n\|$ and $\angle_n = \angle(a_n, b_n)$. Hence

$$\angle_{n+1} = \angle\left(\frac{\eta_n}{\|\eta_n\|} + \frac{\xi_n}{\|\xi_n\|}, \frac{\xi_n}{\|\xi_n\|} \right) \leq \angle(a_n + b_n, b_n) \leq \angle(a_n, b_n) = \angle_n. \tag{3}$$

Considering these facts, by comparing the triangle of the hypotenuse q with the angle $\angle_n$ and the triangle of the hypotenuse $\|a_n + b_n\|$ with the angle $\angle(a_n + b_n, b_n)$, geometrically it is easy to check that

$$q \; sin\angle_n = \|a_n + b_n\| \; sin\angle(a_n + b_n, b_n).$$

Therefore

$$sin\angle(a_n + b_n, b_n) = \frac{q}{\|a_n + b_n\|} \; sin\angle_n. \tag{4}$$

On the other hand, the assumption that $0 < \angle(\eta, \xi) \leq \frac{\pi}{2}$ implies $0 < \angle_n \leq \frac{\pi}{2}$ and deploying cosinus law gives $\sqrt{p^2 + q^2} \leq \|a_n + b_n\|$. Note that if $\angle(\eta, \xi) > \frac{\pi}{2}$, then just take $-\xi$ to get $\angle(\eta, -\xi) \leq \frac{\pi}{2}$. Combining these results (Inequality 3 and Equation 4) leads us to get

$$sin\angle_{n+1} \leq sin\angle(a_n + b_n, b_n) \leq \frac{q}{\sqrt{p^2 + q^2}} \; sin\angle_n.$$

The last inequality gives $\lim_{n\to\infty} \angle_n = \lim_{n\to\infty} \angle\left(\eta_n, \xi_n\right) = 0$. By Definition 2.2, this is possible if and only if $\left\|\frac{\eta_n}{\|\eta_n\|} - \frac{\xi_n}{\|\xi_n\|}\right\| \to 0$, that is there exist subsequences $\frac{\eta_{n_i}}{\|\eta_{n_i}\|}$ and $\frac{\xi_{n_i}}{\|\xi_{n_i}\|}$ that converge to norm-one vector in A and $\ell\left(\eta_{n_i}, \xi_{n_i}\right) \to 0$. The definition of mapping ℓ and the fact we get in Equation 1, together with the discussion of Lemma 2.3 give

$$\begin{aligned}
\ell\left(\eta_{n+1}, \xi_{n+1}\right) &= \ell\left(\frac{\eta_n}{\|\eta_n\|} + \frac{\xi_n}{\|\xi_n\|}, \frac{\xi_n}{\|\xi_n\|}\right) \\
&\leq \left\|\frac{\eta_n}{\|\eta_n\|} - \frac{\xi_n}{\|\xi_n\|}\right\| \\
&= \ell\left(\eta_n, \xi_n\right).
\end{aligned}$$

This shows that $\ell\left(\eta_n, \xi_n\right)$ is non-increasing and hence $\ell\left(\eta_n, \xi_n\right) \to 0$. □

Now we can state the main theorem of this paper.

Theorem 3.3. *Let X and Y be real Banach spaces and $f_0 : S_X \to S_Y$ be an ε-isometry mapping. $f : X \to Y$ is a linear ε-isometry if and only if $f\left(\eta_0 + \xi_0\right) = f\left(\eta_0\right) + f(\xi_0)$ for all $\eta_0, \xi_0 \in S_X$.*

Proof. The first part is just the consequence of the definition of linear mapping, thus we just prove the second part. Assume that $f\left(\eta_0 + \xi_0\right) = f\left(\eta_0\right) + f(\xi_0)$ for all $\eta_0, \xi_0 \in S_X$. Take $\eta, \xi \in X$. If $\|\eta\| = \|\xi\| = 0$, then the proof is just the consequence of the Mazur-Ulam theorem [15]. Assume that $\|\eta\| = \|\xi\| > 0$.

$$\begin{aligned}
f\left(\eta\right) + f\left(\xi\right) &= \|\eta\| f_0\left(\frac{\eta}{\|\eta\|}\right) + \|\xi\| f_0\left(\frac{\xi}{\|\xi\|}\right) \\
&= \|\eta\| f_0\left(\frac{\eta}{\|\eta\|} + \frac{\xi}{\|\xi\|}\right) \\
&= f\left(\eta + \xi\right)
\end{aligned}$$

which shows the linearity of f. Therefore, it remains only to show that f is a linear mapping for $\|\xi\| > \|\eta\| > 0$. In this case

$$
\begin{aligned}
f(\eta)+f(\xi) &= f(\eta)+f\left(\frac{\|\eta\|}{\|\xi\|}\xi\right)-f\left(\frac{\|\eta\|}{\|\xi\|}\xi\right)+f(\xi)\\
&= \|\eta\|\, f_0\left(\frac{\eta}{\|\xi\|}\right)+\|\eta\|\, f_0\left(\frac{\xi}{\|\xi\|}\right)-\|\eta\|\, f_0\left(\frac{\xi}{\|\xi\|}\right)+\|\xi\|\, f_0\left(\frac{\xi}{\|\xi\|}\right)\\
&= \|\eta\|\, f_0\left(\frac{\eta}{\|\xi\|}+\frac{\xi}{\|\xi\|}\right)+(\|\xi\|-\|\eta\|)f_0\left(\frac{\xi}{\|\xi\|}\right)\\
&= f\left(\eta+\frac{\|\eta\|}{\|\xi\|}\xi\right)+f\left(\xi-\frac{\|\eta\|}{\|\xi\|}\xi\right)
\end{aligned}
$$

Take a pair sequence $\eta_{n+1}=\eta_n+\frac{\|\eta_n\|}{\|\xi_n\|}\xi_n$ and $\xi_{n+1}=\xi_n-\frac{\|\eta_n\|}{\|\xi_n\|}\xi_n$ with $\|\eta_n\|\le\|\xi_n\|$ as in Lemma 3.2. Hence $f(\eta)+f(\xi)=f(\eta_1)+f(\xi_1)$. Similarly,

$$
\begin{aligned}
f(\eta_1)+f(\xi_1) &= f(\eta_1)+f\left(\frac{\|\eta_1\|}{\|\xi_1\|}\xi_1\right)-f\left(\frac{\|\eta_1\|}{\|\xi_1\|}\xi_1\right)+f(\xi_1)\\
&= \|\eta_1\|\, f_0\left(\frac{\eta_1}{\|\xi_1\|}\right)+\|\eta_1\|\, f_0\left(\frac{\xi_1}{\|\xi_1\|}\right)-\|\eta_1\|\, f_0\left(\frac{\xi_1}{\|\xi_1\|}\right)+\|\xi_1\|\, f_0\left(\frac{\xi_1}{\|\xi_1\|}\right)\\
&= \|\eta_1\|\, f_0\left(\frac{\eta_1}{\|\xi_1\|}+\frac{\xi_1}{\|\xi_1\|}\right)+(\|\xi_1\|-\|\eta_1\|)f_0\left(\frac{\xi_1}{\|\xi_1\|}\right)\\
&= f\left(\eta_1+\frac{\|\eta_1\|}{\|\xi_1\|}\xi_1\right)+f\left(\xi_1-\frac{\|\eta_1\|}{\|\xi_1\|}\xi_1\right)\\
&= f(\eta_2)+f(\xi_2)
\end{aligned}
$$

Repeating the process for all n we have $f(\eta)+f(\xi)=f(\eta_n)+f(\xi_n)$. Thus, the construction of η_n, ξ_n and Lemma 3.1 give

$$
\begin{aligned}
\|f(\eta)+f(\xi)-f(\eta+\xi)\| &= \|f(\eta_n)+f(\xi_n)-f(\eta_n+\xi_n)\|\\
&\le (4+4\varepsilon)\min\{\|\eta_n\|,\|\xi_n\|\}\;\ell(\eta_n,\xi_n)\\
&\le (4+4\varepsilon)(\|\eta_n\|+\|\xi_n\|)\,\ell(\eta_n,\xi_n)
\end{aligned}
$$

and hence by Lemma 3.2 we have $f(\eta)+f(\xi)=f(\eta+\xi)$. □

4. Conclusion

In this paper, we show that linear ε-isometry exists as an extension of ε-isometry on the unit sphere of a Banach spaces. This important result is in line with some of the results that have been found previously.

Acknowledgement

The authors thank OMU Functional Analysis and Function Theory Research Group (OFAFTReG) for meaningful discussions in this study. Special thank to Prof. Cenap Duyar, Prof. Birsen Duyar, and Dr. Halil Mutuk for their suggestions. The authors also thank the reviewers for their comments and suggestions in this article.

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